Limiting density and free structures

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Group presentations

Definition. A *presentation* for a group G has the form $\bar{a}|R$, where $\bar{a}$ is a generating tuple and R is a set of words $w(\bar{a})$, *relators*, s.t. the equalities true of $\bar{a}$ in G are just the ones that follow logically from the group axioms and the identities $w(\bar{a}) = e$, for $w(\bar{a}) \in R$.

Gromov

Question (Gromov, 1987). What does a typical group look like?

Gromov had in mind finitely presented groups. He described informally, in terms of limiting density, what it might mean for the typical group to have some property.

Gromov stated that the typical group is "hyperbolic." Ol'shanskii (1992) made Gromov's notion of limiting density precise and gave a precise proof.

Allowable presentations

There are different ways to make precise the notion of limiting density. We must say what are allowable presentations. We fix the number of generators. We also put conditions on the relators:

1. We restrict the *number* of relators—without this, we risk that the typical group will be trivial,

2. We restrict the *form* of the relators—to make calculations easier.

Allowable presentations

Ol'shanskii: Allowable presentations have m generators, k reduced relators, for fixed $m, k \geq 1$.

Kapovich and Schupp (2005): Allowable presentations have m generators, for fixed $m \geq 1$, and a single reduced relator.

Limiting density

Fix the set P of allowable presentations. Let $Q \subseteq P$—usually Q is the set of presentations for which the resulting groups have some interesting property.

Definition.

1. Let P_s be the set of presentations for which the relators all have length at most s.
2. Let $P_s(Q) = P_s \cap Q$.
3. The *limiting density for* Q is $\lim_{s\to\infty} \frac{|P_s(Q)|}{|P_s|}$, if this exists.

The typical group resembles a free group

By the result of Gromov, Ol'shanskii, the typical group is hyperbolic. Kapovich and Schupp showed that in the typical group, all minimal generating tuples are "Nielsen equivalent." These are properties of free groups.

Off-hand comment (Fine). In the limiting density sense, all groups look free.

Fine's comment can be made precise, for properties that can be expressed by elementary first order sentences—we identify φ with the property of satisfying φ.

Conjecture

Conjecture (2013). Let P consist of group presentations with $m \geq 2$ generators, one reduced relator. For any elementary first order sentence φ,

1. The limiting density for φ exists,
2. the value is always 0 or 1,
3. the value is 1 iff φ is true in the non-Abelian free groups.

Remark. Sela (2001-2006) showed that the elementary first order theory of F_m (the free group on m generators) is the same for all $m \geq 2$—see also Kharlampovich and Miasnikov (2006).

Evidence for the Conjecture

Schupp. "It looks plausible."

Theorem (Coulon, Ho, Logan). For all $m \geq 2$ and $k \geq 1$, if an existential sentence φ has limiting density 1, then it is true in the non-Abelian free groups.

Proof involves machinery of Sela.

Algebraic varieties

Definition. An *algebraic language* consists of function symbols and constants.

Definition. A *variety* $\mathcal{V}$ is the class of structures for an algebraic language satisfying a set of axioms of the form $(\forall \bar{x}) t(\bar{x}) = t'(\bar{x})$.

Definition. A *presentation* for $\mathcal{A} \in \mathcal{V}$ has form $\bar{a}|R$, where $\bar{a}$ is a generating tuple and R is a set of identities on $\bar{a}$ s.t. the identities true in $\mathcal{A}$ are the ones proved from R and the axioms for $\mathcal{V}$.

For an m-tuple $\bar{a}$, there is a well-defined free structure in $\mathcal{V}$ with generators $\bar{a}$. Any member of the variety that is generated by $\bar{a}$ is a homomorphic image of the free structure.

Analogue of Gromov's question

For an algebraic variety $\mathcal{V}$, we may ask the following.

Question: What does a typical member of $\mathcal{V}$ look like?

Very general lemmas

Lemma 1. $Q_1 \cup Q_2$ has limiting density 0 iff Q_1, Q_2 both have limiting density 0. Similarly, for any finite union.

Proof. $\frac{|P_s(Q_i)|}{|P_s|} \leq \frac{|P_s(Q_1 \cup Q_2)|}{|P_s|} \leq \frac{|P_s(Q_1)|}{|P_s|} + \frac{|P_s(Q_2)|}{|P_s|}$.

