THE DETERMINED PROPERTY OF BAIRE IN REVERSE MATH

ERIC P. ASTOR, DAMIR DZHAFAROV, ANTONIO MONTALBÁN, REED SOLOMON, AND LINDA BROWN WESTRICK

ABSTRACT. We define the notion of a determined Borel code in reverse math, and consider the principle DPB, which states that every determined Borel set has the property of Baire. We show that this principle is strictly weaker than ATR_0. Any ω-model of DPB must be closed under hyperarithmetic reduction, but DPB is not a theory of hyperarithmetic analysis. We show that whenever $M \subseteq 2^\omega$ is the second-order part of an ω-model of DPB, then for every $Z \in M$, there is a $G \in M$ such that G is Δ^1_1-generic relative to Z.

The program of reverse math aims to quantify the strength of the various axioms and theorems of ordinary mathematics by assuming only a weak base theory (RCA_0) and then determining which axioms and theorems can prove which others over that weak base. Five robust systems emerged, (in order of strength, $\mathsf{RCA}_0, \mathsf{WKL}, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1\text{-}\mathsf{CA}$) with most theorems of ordinary mathematics being equivalent to one of these five (earning this group the moniker "the big five"). The standard reference is [Sim09]. In recent decades, most work in reverse math has focused on the theorems that do not belong to the big five but are in the vicinity of ACA_0. Here we discuss two principles which are outside of the big five and located in the general vicinity of ATR_0: the *property of Baire for determined Borel sets* (DPB) and the *Borel dual Ramsey theorem for 3 partitions and ℓ colors* ($\mathsf{Borel}\text{-}\mathsf{DRT}^3_\ell$). Both principles involve Borel sets.

Our motivation is to make it possible to give a meaningful reverse math analysis of theorems whose statements involve Borel sets. The way that Borel sets are usually defined in reverse math forces many theorems that even mention a Borel set to imply ATR_0, in an unsatisfactory sense made precise in [DFSW17]. Here we propose another definition for a Borel set in reverse math, distinguished from the original by the terminology *determined Borel set*, and to put bounds on the strength of the statement

$$\mathsf{DPB}: \text{“Every determined Borel set has the property of Baire”}$$

This statement should be compared with the usual "Every Borel set has the property of Baire", which [DFSW17] showed is equivalent to ATR_0 for aforementioned empty reasons. In contrast, working with DPB requires working with hyperarithmetic generics, giving this theorem more thematic content. While we do not claim that DPB is the "right" formalization of the principle that Borel sets have the Baire property, it is a step in that direction.

We show that over RCA_0, DPB is implied by ATR_0 and implies $\mathsf{L}_{\omega_1,\omega}\text{-}\mathsf{CA}$. Our first main theorems say that both implications are strict.

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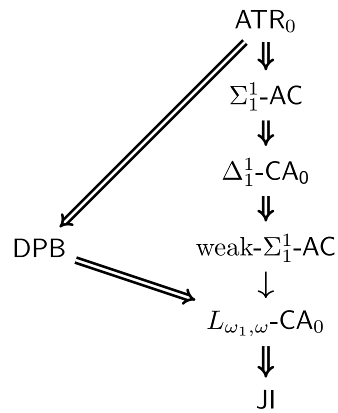

FIGURE 1. DPB, ATR_0, and some theories of hyperarithmetic analysis. The new results are those concerning DPB. A double arrow indicates a strict implication.

Theorem 0.1. *There is an ω-model of* DPB *in which* ATR_0 *fails.*

Theorem 0.2. *There is an ω-model of* $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ *in which* DPB *fails. In fact, HYP is such an ω-model.*

This establishes that DPB is located in the general vicinity of the *theories of hyperarithmetic analysis*, a mostly linearly ordered collection of logical principles which are strong enough to support hyperarithmetic reduction, but too weak to imply the existence of jump hierarchies. With the exception of Jullien's indecomposability theorem [Mon06], no theorems of ordinary mathematics are known to exist in this space. And the only known statement of hyperarithmetic analysis that is not linearly ordered with the others is the arithmetic Bolzano-Weierstrass theorem (see [Fri75], [Con12]). Now, DPB is not a theory of hyperarithmetic analysis because it does not hold in HYP. But these theories of hyperarithmetic analysis are the closest principles to DPB that have already been studied.

To elaborate on the factors preventing to DPB from being a theory of hyperarithmetic analysis, we prove the following generalization of Theorem 0.2 above, establishing that hyperarithmetic generics must appear in any ω-model of DPB.

Theorem 0.3. *If $\mathcal{M}$ is an ω-model of* DPB*, then for any $Z \in M$, there is a $G \in M$ that is $\Delta^1_1(Z)$-generic.*

As an application, we use DPB to analyze the theorem $\mathsf{Borel\text{-}DRT}^3_\ell$, whose statement contains no concept of mathematical logic apart from that of Borel sets. (The statement of this theorem can be found in Section 7.) We show that, under appropriate formalization, $\mathsf{Borel\text{-}DRT}^3_\ell$ is strictly weaker than ATR_0 and shares some properties with the theories of hyperarithmetic analysis. It is left open whether $\mathsf{Borel\text{-}DRT}^3_\ell$ is a statement of hyperarithmetic analysis.

Theorem 0.4. *For any finite $\ell \geq 2$, the principle* $\mathsf{Borel\text{-}DRT}^3_\ell$ *is strictly implied by* ATR_0*. Any ω-model of* $\mathsf{Borel\text{-}DRT}^3_\ell$ *is closed under hyperarithmetic reduction.*

The first section gives the preliminaries. In Section 2 we give the definition of a determined Borel code and prove its basic properties. In Section 3 we construct an ω-model to separate DPB from ATR_0. In Section 4 we develop the machinery of *decorating trees* which will be used in Sections 5 and 6. In Section 5, we prove that DPB does not hold in HYP. In Section 6, we prove Theorem 0.3. This is a strictly stronger theorem than the one proved in Section 5, but also a bit longer to prove, so Section 5 could be regarded as a warm-up. In Section 7 we discuss the implications of all this for the Borel dual Ramsey theorem. Section 8 has the open questions.

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1. Preliminaries

1.1. **Notation, Borel sets and Borel codes.** We typically denote elements of $\omega^{<\omega}$ by σ, τ and elements of $2^{<\omega}$ by p, q. We write $p \preceq q$ to indicate that p is an initial segment of q, with $\prec$ if $p \neq q$. The empty string is denoted by λ. A string with a single component of value $n \in \omega$ is denoted by $\langle n \rangle$. String concatenation is denoted by $\sigma^\frown\tau$. Usually we write $\sigma^\frown n$ instead of the more technically correct but uglier $\sigma^\frown\langle n\rangle$.

If U is a set of strings (for example, a tree, or a coded open subset of 2^ω), and σ is any string, we write $\sigma^\frown U$ to mean $\{\sigma^\frown\tau : \tau \in U\}$. If T is a tree and $\sigma \in T$, we write T_σ to mean $\{\tau : \sigma^\frown\tau \in T\}$.

The *Borel* subsets of a topological space are the smallest collection which contains the open sets and is closed under complements and countable unions (and thus countable intersections).

A *Borel code* is a well-founded tree $T \subseteq \omega^{<\omega}$ whose leaves are labeled by basic open sets or their complements, and whose inner nodes are labeled by $\cup$ or $\cap$. The Borel set associated to a Borel code is defined by induction, interpreting the labels in the obvious way. Any Borel set can be represented this way, by applying DeMorgan's laws to push any complementation out to the leaves.

1.2. **Ordinal notations, alternating and ranked trees.** We assume the reader is familiar with ordinal notations and overflow arguments. A standard reference is [Sac90a]. Recall that there is a computable procedure p which, on input $a \in \mathcal{O}$, gives an index for $W_{p(a)} = \{b \in \mathcal{O} : b <_{\mathcal{O}} a\}$. By overflow, $W_{p(a)}$ also produces reasonable-looking stuff for many $a \notin \mathcal{O}$.

Definition 1.1. *For arbitrary $a, b \in \mathbb{N}$, define $b <_* a$ to mean $b \in W_{p(a)}$. Let $\mathcal{O}^* \subseteq \mathbb{N}$ be defined by*

$$\mathcal{O}^* = \{a \in \mathbb{N} : 1 \in W_{p(a)}, W_{p(a)} \text{ is linearly ordered by } <_* \text{ and } W_{p(a)} \text{ has no hyperarithmetic } <_*\text{-descending sequences}\}.$$

If $a^* \in \mathcal{O}^* \setminus \mathcal{O}$, then $W_{p(a^*)}$ has an initial segment $\{b \in \mathcal{O} : b <_* a^*\}$ of length ω_1^{ck}.

According to the conventions governing ordinal notations, if $b \in \mathcal{O}$, the successor of b is 2^b. This is cumbersome when adding a finite number to the ordinal represented by b, so instead we write $b + k$ to mean take the kth successor of b. It does not make sense to ever add ordinals using the ordinary addition on $\mathbb{N}$, so this should not create confusion. Also, sometimes we will take a fixed but unspecified number of successors of b, and the result is denoted $b + O(1)$.

