A theorem from Rival and Sands and reverse mathematics

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Theorem (Rival Sands)

Let $(P, <_P)$ be an infinite poset of finite width. Then there exists an infinite chain C such that every element of P is comparable to none or to infinitely many elements of C.

Moreover, if P is countable, C may be chosen so that each $p \in P$ is comparable to none or to cofinitely many elements of C.

I. Rival, B. Sands
On the adjacency of vertices to the vertices of an infinite subgraph.
J. London Math. Soc., 1980

A poset has finite width if there exists $k \in \mathbb{N}$ such that each antichain in P has size at most k

In order to analyse the previous theorem in reverse mathematics it is convenient to introduce these families of principles

RSpo_k	Each countable poset $(P, <_P)$ of width k contains an infinite chain C such that every element in P is comparable to none or to infinitely many elements in C.
sRSpo_k	Each countable poset $(P, <_P)$ of width k contains an infinite chain C such that every element in P is comparable to none or to <u>cofinitely</u> many elements in C.

Ramsey-theoretic principle?

Three measures of proximity of Rival-Sands theorem and Ramsey's theorem

1. Rival and Sands present their theorems as trade-off of Ramsey's theorem for pairs RT^2_2
2. sRSpo_k is a Ramsey-type statement
3. we prove some equivalences with ADS, the ascending/descending sequence principle, a consequence of Ramsey's theorem for pairs

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RT^2_2 | For each countable graph there exists an infinite subgraph which is either complete or totally disconnected

- instances of RSpo_k and sRSpo_k, i.e. countable comparability graphs such that the size of their totally disconnected subgraphs is bounded by some k, form a subclass of instances of RT^2_2

  RSpo_k is a (combinatorial) improvement of a more general statement proved by Rival and Sands in the same paper, whose instances are countable graphs

- solutions to RSpo_k and sRSpo_k form a subclass of the solutions of RT^2_2 (even restricted to instances of RSpo_k and sRSpo_k)

  solutions to RSpo_k and sRSpo_k give information about the adjacency relation between the inside and the outside of the solution. While solutions to RT^2_2 carry information about the adjacency relation only of the inside

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A statement of the form $\forall P(\varphi(P) \Rightarrow \exists C \psi(P, C))$ is said to be of Ramsey-type when it has the following properties:

- if $\varphi(P)$ and $\psi(P, C)$, then C must be infinite,
- if $\varphi(P)$, $\psi(P, C)$, and $D \subseteq C$ is infinite, then $\psi(P, D)$.

sRSpo_k	Each countable poset $(P, <_P)$ of width k contains an infinite chain C such that every element in P is comparable to none or to COFINITELY many elements in C.

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RSpo_k is NOT in general a Ramsey-type principle.

sRSpo_k is a Ramsey-type principle: let $(P, <_P)$ be a poset of width k, C a solution and $D \subseteq C$ be infinite. If $p \in P$ is comparable with only finitely many elements of D, then p is comparable with some elements of C and incomparable with infinitely many elements of C, contrary to the fact that C is a solution.

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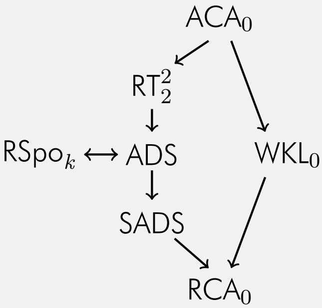
ADS | Each countable linear order $(L, <_L)$ contains an ω or an ω^* chain.

Theorem

Over RCA_0, for each $k \geq 3$, ADS is equivalent to RSpo_k



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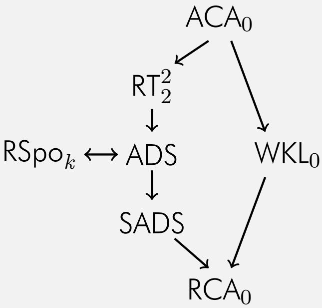


- RSpo_k is the first theorem from ordinary mathematics proved to be equivalent to ADS
- our result improves considerably the upper bound, $\Pi^1_1-\mathsf{CA}_0$, given by the original proof of RSpo_k
- we gave proofs of RSpo_k which are different combinatorially from the original proof

Key elements of the proof

1. Chain decomposition

   Kierstead proved that for each k and for each computable poset $(P, <_P)$ of width k, there exist $h \leq 5^k$ and $C_0, \ldots, C_h$ computable chains such that $P = \bigcup_{i<h} C_i$. His proof can be formalised in RCA_0

2. ADS is used only once at the very beginning of the proof to get an ascending or a descending chain A
3. starting from A, searching iteratively witnesses of the fact that ascending or descending chains are not solutions to RSpo_k, we get a solution to RSpo_k

Theorem

Over WKL_0, SADS is equivalent to RSpo_2

WKL_0 is used in the proof only to decompose a poset of width two into two chains.