Lemma 2. Q has limiting density 0 iff Q^c has limiting density 1, where $Q^c = P - Q$.

Proof. $\frac{|P_s(Q)|}{|P_s|} + \frac{|P_s(Q^c)|}{|P_s|} = 1$.

Bijective structures

The language of *bijective structures* consists of two unary operation symbols S, S^{-1}.

The axioms are:

1. $(\forall x) SS^{-1}(x) = x$,
2. $(\forall x)(S^{-1}S(x) = x$.

Important sentences

Lemma. In bijective structures, every sentence is equivalent to a Boolean combination of the following:

1. $\alpha(n, k)$, saying that there are at least k cycles of size n—the cycles consist of distinct elements,
2. $\beta(n)$, saying that there is a chain of length at least n.

To prove this, we think how to distinguish the countable saturated models. (We count finite models as saturated.)

First case, 1 generator

We consider bijective structures with a single generator a.

Fact: These have one of two forms:

1. Z—an infinite chain,
2. Z_m—a cycle of size m.

Note: Any element serves as a generator.

Truth of basic sentences

Recall: $\alpha(n,k)$—there are k cycles of size n,
$\beta(n)$—there is a chain of length n.

Lemma.

1. For $k > 1$, $\alpha(n,k)$ is not true in Z or Z_m,
2. $\alpha(n,1)$ is true only in Z_n,
3. $\beta(n)$ is true in Z; it is true in Z_m iff $m > n$.

Fact: In bijective structures with 1 generator, every sentence is equivalent to a Boolean combination of sentences $\alpha(n,1)$. In particular, $\beta(n)$ is equivalent to $\neg\bigvee_{m\leq n}\alpha(m,1)$.

Conditions on identity

We consider presentations with a single identity. Moreover, we require that the identity have form $t(a) = a$—all occurrences of S, S^{-1} on one side.

At first, we require that the identity be *reduced*—no occurrence of S next to S^{-1}.

Lemmas

Lemma. There is a unique allowable presentation for Z. For each $n \geq 1$, there are exactly two reduced identities that give Z_n; namely, $S^n(a)$ and $S^{-n}(a)$.

Lemma. The sentences $\alpha(n, 1)$ have limiting density 0. (They are false in Z.)

Then the general lemmas on disjunctions and negations give the following.

Theorem. For all sentences φ, the limiting density is 1 if $Z \models \varphi$ and 0 otherwise.

Identity not reduced

Let P be the set of presentations with a single identity and a single identity, not necessarily reduced.

Theorem. For all sentences φ, the limiting density is 1 iff $Z \models \varphi$.

Ingredients of Proof: Again we can show that $\alpha(n, 1)$ has limiting density 0. Here $|P_s(\alpha(n, 1))|$ involves a sum of combinations. However, the last term turns out to be greater than the sum of the earlier terms. We use Pascal's Identity, plus an approximation of $\binom{2k}{k}$.

m generators, k reduced identities

Let $\bar{a}$ be an m-tuple of generators. Consider k identities, all of form $S^{\pm n}(a_i) = a_j$.

Definition. $a_i \sim a_j$ if some identity (provable from the chosen ones) has form $a_i = a_j$ or $S^{\pm n}(a_i) = a_j$ or $S^{\pm n}(a_j) = a_i$.

Each equivalence class gives a single Z or a finite cycle. We get an n-cycle if among the identities provable from the chosen ones on the equivalence class is $S^n(a_i) = a_i$ for some i. (The chosen identities might be $S^{n_1}(a_{i_1}) = a_{i_2}, S^{n_2}(a_{i_2}) = a_{i_3}, \ldots, S^{n_k}(a_{i_k}) = a_{i_1}$, where $n = n_1 + n_2 + \ldots + n_i$.) An equivalence class gives rise to a copy of Z if it does not give a finite cycle.

Remarks: The sentence $\alpha(n, 1)$ is true if at least one equivalence class gives an n-cycle. The free structure is Z^m. The sentences $\alpha(n, 1)$ are all false in Z^m.

Sets of presentations

Fact: The sentence $\alpha(n, 1)$ cannot be true unless the presentation includes at least one identity of form $S^r(a_i) = a_j$, where $1 \leq r \leq n$.