The following definition of a ranking for a tree is looser than given by some authors. We only require that the notations decrease, rather than the strong requirement that $\rho(\sigma) = \sup_n(\rho(\sigma^\frown n) + 1)$. Additionally, it is technically convenient for us to assume that leaves have the smallest possible rank, but nothing serious hinges on this.

Definition 1.2. *If $T \subseteq \omega^{<\omega}$ is any tree, and $\rho : T \to \mathcal{O}^*$, we say that ρ* ranks *T if*

(1) for all σ and n such that $\sigma^\frown n \in T$, we have $\rho(\sigma^\frown n) <_* \rho(\sigma)$, and
(2) for each leaf $\sigma \in T$, $\rho(\sigma) = 1$.

If T is ranked by ρ and $\rho(\lambda) = a$, we say that T is a-ranked *by ρ.*

If T is a ranked tree and the name of the ranking function is not explicitly given, then its name is ρ_T.

Trees appear for us in two contexts: as codes for formulas of $L_{\omega_1,\omega}$ and codes for Borel sets. In both cases, interior nodes are labeled with one of $\{\cap, \cup\}$. The nicest codes alternate these.

Definition 1.3. *If $T \subseteq \omega^{<\omega}$ is a tree with a labeling function ℓ then we say (T, ℓ)* alternates *if for every $\sigma^\frown n \in T$, we have $\ell(\sigma) \neq \ell(\sigma^\frown n)$.*

The main point about alternating trees is that it is always safe to assume that we have them. If we start with a labeled, a-ranked tree, we can effectively transform it into an alternating a-ranked tree, with no effect on the logic of the tree (assuming that whatever model we are working in does not contain any paths, if the tree is truly ill-founded.)

1.3. **Borel sets in reverse math.** In reverse math, open subsets of 2^ω are represented by sets of strings $p \in 2^{<\omega}$. If U is such a code, we will abuse notation and write $X \in U$ to mean that for some $p \in U$, $p \prec X$. This is in addition to also sometimes speaking of the strings $p \in U$. Context will tell which usage is meant.

For arbitrary Borel sets, we will make a more careful distinction between code and object. We restrict attention to Borel subsets of 2^ω. A clopen subset C of 2^ω is represented by an element of ω which canonically codes a finite subset $F \subseteq 2^{<\omega}$. As above, for $X \in 2^\omega$, we say $X \in C$ if and only if $p \prec X$ for some $p \in F$. A code for C as a clopen set gives more information about C than an open code for the same set, because the number of elements of F is computable from the code. Effectively in a standard code for a clopen set, one can find a standard code for its complement.

We take the following as the definition of a (labeled) Borel code in reverse math.

Definition 1.4. *A* labeled Borel code *is a well-founded tree $T \subseteq \omega^{<\omega}$, together with a function ℓ whose domain is T, such that if σ is an interior node, $\ell(\sigma)$ is either $\cup$ or $\cap$, and if σ is a leaf, $\ell(\sigma)$ is a standard code for a clopen subset of 2^ω.*

We call this a *labeled Borel code* instead of a *Borel code*, because we have added a labeling function to the original definition to improve readability.[1] If $\ell(\sigma) = \cup$

[1]The original definition of a Borel code in reverse math [Sim09] is a well-founded tree T such that for exactly one $m \in \omega$, $\langle m \rangle \in T$.

Some conventions are then adopted: if $\langle m \rangle \in T$ is a leaf, then T represents a clopen set coded by m according to a standard computable look-up; if $\langle m \rangle$ is not a leaf, then T represents a union we may simply say "σ is a union node", and similarly for $\cap$. We will also usually suppress mention of ℓ, in an abuse of notation.

If T is a labeled Borel code and $X \in 2^\omega$, the existence of an *evaluation map* is used to determine whether X is in the set coded by T.

Definition 1.5. *If T is a labeled Borel code and $X \in 2^\omega$, an* evaluation map *for $X \in T$ is a function $f : T \to \{0,1\}$ such that*

- *If σ is a leaf, $f(\sigma) = 1$ if and only if X is in the clopen set coded by $\ell(\sigma)$.*
- *If σ is a union node, $f(\sigma) = 1$ if and only if $f(\sigma^\frown n) = 1$ for some $n \in \omega$.*
- *If σ is an intersection node, $f(\sigma) = 1$ if and only if $f(\sigma^\frown n) = 1$ for all $n \in \omega$.*

We say that X is in the set coded by T, denoted $X \in |T|$, if there is an evaluation map f for X in T such that $f(\lambda) = 1$.

Note that $X \in |T|$ is a Σ^1_1 statement. Because evaluation maps are naturally constructed by arithmetic transfinite recursion, ATR_0 proves that if T is a Borel code and $X \in 2^\omega$, there is an evaluation map f for X in T. Furthermore, ACA_0 proves that if an evaluation map exists, then it is unique.

Because we are considering these definitions in the context of reverse math, there will sometimes be an ill-founded T which a model thinks is well-founded. In these cases, the statement $X \in |T|$ is meaningful inside the model, or in the context of a proof inside second order arithmetic, but is not meaningful outside a model. However, the criteria defining what it means to be an evaluation map are absolute, so we can and will construct evaluation maps on ill-founded but otherwise coherent labeled Borel codes. If T is ill-founded, we will never use the notation $|T|$ outside of a model. But if T is well-founded, then every X has a unique evaluation map in T. In that case we give the notation "$|T|$" the obvious meaning of

$$\{X : \text{the unique evaluation map } f \text{ for } X \text{ in } T \text{ satisfies } f(\lambda) = 1\}$$

when we refer to it outside the context of a model.

Operations on Borel sets are carried out easily. Observe that the operation which corresponds to complementation on a labeled Borel code is primitive recursive: just swap all the $\cup$ and $\cap$ labels, and replace every clopen leaf label with its complementary label.

Definition 1.6. *If (T, ℓ) is a labeled Borel code, let (T, ℓ^c) denote the labeled Borel code whose tree is the same, and whose labeling ℓ^c is complementary to ℓ as described above.*

Continuing the abuse of notation, if T is used to refer to some (T, ℓ), then T^c will be shorthand for (T, ℓ^c). Observe that RCA_0 proves that if T is a labeled Borel code, then T^c is a labeled Borel code. Similarly, if $(T_n)_{n\in\omega}$ is a sequence of labeled Borel codes, in RCA_0 we can construct a code for the intersection or union of these sets in the obvious effective way, and RCA_0 will prove that the result is a labeled Borel code.

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or intersection according to the parity of m, and the sets to be thus combined are those coded by the subtrees $T_n = \{\langle n\rangle^\frown\sigma : \langle m, n\rangle^\frown\sigma \in T\}$. Classically, one can translate easily between this definition and the definition of Borel code given above, but one direction of the translation requires ACA_0 because one cannot effectively determine when a node is a leaf. All the principles considered in this paper will imply ACA_0 over RCA_0, so nothing will be muddled, but for the sake of fastidious readers, we will always call these *labeled Borel codes* to acknowledge the distinction.

1.4. **On the maxim that "Borel sets need ATR_0".** Because making meaning out of a standard (labeled) Borel code requires evaluation maps to be around, ATR_0 is typically taken as the base theory when evaluating theorems involving Borel sets. Even when ATR_0 is not taken as the base theory, theorems involving Borel sets tend to imply ATR_0. The probable reason for this was observed in [DFSW17].

Theorem 1.7 ([DFSW17])**.** *In RCA_0, the statement "For every Borel code T, there exists X such that $X \in |T|$ or $X \in |T^c|$" implies ATR_0.*

The strength comes from the fact that this statement is asserting the existence of an evaluation map for X in T. If f is an evaluation map for X in T, then $1 - f$ is an evaluation map for X in T^c.[2]

Restatement (of Theorem 1.7)**.** *The statement "For every Borel set, either it or its complement is nonempty" is equivalent to ATR_0 over RCA_0.*

This can make the reverse math of some standard theorems about Borel sets feel rather empty. Here is an example. Recall that a set $A \subseteq 2^\omega$ has the *property of Baire* if it differs from an open set by a meager set. That is, there are open sets U and $\{D_n\}_{n\in\omega}$ such that each D_n is dense, and for all $X \in \cap_n D_n$, $X \in U \Leftrightarrow X \in A$. A basic proposition is that every Borel set has the property of Baire, but what is the strength of that proposition in reverse math? In [DFSW17], the relevant notions were formalized as follows.

Definition 1.8. *A* Baire code *is a collection of open sets $U, V, \{D_n\}_{n\in\omega}$ such that $U \cap V = \emptyset$ and the sets $U \cup V$ and D_n are dense.*

The statement BP below formalizes the proposition "Every Borel set has the property of Baire."