Hence, over RCA_0, SADS is equivalent to the following variant of RSpo_2, $\mathsf{RSpo}_2^{\mathsf{Dec}}$:

> for each countable poset $(P, <_P)$ such that there exist two chains D and E such that $P = D \cup E$, there exists an infinite chain C such that every point in P is comparable to none or to infinitely many elements in C.

SADS | For each countable linear order $(L, <_L)$ of order type ω or ω^* or $\omega + \omega^*$ there exists an ω or ω^* chain in L.

The strong version sRSpo_2

Theorem
Over RCA_0, sRSpo_2 implies ADS

Over RCA_0, ADS is strictly stronger than RSpo_2.

Thus, sRSpo_2 is strictly stronger than RSpo_2

sRSpo_2	Each countable poset $(P, <_P)$ of width 2 contains an infinite chain C such that every element in P is comparable to none or to cofinitely many elements in C.

Theorem

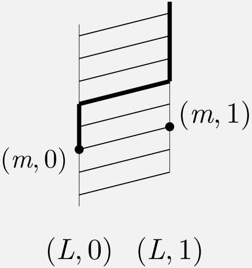
Over RCA_0, sRSpo_2 implies ADS

Proof.

Let $(L, \leq_L)$ be a linear order and let $P = (L \times 2, \leq_P)$ the order on the Cartesian product of L, so that $(\ell, i) \leq_P (m, j) \Leftrightarrow \ell \leq_L m \wedge i \leq j$. Such a poset has clearly width two, so let $C \subseteq P$ be a solution. For each $i < 2$ set $C_i = C \cap (L \times i)$.

We claim that if C_0 is infinite, then C_0 has no minimum, and can thus be refined to a descending chain. Suppose on the contrary that C_0 is infinite and that $(m, 0)$ is minimum in C_0. By definition of $<_P$ it holds that $(m, 0) <_P (m, 1)$ and $(n, 0) \mid_P (m, 1)$, for each $n >_L m$. It follows that $(m, 1)$ is incomparable with infinitely many elements of C, contrary to the assumption that C is a solution.

Similar reasoning allows us to prove that if C_1 is infinite, then C_1 has no maximum, and hence that L contains an ascending chain. □



Counterexample to an ω chain

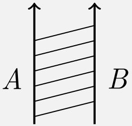
Let $(P, <_P)$ be a poset of width two.
Let A be an ω chain whose tails are not solutions.
There exists an ω chain B such that each $b \in B$ is above some $a \in A$ and incomparable with a tail of A.

- since P has width two it is enough to look at $B = \langle b_n \mid n \in \mathbb{N} \rangle$ such that

$$\forall n \exists m > n \exists \ell > n \, (a_m <_P b_\ell \wedge a_{m+1} \mid_P b_\ell)$$

  We call such B a counterexample to A

- if P has width two there exists $f\colon \mathbb{N} \to \mathbb{N}$ which enumerates B. This f allows to find counterexamples to ascending chains uniformly.

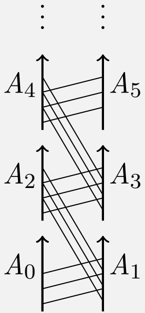


Theorem

Over RCA_0, ADS + (some) arithmetical induction implies sRSpo_2.

Idea of the proof.

Let $(P, <_P)$ be a poset of width two. Assume that P does not have any solution to sRSpo_2. Assume A_0 is an ascending chain in P (if A_0 is descending consider $(P, >_P)$).



Since A_0 is not a solution, there exists a counterexample A_1. Since A_1 is not a solution, so there exists a counterexample A_2. And so on. We thus get a sequence $\langle A_n \mid n \in \mathbb{N} \rangle$ of ascending chains as in the picture.
To make sure the sequence itself exists in RCA_0 we define a uniform procedure to define it. To this end it is crucial that for each n and m the m^{th} element of A_n can be chosen after inspecting only initials segments of $A_0, \ldots, A_{n-1}$.
Let S be an ascending chain with one element from each A_n. Then S is a solution: each element of P is comparable with cofinitely many elements of S. □

Question
What is the strength of sRSpo_k?

$\Pi^1_1 - \mathrm{CA}_0$ is the upper bound and ADS the lower bound.

The proof for sRSpo_2 sketched above exploits crucially the fact that the poset has width two