There are only finitely many such identities. We show that the typical presentation avoids them all.

Limiting density for $\alpha(n, 1)$

Lemma.

1. The set of presentations that include a specific identity has limiting density 0.
2. The set of presentations that involve at least one of a finite set of identities has limiting density 0.
3. The set of presentations for structures satisfying $\alpha(n, 1)$ has limiting density 0.

Consequences

Lemma.

1. The limiting density for $\alpha(n, k)$ is 0 for all n, k.

2. The limiting density for $\beta(n)$ is 1 for all n.

Theorem. For m generators and k identities, for all sentences φ, the limiting density is 1 iff φ is true in the free structure on m generators.

Abelian groups

The Abelian groups form a variety.

For 1 generator, the structures we get are the same as for the variety of all groups.

The free Abelian group on m generators is Z^m.

Important sentences

Szmielew (1955) arrived at elementary invariants for Abelian groups via elimination of quantifiers. Eklof and Fisher (1972) obtained invariants more simply, using saturation. Here is a complete set of elementary invariants. We write $G[p]$ for the set $\{x \in G : px = 0\}$. For a prime p, this consists of the identity and the elements of order p.

1. $\alpha(p, n, k)$, saying $|p^n G| \geq k$,
2. $\beta(p, n, k)$, saying $dim(p^n G/_{p^{n+1}G}) \geq k$,
3. $\gamma(p, n, k)$, saying $dim(p^n G[p]) \geq k$,
4. $\delta(p, n, k)$, saying $dim(p^n G[p]/_{p^{n+1}G[p]}) \geq k$.

One generator, one reduced relator

The free Abelian group on one generator is Z—same as for groups as a whole.

We focus on the sentences $\beta(p, n, 1)$, saying essentially that there is an element divisible by p^n and not by p^{n+1}.

Lemma.

1. $Z \models \beta(p, n, 1)$
2. $C_m \models \beta(p, n, 1)$ iff $p^{n+1}|m$.

Results

The Conjecture fails. The Σ_2-sentences $\beta(p, n, 1)$ are true in the free Abelian group Z. The limiting density is not 1.

Proposition. For all primes p and all natural numbers n, $\beta(p, n, 1)$ has limiting density $\frac{1}{p^{n+1}}$.

Proof: We have $|P_s| = 1 + 2s$, and $|P_s(\beta(p, n, 1))| = 1 + 2d$, where $d(s) = \lfloor \frac{s}{p^{n+1}} \rfloor$. As $s \to \infty$, $\frac{1}{s}\lfloor \frac{s}{p^{n+1}} \rfloor \to \frac{1}{p^{n+1}}$.

Relator not reduced

Proposition. The limiting density for $\beta(p, n, 1)$ is still $\frac{1}{p^{n+1}}$.

Idea of proof: Partition the set of relators as follows:

A—those that give Z,

B_r—those that give C_m for some $m \equiv_{p^{n+1}} r$, $r < p^{n+1}$.

We can show that A has density 0, and for all $r < p^{n+1}$, the limiting density of B_r is the same $\frac{1}{p^{n+1}}$. The limiting density for $\beta(p, n, 1)$ is that of B_0.

Two generators

Consider Abelian groups with two generators a, b, and a single reduced relator.

The sentence $\beta(p, n, 1)$ is true in all of these groups. The conjecture fails for the sentences $\beta(p, n, 2)$.

Proposition. The sentence $\beta(p, n, 2)$ is true in the free Abelian group $Z \oplus Z$, but it has limiting density 0.

Unary functions

We consider the variety $\mathcal{V}$ of unary functions, in the language with a single function symbol F, no axioms.

One generator, one identity

We consider a single generator a, and a single identity of form $F^{(n)}(a) = F^{(m)}(a)$. The elements of $\mathcal{V}$ that we get have the following forms:

1. An ω-chain,
2. A finite cycle,
3. A finite chain leading to a finite cycle.

Result

The sentence φ saying that the function is $1-1$ is true in the free structure ω and in the finite cycles, but false in the finite chains leading to finite cycles.

Proposition. The sentence φ is $1-1$ is true in the free structure ω, but it has limiting density 0.

Question

Question: Under what conditions on a variety and allowable presentations is it true that for all φ, the limiting density is 1 iff φ is true in the free structure and 0 otherwise?