Definition 1.9. *If T is a Borel code and $U, V, \{D_n\}$ is a Baire code, we say that $U, V, \{D_n\}$ is a* Baire approximation *to T if for all $X \in \cap_n D_n$, $X \in U \Rightarrow X \in |T|$ and $X \in V \Rightarrow X \in |T^c|$.*

Definition 1.10. *Let BP denote the statement "Every Borel code has a Baire approximation."*

Proposition 1.11. [DFSW17] *In RCA_0, ATR_0 is equivalent to BP.*

Proof. ($\Rightarrow$) The standard proof uses arithmetic transfinite recursion.
($\Leftarrow$) If a set has the property of Baire, either it or its complement is nonempty. $\square$

The reverse direction of this proof is highly unsatisfactory. The purpose of this paper is to propose a variant on the definition of a Borel set which avoids this and similar unsatisfactory reversals to ATR_0.

1.5. **Some landmarks between ATR_0 and JI.** We will end up placing a variant of BP somewhere in a zoo which exists just below ATR_0. Much of what is known about this region concerns theories, such as Δ^1_1-CA_0, whose ω-models are closed under join, hyperarithmetic reduction, and not much more.

Definition 1.12. *A* statement of hyperarithmetic analysis *is any statement S such that*

[2] The statement in [DFSW17] is for original Borel codes, but the proof of the theorem remains valid for labeled Borel codes.

(1) *whenever $\mathcal{M}$ is an ω-model which satisfies S, its second-order part M is closed under hyperarithmetic reduction.*
(2) *For every Y, $HYP(Y)$ is the second-order part of an ω-model of S, where $HYP(Y) = \{X : X \leq_h Y\}$.*

A theory of hyperarithmetic analysis *is any theory which satisfies the same requirements as above.*

It would be tempting to hope that there would be some theory of hyperarithmetic analysis whose ω-models are *exactly* the Turing ideals which are closed under hyperarithmetic reduction, in analogy to the theorems characterizing the ω-models of RCA_0 as the Turing ideals, the ω-models of WKL_0 as the Scott ideals, and the ω-models of ACA_0 as the Turing ideals closed under arithmetic reduction. However, no such theory can exist.

Theorem 1.13. [VW77] *For every theory T, all of whose ω-models are closed under hyperarithmetic reduction, there is a strictly weaker theory T', all of whose ω-models are also closed under hyperarithmetic reduction, and which has more ω-models than T.*

Therefore, we are stuck with an infinitely descending zoo of statements/theorems of hyperarithmetic analysis.

One theory of hyperarithmetic analysis is most relevant to us. Recall that a formula of $L_{\omega_1,\omega}$ is a formula constructed from the usual building blocks of first-order logic, together with countably infinite conjunctions and disjunctions. In a language which contains no atomic formulas other than `true` and `false`, a formula of $L_{\omega_1,\omega}$ is just a well-founded tree whose interior nodes are labeled with either $\cup$ (infinite disjunction) or $\cap$ (infinite conjunction), and whose leaves are labeled with either `true` or `false`. An *evaluation map* for a formula of $L_{\omega_1,\omega}$ is defined the same as an evaluation map for an element X in a Borel code T, except that the evaluation map must satisfy $f(\sigma) = 1$ if $\ell(\sigma) = \texttt{true}$ and $f(\sigma) = 0$ if $\ell(\sigma) = \texttt{false}$. A formula of $L_{\omega_1,\omega}$ is *determined* if it has an evaluation map. Classically, every formula of $L_{\omega_1,\omega}$ is determined, but in weaker theories the witnessing function could fail to exist. A formula is called *true* if it has a witnessing function which maps the formula itself to `true`.

The following definition and result essentially appear in [Mon06], where $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ goes by the name CDG-CA, and is stated in terms of games. The name $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ and the definition given here were introduced in [Mon09].

Definition 1.14 (similar to [Mon06])**.** *The principle $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ is this statement: If $\{\phi_i : i \in \mathbb{N}\}$ is a sequence of determined $L_{\omega_1,\omega}$ formulas, then the set $X = \{i : \phi_i \text{ is true}\}$ exists.*

Theorem 1.15 (essentially [Mon06])**.** *The principle $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ is a statement of hyperarithmetic analysis.*

2. Determined Borel codes

We propose the following variation on the definition of a Borel code. We shall see that when this variant is used, the unsatisfactory shortcut in Proposition 1.11 vanishes, and indeed the reversal no longer holds.

Definition 2.1. *A labeled Borel code T is called* determined *if every $X \in 2^\omega$ has an evaluation map in T. A* determined Borel code *is a labeled Borel code that is determined.*

When we formalize statements in reverse math, in order to not conflict with existing convention, we will say *determined Borel set* to indicate when the formalized version of the statement should call for a determined Borel code.

The following facts are immediate.

Proposition 2.2. *In RCA_0,*

(1) If T is a determined Borel code, then T^c is also a determined Borel code.
(2) For every determined Borel set A and $X \in 2^\omega$, either $X \in A$ or $X \notin A$.

With only a slight amount of effort, we also have the following.

Proposition 2.3. *In RCA_0, if A is a determined Borel set and $h : 2^\omega \to 2^\omega$ is continuous, then $h^{-1}(A)$ is a determined Borel set.*

Proof. Let T be a determined Borel code and $h : 2^\omega \to 2^\omega$ a continuous function. Define S by starting with $S = T$ and modifying each leaf $\sigma \in T$ as follows:

(1) In S, σ is a union.
(2) For each n, $\sigma^\frown n \in S$ and is a leaf.
(3) If U is the clopen set attached to σ in T, let $\sigma^\frown n$ be labeled with a code for the clopen subset of $h^{-1}(U)$ inferrable from the first n bits of partial information about h.

We claim that S is determined and $X \in |S|$ if and only if $h(X) \in |T|$. Let f be an evaluation map for $h(X)$ in T. We claim that f can be extended to an evaluation map for X in S by adding $f(\sigma^\frown n) = 1$ if and only if X is in the clopen set attached to $\sigma^\frown n$ in S. One only needs to check that the logic of the evaluation map is correct at each σ which was a leaf in T. □

The fact that Borel sets are closed under countable union, which was trivial using the standard definition of a Borel set, has quite some power for determined Borel sets.

Proposition 2.4. *In RCA_0, the statement "A countable union of determined Borel sets is a determined Borel set" is equivalent to $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$.*

Proof. If $\{T^k : k \in \mathbb{N}\}$ are determined Borel codes, and $T = \{\lambda\} \cup \{\langle k\rangle^\frown \sigma : \sigma \in T^k\}$, we claim that, assuming $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$, T is determined. Fixing X, let $\phi_{k,\sigma}$ be the formula obtained by replacing each clopen set at each leaf of T^k_σ by `true` or `false` according to whether X is in each clopen set. Any evaluation map for X in T^k can be restricted to an evaluation map for X in T^k_σ, which is an evaluation map for $\phi_{k,\sigma}$, so all these formulas are determined. One obtains an evaluation map for X in T by letting $f(\sigma) = 1$ if and only if $\phi_{k,\sigma}$ is true, and the non-uniformly filling in $f(\lambda)$ to its unique correct value.

Conversely, if $\{\phi_k : k \in \mathbb{N}\}$ are determined, these formulas can be modified at the leaves to become determined Borel codes T^k for $\emptyset$ or 2^ω according to whether they are true or false. Defining T as above, any evaluation map f for T satisfies $f(\langle k\rangle) = 1$ if and only if ϕ_k is true. □

Now we consider the determined variant of BP.

Definition 2.5. *Let* DPB *be the statement "Every determined Borel set has the property of Baire."*

Our main question is: what is the reverse math strength of DPB?

Proposition 2.6. *In* RCA_0*,* DPB *implies* $\mathsf{L}_{\omega_1,\omega}$*-*$\mathsf{CA}$*.*

Proof. Any sequence $\{\phi_k : k \in \mathbb{N}\}$ of determined formulas of $L_{\omega_1,\omega}$ can be modified at the leaves to produce a sequence of determined Borel codes which code either $[0^k1]$ or $\emptyset$ depending on whether ϕ_k is true or false. The union of these remains determined because each X passes through at most one of these sets. Any Baire approximation to $\cup_{k:\phi_k \text{ is true}} [0^k1]$ computes $\{k : \phi_k \text{ is true}\}$. □

This places DPB somewhere in the general area of ATR_0 and the theories of hyperarithmetic analysis. If DPB were equivalent to $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$, our variant would be subject to the same kinds of critique that we made of the original definition (all the strength of the theorem coming essentially from Proposition 2.4). However, it turns out DPB is equivalent to none of the principles mentioned so far.

When considering how to show that DPB is strictly weaker than ATR_0, it is informative to consider the usual proof that every Borel set has the property of Baire. This proof uses arithmetic transfinite recursion on the Borel code of the given set to construct not only a Baire code for given set, but along the way also it constructs, in a uniform way, Baire codes for all Borel sets used to build up the given one (these ancillary sets are coded by $T_\sigma := \{\tau : \sigma^\frown\tau \in T\}$ for each $\sigma \in T$). Below, we give the name *Baire decomposition* to this extended object that ATR_0 would have created. Superficially, DPB would seem weaker than the statement "every determined Borel set has a Baire decomposition", and one might wonder whether the additional information in the Baire decomposition carries any extra strength. The purpose of the rest of this section is to show that it does not (Proposition 2.8), and to mention exactly how a Baire approximation is constructively obtained from a Baire decomposition (Proposition 2.9).

The point is that any model separating DPB from ATR_0 will need another method of producing an entire Baire decomposition, not just the Baire approximation.

Definition 2.7. *Let T be a determined Borel code. A* Baire decompositon *for T is a collection of open sets U_σ and V_σ for $\sigma \in T$ such that for each $\sigma \in T$ and each $p \in 2^{<\omega}$,*

(1) $U_\sigma \cup V_\sigma$ is dense and $U_\sigma \cap V_\sigma = \emptyset$,
(2) if σ is a leaf, then $U_\sigma =$ the clopen set coded by σ and $V_\sigma = \overline{U}_\sigma$,
(3) if σ is a union node, then U_σ is dense in $\bigcup_n U_{\sigma^\frown n}$ and $\bigcup_n U_{\sigma^\frown n}$ is dense in U_σ,
(4) if σ is an intersection node, then V_σ is dense in $\bigcup_n V_{\sigma^\frown n}$ and $\bigcup_n V_{\sigma^\frown n}$ is dense in V_σ.

Proposition 2.8 (ACA_0)**.** DPB *implies that every determined Borel set has a Baire decomposition.*

Proof. Let T be a determined Borel code. Informally, we partition the space into countably many disjoint clopen pieces (plus one limit point) and put an isomorphic copy of the set coded by T_σ in the σth piece. Then we show that a Baire approximation to this disintegrated set can be translated back to a Baire decomposition for the original set coded by T.

More formally, for any $p \in 2^{<\omega}$, let $T[p]$ denote the labeled Borel code for $\{p^\frown X : X \in |T|\}$. This is an effective operation on codes. Recall that each leaf codes a clopen set by a finite list $F \subseteq 2^{<\omega}$. By replacing each such F with $\{p^\frown q : q \in F\}$, we achieve the desired effect.

For any $\sigma \in \omega^{<\omega}$, let $\ulcorner\sigma\urcorner$ be a natural number which codes σ in a canonical way. Define S to be the labeled Borel code

$$S = \{\lambda\} \cup \{\ulcorner\sigma\urcorner^\frown\tau : \tau \in T_\sigma[0^{\ulcorner\sigma\urcorner}1], \sigma \in T\}$$

where λ is a $\cup$ and all other labels are inherited from the $T_\sigma[0^{\ulcorner\sigma\urcorner}1]$. Then S is determined: for any X, if $X = 0^\omega$, then the identically zero map is an evaluation map for X; if $X = 0^n1^\frown Y$, then if f is an evaluation map for Y in T and $n = \ulcorner\sigma\urcorner$, an evaluation map g for X in S can be defined by letting $g(\ulcorner\sigma\urcorner^\frown\tau) = f(\sigma^\frown\tau)$ on

$$\{\ulcorner\sigma\urcorner^\frown\tau : \tau \in T_\sigma[0^{\ulcorner\sigma\urcorner}1]\},$$

$g(\lambda) = f(\sigma)$, and g identically zero elsewhere. Therefore, for all Y and σ,

$$0^{\ulcorner\sigma\urcorner}1^\frown Y \in |S| \iff Y \in |T_\sigma|.$$

Now suppose that $(U, V, \{D_k\}_{k\in\omega})$ is a Baire approximation for S. Then define $U_\sigma = \{q : 0^{\ulcorner\sigma\urcorner}1^\frown q \in U\}$ and $V_\sigma = \{q : 0^{\ulcorner\sigma\urcorner}1^\frown q \in V\}$. Properties (1) and (2) of a Baire decomposition are clear. For property (3), suppose that $\cup_n U_{\sigma^\frown n}$ is not dense in $[p]$. Let q extend p such that for all n, $U_{\sigma^\frown n} \cap [q] = \emptyset$. Then define Y so that $q \prec Y$ and the following collection of comeager events occur:

(i) For all n, $Y \in V_{\sigma^\frown n}$
(ii) For all n, $0^{\ulcorner\sigma^\frown n\urcorner}1^\frown Y \in \cap_k D_k$
(iii) $Y \in U_\sigma \cup V_\sigma$
(iv) $0^{\ulcorner\sigma\urcorner}1^\frown Y \in \cap_k D_k$

The first comeager event guarantees that $Y \in V_{\sigma^\frown n}$ for all n. Together with second comeager event this implies that $0^{\ulcorner\sigma^\frown n\urcorner}1^\frown Y \notin |S|$, and therefore $Y \notin |T_{\sigma^\frown n}|$. Therefore, $Y \notin |T_\sigma|$. In the third dense event, if we had $Y \in U_\sigma$, the fourth comeager event would imply that $Y \in |T_\sigma|$; therefore it must be that $Y \in V_\sigma$, and so U_σ is not dense in $[p]$. On the other hand, if U_σ is not dense in $[p]$, then assuming $\cup_n U_{\sigma^\frown n}$ is dense in $[p]$ leads to a contradiction, for we may similarly define Y to meet $V_\sigma \cap [p]$ and $\cup_n U_{\sigma^\frown n}$, while also satisfying (ii) and (iv).

The proof of (4) is similar to the proof of (3). □

Turning a Baire decomposition into a Baire approximation involves extracting the comeager set on which the approximation should hold. The following proposition gives a canonical sequence of dense open sets which suffices for this.

Proposition 2.9 (ACA_0). *Let T be a determined Borel code and $(U_\sigma, V_\sigma)_{\sigma\in T}$ be a Baire decomposition for T. Let $\{D_n\}_{n\in\omega}$ consist of the following dense open sets:*

(1) $U_\sigma \cup V_\sigma$ for $\sigma \in T$,
(2) $V_\sigma \cup \bigcup_n U_{\sigma^\frown n}$ for $\sigma \in T$ a union node, and
(3) $U_\sigma \cup \bigcup_n V_{\sigma^\frown n}$ for $\sigma \in T$ an intersection node.

Then, $(U_\lambda, V_\lambda, \{D_n\}_{n\in\omega})$ is a Baire approximation for T.

Proof. The properties of a Baire decomposition suffice to ensure that $(U_\lambda, V_\lambda, \{D_n\}_{n\in\omega})$ is a Baire code. We must show that if $X \in \cap_n D_n$, then $X \in U_\lambda \implies X \in |T|$ and $X \in V_\lambda \implies X \notin |T|$. Fix such X. We prove by arithmetic transfinite induction that for all $\sigma \in T$, if $X \in U_\sigma$ then $X \in |T_\sigma|$ and if $X \in V_\sigma$ then $X \notin |T_\sigma|$. This holds when σ is a leaf.

If σ is a union node, suppose $X \in U_\sigma$. Then $X \notin V_\sigma$, but $X \in V_\sigma \cup \bigcup_n U_{\sigma^\frown n}$, so $X \in U_{\sigma^\frown n}$ for some n. Then the induction hypothesis gives us $X \in |T_{\sigma^\frown n}|$, so $X \in |T_\sigma|$ since σ is a union.

On the other hand, if $X \in V_\sigma$, let $p \prec X$ with $p \in V_\sigma$. Then $U_\sigma \cap [p] = \emptyset$, so $\cup_n U_{\sigma^\frown n} \cap [p] = \emptyset$. So for each n, $V_{\sigma^\frown n}$ is dense in $[p]$. Therefore, X meets each $V_{\sigma^\frown n}$, so by induction $X \notin |T_{\sigma^\frown n}|$ holds for all n. Therefore, $X \notin |T_\sigma|$.

The case where σ is an intersection node is similar. □

3. DPB does not imply ATR_0

Our non-ATR_0 method of producing a Baire decomposition involves polling sufficiently generic X to see whether they are in or out of a given set.

To say just how generic, we assume a general familiarity with hyperarithmetic theory, and refer the reader to [Sac90a] for definitions and details. For $G \in 2^\omega$, it is well-known that an element X of 2^ω is $\Delta^1_1(G)$ if and only if it is $HYP(G)$, if and only if there is some $b \in \mathcal{O}^G$ such that $X \leq_T H^G_b$.

Recall that if Γ is a pointclass, $X \in 2^\omega$ is called Γ-generic if X meets or avoids every open set U with a code in Γ. We are interested in Δ^1_1-generics G with the additional property that $\omega_1^{ck} = \omega_1^G$. By [GM17], these are precisely the Σ^1_1-generics.

The following three propositions must be folklore, but we give their proofs here. Recall that A and B are relatively Γ-generic if A is $\Gamma(B)$-generic and B is $\Gamma(A)$-generic.

Proposition 3.1. *For $G_0, G_1 \in 2^\omega$, we have $G_0 \oplus G_1$ is Σ^1_1-generic if and only if G_0 and G_1 are relatively Σ^1_1-generic.*

Proof. Consider the argument in [DH10, Thm. 8.20.1] (originally due to [**?**]), where it is shown that $A \oplus B$ is n-generic if and only if A and B are relatively n-generic. Observe that at no point do they make use of the fact that n is finite, and the same argument goes through if n is replaced with any $a \in \mathcal{O}$. Therefore, if we define a-generic to mean Σ^0_a-generic[3], the same argument shows that $A \oplus B$ is a-generic if and only if A and B are relatively a-generic. Observe that A is Δ^1_1-generic if and only if A is a-generic for all $a \in \mathcal{O}$.

Now suppose that $G_0 \oplus G_1$ is Σ^1_1-generic. We will show that G_0 is $\Sigma^1_1(G_1)$-generic. We have $\omega_1^{G_0 \oplus G_1} = \omega_1^{ck} = \omega_1^{G_1}$, so it suffices to show that G_0 is $\Delta^1_1(G_1)$-generic, or equivalently, that G_0 is a-generic relative to G_1 for all $a \in \mathcal{O}$ (here we use the fact that $\omega_1^{G_1} = \omega_1^{ck}$). This follows from the previous paragraph because $G_0 \oplus G_1$ is a-generic.

On the other hand, if G_0 and G_1 are relatively Σ^1_1-generic, then in particular each is Σ^1_1-generic, so $\omega_1^{G_0} = \omega_1^{G_1} = \omega_1^{ck}$, and by relative Σ^1_1-genericity, we also have $\omega_1^{G_0 \oplus G_1} = \omega_1^{ck}$. Therefore it suffices to show that $G_0 \oplus G_1$ is Δ^1_1-generic, or equivalently, that it is a-generic for all $a \in \mathcal{O}$. This follows because G_0 and G_1 are relatively a-generic for all $a \in \mathcal{O}$. □

Proposition 3.2. *If $G_0 \oplus G_1$ is Σ^1_1-generic, then $\Delta^1_1(G_0) \cap \Delta^1_1(G_1) = \Delta^1_1$.*

[3] A Σ^0_a set or relation is one given by a computable a-ranked formula of $L_{\omega_1,\omega}$, where the leaves are labeled with computable formulas instead of just the symbols `true` or `false`. An object belongs in such a set if the formula evaluates to `true` when that object is given as input at all the leaves.

Proof. If $X \in \Delta^1_1(G_0) \cap \Delta^1_1(G_1)$, then since $\omega_1^{G_0} = \omega_1^{G_1} = \omega_1^{ck}$, there are $a, b \in \mathcal{O}$ and indices e and f such that $X = \phi_e(H_a^{G_0}) = \phi_f(H_b^{G_1})$. Consider the set

$$W = \{Y \oplus Z : \phi_e(H_a^Y) = \phi_f(H_b^Z)\}.$$

This set is Δ^1_1, so it has the property of Baire, and in particular there is a Δ^1_1 open set V such that every sufficiently generic $Y \oplus Z$ is an element of V if and only if it is an element of W. Here the amount of genericity needed is not full Δ^1_1-genericity, but rather c-genericity, where c is some element of $\mathcal{O}$ that can be determined from a, b and the definition of W. Since $G_0 \oplus G_1$ is Δ^1_1-generic and in W, it is in V. Let $p, q \in 2^{<\omega}$ be such that $p \prec G_0, q \prec G_1$ and $p \oplus q \in V$. Now let Y be any c-generic, hyperarithmetic real with $p \prec Y$. Then since G_1 is Δ^1_1-generic, it is Δ^1_1 generic relative to Y, so in particular it is c-generic relative to Y, so $Y \oplus G_1$ is c-generic, and meets V. Therefore, $Y \oplus G_1 \in W$, and we obtain a Δ^1_1 formula for X, that is, $X = \phi_e(H_a^Y)$. □

Proposition 3.3. *Let G_0 be Σ^1_1-generic and P a hyperarithmetic predicate. If there is a $Y \in \Delta^1_1(G_0)$ such that $P(Y)$ holds, then for all Δ^1_1-generic G_1, there is a $Y \in \Delta^1_1(G_1)$ such that $P(Y)$ holds.*

Proof. Since $\omega_1^{ck} = \omega_1^{G_0}$, there is some $a \in \mathcal{O}$ and an index e such that $Y = \phi_e(H_a^{G_0})$. Then $R(X) := \exists e P(\phi_e(H_a^X))$ is a hyperarithmetic predicate that holds of G_0, and holds of $p^\frown G_0$ for any $p \in 2^{<\omega}$. Therefore, for any Δ^1_1-generic G_1, $R(G_1)$ holds. □

Let $G = \bigoplus_i G_i$ be a Σ^1_1 generic. Let $\mathcal{M} = \bigcup_n \Delta^1_1(\bigoplus_{i<n} G_i)$. This is the model which will be used to separate DPB and ATR_0. But first, some lemmas.

Lemma 3.4. $\mathcal{M} \models \mathsf{L}_{\omega_1,\omega}\text{-}\mathsf{CA}$*. Furthermore, whenever $F \subseteq \omega$ is finite and the determined sequence of formulas $\{\phi_k : k \in \mathbb{N}\}$ is in $\Delta^1_1(\bigoplus_{i \in F} G_i)$, we also have*

$$\{k : \phi_k \textit{ is true in } M\} \in \Delta^1_1\left(\bigoplus_{i \in F} G_i\right).$$

Proof. We begin with three facts. First, applying Proposition 3.1 to the decomposition $G = \bigoplus_{i \in F} G_i \oplus \bigoplus_{i \notin F} G_i$, we conclude that $\bigoplus_{i \notin F} G_i$ is $\Sigma^1_1(\bigoplus_{i \in F} G_i)$-generic.

Second, fix $j \notin F$. Applying Proposition 3.1 to $G = G_j \oplus \bigoplus_{i \neq j} G_i$, we have that G_j is $\Sigma^1_1(\bigoplus_{i \neq j} G_i)$-generic and hence G_j is $\Sigma^1_1(\bigoplus_{i \in F} G_i)$-generic.

Third, fix $j_0, j_1 \notin F$ with $j_0 \neq j_1$. By the same argument, we have that G_{j_0} is $\Sigma^1_1(G_{j_1} \oplus \bigoplus_{i \in F} G_i)$-generic and that G_{j_1} is $\Sigma^1_1(G_{j_0} \oplus \bigoplus_{i \in F} G_i)$-generic. By Proposition 3.2 relativized to $\bigoplus_{i \in F} G_i$, it follows that $\Delta^1_1(G_{j_0} \oplus \bigoplus_{i \in F} G_i) \cap \Delta^1_1(G_{j_1} \oplus \bigoplus_{i \in F} G_i) = \Delta^1_1(\bigoplus_{i \in F} G_i)$.

We now apply Proposition 3.3 relativized to $\bigoplus_{i \in F} G_i$. Fix $j \notin F$ and $k \in \omega$. Since $\bigoplus_{i \notin F} G_i$ is $\Sigma^1_1(\bigoplus_{i \in F} G_i)$-generic, G_j is $\Sigma^1_1(\bigoplus_{i \in F} G_i)$-generic and there is a $\Delta^1_1(G)$ evaluation map for ϕ_k, it follows that ϕ_k is determined in $\Delta^1_1(G_j \oplus \bigoplus_{i \in F} G_i)$. Because this holds for any $j \notin F$, ϕ_k is determined in $\Delta^1_1(\bigoplus_{i \in F} G_i)$ by the third fact above. Since $\mathsf{L}_{\omega_1,\omega}\text{-}\mathsf{CA}$ is a theory of hyperarithmetic analysis, the conclusion follows. □

Proposition 3.5. $\mathcal{M} \not\models \mathsf{ATR}_0$.

Proof. Let $a^* \in \mathcal{O}^*$. Then $\mathcal{M}$ believes that a^* is an ordinal. For if there were a $\Delta^1_1(G)$-computable descending sequence in a^*, then for some $b \in \mathcal{O}$ (here we use the fact that $\omega_1^{ck} = \omega_1^G$) the statement $R(X)$: "H_b^X computes a descending sequence in a^*" is a hyperarithmetic predicate which holds of G. As R holds of $p^\frown G$ for any $p \in 2^\omega$, the set of X for which R holds is comeager. Furthermore, $R(X)$ is of Borel class $b + O(1)$, so $R(X)$ holds for any X which is $b + O(1)$-generic. There is a hyperarithmetic such X. But then H_b^X is also hyperarithmetic, contradicting that a^* has no hyperarithmetic descending sequence. So a^* is well-founded, according to $\mathcal{M}$.

For contradiction, suppose there were a jump hierarchy on a^* in $\Delta^1_1(G)$. Then for some $b \in \mathcal{O}$, $R(X) :=$ "H_b^X computes a jump hierarchy on a^*" is again a hyperarithmetic predicate of Borel class $b + O(1)$, where R holds of G. (Recall that being a jump hierarhcy on a^* is just a Π^0_2 property). Arguing as above, hyperarithmetically in any $b+O(1)$-generic X, we would have a jump hierarchy on a^*, which is impossible since a^* has no hyperarithmetic jump hierarchy. ☐

Below, the way that $\mathcal{M}$ can produce a Baire decomposition without resorting to arithmetic transfinite recursion is by polling a sufficiently generic element G_i about whether $p^\frown G_i \in |T|$ while varying $p \in 2^{<\omega}$ to get a complete picture of the comeager behavior of T.

Theorem 3.6. *There is an ω-model of* DPB *that does not satisfy* ATR_0.

Proof. Let $\mathcal{M}$ be as above. Let $T \in M$ be a labeled Borel code which is determined in M. We consider the case where $T \in \Delta^1_1$; the case where $T \in \Delta^1_1(\bigoplus_{i<n} G_i)$ follows by relativization. Since T is determined, for each G_i and each $p \in 2^{<\omega}$, the statements $p^\frown G_i \in |T_\sigma|$ can be understood as a determined formulas of $L_{\omega_1,\omega}$ (by replacing the leaves of T_σ with 0 or 1 according to whether $p^\frown G_i$ is in those sets). These formulas are uniformly $\Delta^1_1(G_i)$. Therefore, by Lemma 3.4, we have

$$\{(\sigma, p) : p^\frown G_i \in |T_\sigma|\} \in \Delta^1_1(G_i)$$

Therefore,for each i, $\Delta^1_1(G_i)$ contains the sequence $(U^i_\sigma, V^i_\sigma)_{\sigma\in T}$ defined by

$$U^i_\sigma = \{p : \forall q \succeq p, q^\frown G_i \in |T_\sigma|\}, \qquad V^i_\sigma = \{p : \forall q \succeq p, q \notin U^i_\sigma\}.$$

We claim that for each $i \neq j$ and for each $\sigma \in T$, the collections

$$(U^i_{\sigma^\frown\tau}, V^i_{\sigma^\frown\tau})_{\tau\in T_\sigma}, \qquad (U^j_{\sigma^\frown\tau}, V^j_{\sigma^\frown\tau})_{\tau\in T_\sigma}$$

are Baire decompositions for T_σ, and are equal. The proof (for fixed i, j) is carried out inside of $\mathcal{M}$ by arithmetic transfinite induction on the rank of σ in T. Specifically, we claim that

(1) If σ is a leaf, then $U^i_\sigma =$ the clopen set coded by σ and $V_\sigma = \overline{U}^i_\sigma$.
(2) If σ is a union node, then for all $p \in 2^{<\omega}$, $p \in U^i_\sigma$ if and only if $\bigcup_n U^i_{\sigma^\frown n}$ is dense in $[p]$.
(3) If σ is an intersection node, then for all $p \in 2^{<\omega}$, $p \in V^i_\sigma$ if and only if $\bigcup_n V^i_{\sigma^\frown n}$ is dense in $[p]$.
(4) $U^j_\sigma = U^i_\sigma$ (and thus $V^j_\sigma = V^i_\sigma$).

Note that the definition of the V^i_σ in term of U^i_σ guarantees that $U^i_\sigma \cup V^i_\sigma$ is dense and $U^i_\sigma \cap V^i_\sigma = \emptyset$, and the remaining parts of the claim suffice to establish that we have a Baire decomposition.

When σ is a leaf, it is clear that U^i_σ and U^j_σ consist of precisely those p such that $[p]$ is contained in the clopen set coded by $\ell(\sigma)$.

Now fix an interior node σ. By induction, we can assume that for all $\tau \in T$ properly extending σ, condition (4) holds, so we drop the superscripts and denote these open sets by U_τ and V_τ. Since Properties (1)-(3) hold for ρ extending such τ, we have that $(U_\rho, V_\rho)_{\rho\in T_\tau}$ are a Baire decomposition for T_τ. We let $D_{m,\tau}$ denote the canonical sequence of dense open sets corresponding to this Baire decomposition. Since $(D_{m,\tau})_m \in \Delta^1_1(G_i) \cap \Delta^1_1(G_j)$, so by Proposition 3.2, $(D_{m,\tau})_m \in \Delta^1_1$. Therefore, for all $p \in 2^{<\omega}$, we have $p^\frown G_i, p^\frown G_j \in \cap_m D_{m,\tau}$. Therefore, if $p^\frown G_i \in U^i_\tau$, then $p^\frown G_i \in |T_\tau|$, and if $p^\frown G_i \in V^i_\tau$, then $p^\frown G_i \notin |T_\tau|$, and the same holds for G_j.

Suppose that σ is a union node. To prove ($\Rightarrow$) in (2), fix $q \in U^i_\sigma$. We need to show that $\{r \in 2^{<\omega} : q^\frown r \in \bigcup_n U_{\sigma^\frown n}\}$ is dense. For a contradiction, suppose $[q^\frown r_0] \cap \bigcup_n U_{\sigma^\frown n} = \emptyset$ for some fixed r_0. To obtain a contradiction, we will show that for all n, we have $q^\frown r_0^\frown G_i \notin |T_{\sigma^\frown n}|$. Since σ is a union node, it follows that $q^\frown r_0^\frown G_i \notin |T_\sigma|$ contradicting the fact that $q \in U^i_\sigma$.

Fix n and let $\tau = \sigma^\frown n$. Since τ properly extends σ, we have that $q^\frown r_0^\frown G_i \in \bigcap_m D_{m,\tau}$ by the comments two paragraphs above. Since $U_\tau \cup V_\tau$ is dense, it follows that V_τ is dense in $[q^\frown r_0]$ and therefore $q^\frown r_0^\frown G_i \in V_\tau$. From $q^\frown r_0^\frown G_i \in \bigcap_m D_{m,\tau}$ and $q^\frown r_0^\frown G_i \in V_\tau$, it follows that $q^\frown r_0^\frown G_i \notin |T_\tau|$ as required to complete the contradiction.

To prove ($\Leftarrow$) in (2), assume that $\bigcup_n U_{\sigma^\frown n}$ is dense in $[q]$. We need to show that $q \in U^i_\sigma$. Fix $r_0 \in 2^{<\omega}$. Since $\bigcup_n U_{\sigma^\frown n}$ is dense in $[q]$, it is also dense in $[q^\frown r_0]$. By the induction hypothesis and Proposition 3.2, $\bigcup_n U_{\sigma^\frown n}$ is Δ^1_1. Let $A = \{\tau : \exists n\, (q^\frown r_0^\frown \tau \in U_{\sigma^\frown n})\}$. A is dense and is Δ^1_1. Therefore, G_i meets the set A. Fix $\tau \in A$ such that $\tau \prec G_i$ and fix n such that $q^\frown r_0^\frown \tau \in U_{\sigma^\frown n}$. Then $q^\frown r_0^\frown G_i \in U_{\sigma^\frown n}$. So, as noted above, $q^\frown r_0^\frown G_i \in \bigcap_m D_{m,\sigma^\frown n}$ and so $q^\frown r_0^\frown G_i \in |T_{\sigma^\frown n}|$. As r_0 was arbitrary, this shows that $q \in U^i_\sigma$.

The exact same argument shows that (2) is also satisfied when i is replaced by j. Therefore, U^i_σ and U^j_σ are described by exactly the same condition, so they are equal.

Finally, let σ be an intersection node. First, consider the direction ($\Leftarrow$) of (3): Suppose that $q \notin V^i_\sigma$ and fix r_0 such that $q^\frown r_0 \in U^i_\sigma$. We will show that $q^\frown r_0 \in U_{\sigma^\frown n}$ for all n, so $\bigcup_n V_{\sigma^\frown n}$ is not dense in $[q]$ (it is disjoint from $[q^\frown r_0]$).

Fixing n, consider an arbitrary string p extending $q^\frown r_0$. Since $q^\frown r_0 \in U^i_\sigma$, we know that $p^\frown G_i \in T_\sigma$. Since σ is an intersection node, it follows that $p^\frown G_i \in T_{\sigma^\frown n}$. Since p was an arbitrary string extending $q^\frown r_0$, this implies $q^\frown r_0 \in U_{\sigma^\frown n}$ as required to complete this direction of (3).

To prove ($\Rightarrow$) in (3), assume $\bigcup_n V_{\sigma^\frown n}$ is not dense in $[q]$. We need to show that $q \notin V^i_\sigma$. Fix r_0 such that $[q^\frown r_0] \cap \bigcup_n V_{\sigma^\frown n} = \emptyset$. Therefore, for each n, $U_{\sigma^\frown n}$ is dense in $[q^\frown r_0]$.

Fix an arbitrary string p extending $q^\frown r_0$. We claim that for all n, we have $p^\frown G_i \in U_{\sigma^\frown n}$. First, note that $U_{\sigma^\frown n}$ is dense in $[p]$ and that by the induction hypothesis and Proposition 3.2, $U_{\sigma^\frown n}$ is Δ^1_1. We shift $U_{\sigma^\frown n}$ to a set $A = \{\tau : p^\frown \tau \in U_{\sigma^\frown n}\}$ which is dense and Δ^1_1, so G_i meets A. Let $\tau \in A$ such that $\tau \prec G_i$. Then, $p^\frown \tau \in U_{\sigma^\frown n}$ and so $p^\frown G_i \in U_{\sigma^\frown n}$. Furthermore, as noted above, since $p^\frown G_i \in \bigcap_m D_{m,\sigma^\frown n}$, it follows that $p^\frown G_i \in |T_{\sigma^\frown n}|$. Since this property holds for each n and since σ is an intersection node, it follows that $p^\frown G_i \in |T_\sigma|$. The string p extending $q^\frown r_0$ was arbitrary, so by the definition of U^i_σ, we have $q^\frown r_0 \in U^i_\sigma$, and therefore $q \notin V^i_\sigma$ to complete the proof of (3).

We have actually proved a little more. Inspecting the argument for ($\Rightarrow$) in (3), we see that whenever $[q] \cap \bigcup_n V_{\sigma^\frown n} = \emptyset$, we have $q \in U^i_\sigma$; and inspecting the argument for ($\Leftarrow$) in (3), we see that whenever $q \in U^i_\sigma$, we have $[q] \cap \bigcup_n V_{\sigma^\frown n} = \emptyset$. This gives a definition of U^i_σ that does not depend on i, and indeed the arguments above could be repeated exactly for U^j_σ. Therefore, $U^i_\sigma = U^j_\sigma$ in the case where σ is an intersection as well.

We conclude that $(U_\sigma, V_\sigma)_{\sigma \in T}$ is a Baire decomposition for T, and so T has a Baire approximation in M. Therefore $\mathcal{M}$ satisfies DPB but not ATR_0. $\square$

4. Decorating trees

In order to show that DPB is strictly stronger than $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$, we need to make some techniques for building non-standard Borel codes in a way that ensures they are determined.

A non-standard Borel code is a code that is not actually well-founded, but which the model thinks is well-founded. These fake codes are essential for the strength of DPB. If a Borel code is truly well-founded, then it has a Baire code which is hyperarithmetic in itself. Since any ω-model of $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ is closed under hyperarithmetic reduction, $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ alone would be enough to guarantee the Baire code exists in the case when the Borel code is truly well-founded (at least in ω-models). So now we are going to describe how to construct a non-standard Borel code which makes every effort to be determined.

If we make a Borel code T which is not well-founded, the most likely scenario is that it is also not determined. This is because, in general, it might take a jump hierarchy the height of the rank of T in order to produce an evaluation map. So in this section, we show how to add "decorations" to the tree, which shortcut the logic of the tree to make sure that for a small set of X, there is an evaluation map for X in the decorated tree. In Section 5, "small" is countable, and in Section 6, "small" is meager. This comes at the cost of trashing any information about whether X was in the original set but if that set had a Baire approximation, then its decorated version should have the same Baire approximation, since the set of X whose membership facts were overwritten is small. We use this to show that if the model satisfies DPB, then the "small" set cannot be the entire second-order part of the model.

Suppose that we have a partial computable function h which maps a number $b \in \mathcal{O}^*$ to a pair of b-ranked labeled trees (P_b, N_b). We do not mind if h happens to also make some outputs for $b \notin \mathcal{O}^*$.

The intention is that when $b \in \mathcal{O}$, any $X \in |P_b| \cup |N_b|$ will have an approximately H^X_b-computable evaluation map in the decorated tree, and X will be in the decorated tree if $X \in |P_b|$ and out of the decorated tree if $X \in |N_b|$. (In practice we will always have $|P_b| \cap |N_b| = \emptyset$.)

The computable operation Decorate is defined using the recursion theorem.

Definition 4.1. *The operation* Decorate *is defined as follows. The inputs are an a-ranked labeled tree T and a partial computable function h as above.*

$$\begin{aligned}\text{Decorate}(T,h) = \{\lambda\} \cup &\bigcup_{\langle n\rangle \in T} \langle 2n\rangle^\frown \text{Decorate}(T_{\langle n\rangle}, h)\\ \cup &\bigcup_{b <_* \rho_T(\lambda)} \langle 2b+1\rangle^\frown \text{Decorate}(Q_b, h)\end{aligned}$$

where $Q_b = P_b$ if λ is a $\cup$ in T, and $Q_b = N_b^c$ if λ is a $\cap$ in T.

The rank and label of λ in Decorate(T,h) *are defined to coincide with the rank and label of λ in T.*

Since P_b and N_b are b-ranked, Decorate(T,h) satisfies the local requirements on a ranking. So if T is a-ranked, so is Decorate(T,h).

Similarly, if T and each P_b and N_b are alternating, and each P_b and N_b have an intersection or leaf at their root, then Decorate(T,h) will also be alternating. (Note that in this case, N_b^c has a union at its root).

The following is the essential feature of a decorated tree.

Proposition 4.2. *If $\sigma \in$ Decorate(T,h) has rank b, then for all $d <_* b$,*

$$\text{Decorate}(T,h)_{\sigma^\frown\langle 2d+1\rangle} = \text{Decorate}(Q_d, h),$$

where $Q_d = P_d$ or N_d^c as appropriate.

Proof. By induction on the length of σ. □

Definition 4.3. *A* nice decoration generator *is a partial computable function which maps any $b \in \mathcal{O}^*$ to alternating, b-ranked trees (P_b, N_b), where each P_b and N_b have an intersection or a leaf at their root.*

Lemma 4.4. *Let h be a nice decoration generator. Suppose $b \in \mathcal{O}$, and suppose that $X \notin |P_d| \cup |N_d|$ for any $d <_* b$. Then for any b-ranked tree T, $X \in |\,\text{Decorate}(T,h)|$ if and only if $X \in |T|$.*

Proof. By induction on b. Since $b \in \mathcal{O}$, T is truly well-founded, so there is a unique evaluation map f for X in T. Further, for each $d <_* b$, there are unique evaluation maps $g_{P,d}, g_{N,d}$ for X in Decorate(P_d,h) and Decorate(N_d^c,h). Consider the function $g :$ Decorate$(T,h) \to \{0,1\}$ defined by

$$g(\sigma) = \begin{cases} f(\frac{\sigma}{2}) & \text{if each component of } \sigma \text{ is even}\\ g_{Q,d}(\sigma_1) & \text{if } \sigma = \sigma_0^\frown\langle 2d+1\rangle^\frown\sigma_1 \text{ and each component of } \sigma_0 \text{ is even,}\end{cases}$$

where the division $\sigma/2$ is taken componentwise, and where Q is either P or N depending on whether σ_0 is a union or intersection in Decorate(T,h).

Since $g(\lambda) = f(\lambda)$, it is enough to show that g is an evaluation map for X in Decorate(T,h). Clearly g satisfies the logic of the tree at leaves and at nodes which have an odd component. Consider $\sigma \in$ Decorate(T,h) where σ is a $\cup$ and all components of σ are even. By induction, since P_d is a d-ranked tree, $X \in |\,\text{Decorate}(P_d,h)|$ if and only if $X \in |P_d|$. By hypothesis, $X \notin |P_d|$, so $g_{P,d}(\lambda) = 0$, so by Proposition 4.2, $g(\sigma^\frown\langle 2d+1\rangle) = 0$. Therefore, the nodes of this form can be ignored: we have

$$\exists m(g(\sigma^\frown m) = 1) \iff \exists n(g(\sigma^\frown\langle 2n\rangle) = 1) \iff f(\sigma/2) = 1$$

so $g(\sigma)$ takes the correct value. The argument if σ is a $\cap$ is similar, except that as $X \notin |N_b|$, we have $X \in |N_b^c|$, and therefore $g_{N,d}(\lambda) = 1$, meaning that nodes of the form $\sigma^\frown\langle 2d+1\rangle$ can be safely ignored when taking an intersection. $\square$

Lemma 4.5. *Let $a \in \mathcal{O}^*$ and $b \in \mathcal{O}$ with $b <_* a$. Let T be an alternating, a-ranked tree and let h be a nice decoration generator. Suppose $X \in |P_b| \cup |N_b|$. Then*

(1) X has a unique evaluation map in Decorate(T,h)*.*
(2) This evaluation map is $H^{X\oplus T}_{b+O(1)}$-computable.
(3) If b is $<_*$-minimal such that $X \in |P_b| \cup |N_b|$, and $b <_* \rho_T(\langle n\rangle)$ for all $\langle n\rangle \in T$, and g is the unique evaluation map for X in Decorate(T,h)*, then*
  (a) $X \in |P_b| \setminus |N_b| \implies g(\lambda) = 1$
  (b) $X \in |N_b| \setminus |P_b| \implies g(\lambda) = 0$.

Proof. It suffices to show all three parts in the case when b is $<_*$-minimal such that $X \in |P_b| \cup |N_b|$.

We prove (1) and (2) by showing that for each $\sigma \in$ Decorate(T,h), there is only one possible value for $g(\sigma)$ for any evaluation map g for X in Decorate(T,h) and that $H^{X\oplus T}_{b+O(1)}$ suffices to compute this value. Since these unique values satisfy the internal logic of the tree (which the reader can verify from the description below), they constitute an evaluation function for X in Decorate(T,h), proving (1) and (2).

To show that there is only one possible value for $g(\sigma)$, we break into cases depending on the rank and label of σ in Decorate(T,h) and on whether $X \in |P_b|$ or $X \in |N_b|$. Note that $H^{X\oplus T}_b$ can uniformly determine the appropriate case for each σ.

Case 1. Suppose $\rho(\sigma) \leq_* b$. Since $b \in \mathcal{O}$, Decorate$(T,h)_\sigma$ is truly well-founded. Therefore, there is a unique evaluation map f for X in Decorate$(T,h)_\sigma$ and we have $g(\sigma) = f(\lambda)$. The map f is uniformly $H^{X\oplus T}_b$-computable.

Case 2. Suppose $b <_* \rho(\sigma)$, σ is a union node in Decorate(T,h) and $X \in |P_b|$. In this case, we claim that $g(\sigma) = 1$. By Proposition 4.2, all nodes extending $\sigma^\frown\langle 2b+1\rangle$ have rank b or less. Therefore, there is a unique evaluation map f on Decorate$(T,h)_{\sigma^\frown\langle 2b+1\rangle}$ and so $g(\sigma^\frown\langle 2b+1\rangle) = f(\lambda)$. By Lemma 4.4, $X \in |P_b|$ implies $f(\lambda) = 1$. Therefore, $g(\sigma^\frown\langle 2b+1\rangle) = 1$ and because σ is a union node, $g(\sigma) = 1$.

Case 3. Suppose $b <_* \rho(\sigma)$, σ is an intersection node in Decorate(T,h) and $X \in |P_b|$. Since Decorate(T,h) is alternating, each node $\sigma^\frown m$ is either a union node or a leaf. If $\rho(\sigma^\frown m) \leq_* b$, then the value of $g(\sigma^\frown m)$ is fixed as in Case 1. If $b <_* \rho(\sigma^\frown m)$, then $g(\sigma^\frown m) = 1$ as in Case 2. Together, these values determine $g(\sigma)$ uniquely. $H^{X\oplus T}_b$ suffices to compute the values of $g(\sigma^\frown m)$ and it takes one extra jump to determine if $g(\sigma^\frown m) = 1$ for all m, and hence determine $g(\sigma)$.

Case 4. Suppose $b <_* \rho(\sigma)$, σ is an intersection node in Decorate(T,h) and $X \in |N_b|$. An analogous argument to Case 2 shows that $g(\sigma) = 0$.

Case 5. Suppose $b <_* \rho(\sigma)$, σ is a union node in Decorate(T,h) and $X \in |N_b|$. This case is analogous to Case 3 and the unique value of $g(\sigma)$ can be determined with one extra jump.

These cases are exhaustive, but if $P_b \cap N_b \neq \emptyset$, then more than one case can apply. However, if $X \in P_b \cap N_b$, there is no conflict between the values given in the cases. In this degenerate case, we have that for any σ such that $b <_* \rho(\sigma)$, $g(\sigma) = 1$ if σ is a union node and $g(\sigma) = 0$ if σ is an intersection node. This completes the proof of (1) and (2).

For (3), if $X \in |P_b| \setminus |N_b|$, and if λ is $\cup$, then $g(\lambda) = 1$ just as above. But if λ is $\cap$, then we claim that for each m, $g(\langle m\rangle) = 1$. If $m = 2n$ for some $\langle n\rangle \in T$, or if $m = 2d+1$ for some $d >_* b$, then because $b <_* \rho_T(\langle n\rangle)$ for all n, and each $\langle m\rangle$ is a union, again we have $g(\langle m\rangle) = 1$ for such m. In the remaining case, when $m = 2d + 1$ with $d \leq_* b$, then since b is minimal such that $X \in |P_b| \cup |N_b|$, and $X \notin |N_b|$, we have $X \in |N_d^c|$. So by Lemma 4.4, $X \in |\operatorname{Decorate}(N_d^c, h)|$, so $g(\langle 2d + 1\rangle) = 1$. Since $g(\langle m\rangle) = 1$ for all m, we have $g(\lambda) = 1$ as well. A complementary argument establishes (3b). ☐

5. DPB does not hold in HYP

We now show that DPB is not a theory of hyperarithmetic analysis by showing that DPB fails in the ω-model HYP. In brief, we let E_a code a canonical universal Σ^0_a set. Using overflow, we make a computable code for the set

$$\bigcup_b |E_b| \cap \{X : b \text{ is least s.t. } X \leq_T H_b\}.$$

We decorate the code to give each H_b-computable set an H_b-computable evaluation map. Then we argue that the result is a code which HYP thinks is well-founded and determined, but which can have no HYP Baire code.

Theorem 5.1. DPB *does not hold in HYP.*

Proof. There is a computable procedure which, on inputs $a \in \mathcal{O}$, $e \in \mathbb{N}$, $p \in 2^{<\omega}$, outputs an index for a 2^a-ranked computable $L_{\omega_1,\omega}$ formula $F_{a,e,p}$, which holds true if and only if $p \in W_e^{H_a}$. Transform each formula $F_{a,e,p}$ into a Borel code by swapping `false` for $\emptyset$, and `true` for $[0^e 1^\frown p]$. Then take the union of all of these, obtaining a code E_a of rank $a + O(1)$ such that for all $a \in \mathcal{O}$,

$$|E_a| = \bigcup_{e,p:p\in W_e^{H_a}} [0^e 1^\frown p].$$

By overflow, there is also a pseudo-ordinal a^*, for which $W_{p(a^*)}$ is not well-founded, but has no hyperarithemtic descending sequence, so that HYP believes $W_{p(a^*)}$ is well-founded. Then HYP also believes that E_b is well-founded for any $b <_* a^*$. We may assume that E_b are alternating and $(b+O(1))$-ranked for all $b \leq_* a^*$. For the sake of a later application of Lemma 4.5, note that we can also assume that the rank of E_{a^*} is a successor, so of the form 2^x for some x, and that for each $\langle n\rangle \in E_{a^*}$, the rank of $\langle n\rangle$ in E_{a^*} is x.

Similarly, there is a computable procedure which, for each $b \in \mathcal{O}$, outputs a $(b + O(1))$-ranked Borel code S_b such that

$$|S_b| = \{X \in 2^\omega : X \leq_T H_b \text{ and for all } c <_* b, X \not\leq_T H_c\}.$$

We think of S_b as coding a slice of HYP. By overflow, for any $b < a^*$, HYP thinks that S_b is well-founded.

For each $b <_* a^*$, define P^b and N^b so that they are alternating, and

$$|P^b| = |S_b| \cap |E_b|, \qquad |N^b| = |S_b| \cap |E_b^c|.$$

Observe that P^b and N^b can be both $(b+k)$-ranked, where k is some fixed finite ordinal. Let h be the function which, on input b, outputs $P_b = P^{b-k}$ and $N_b = N^{b-k}$ if the operation $b - k$ can be performed, and outputs a degenerate b-ranked tree coding the empty set, if b is less than k successors from a limit ordinal.

We claim that Decorate(E_{a^*}, h) is determined in HYP. Observe that h is a nice decoration generator. Let $X \in HYP$. Then there is some $b \in \mathcal{O}$ with $b <_* a$ such that $X \leq_T H_b$. Since a^* is nonstandard, $b + O(1) <_* a^*$ is satisfied. By the choice of b we have $X \in |S_b| = |P_{b+k}| \cup |N_{b+k}|$. Therefore, by Lemma 4.5, X has a HYP evaluation map. Therefore, Decorate(E_{a^*}, h) is determined in HYP.

Suppose for contradiction that Decorate(E_{a^*}, h) had a HYP Baire code. Let $b \in \mathcal{O}$ with $b <_* a^*$ and with the Baire code $(U, V, \{D_n\}_{n\in\omega}) \leq_T H_b$. By the recursion theorem, there is an index e such that

$$W_e^{H_b} = \{p : 0^e 1^\frown p \in V\}$$

where H_b is used to compute V. Choose p with $0^e 1^\frown p \in U \cup V$, this is possible as $U \cup V$ is dense. Let $X \in HYP$ be such that

1. $0^e 1^\frown p \prec X$
2. $X \leq_T H_b$ but $X \not\leq_T H_c$ for any $c <_* b$,
3. $X \in D_n$ for all n.

This is possible because the D_n, and the dense sets which need to be met to avoid being computed by H_c for $c <_* b$, are uniformly H_b-computable.

Now $b + k$ is least such that $X \in |P_{b+k}| \cup |N_{b+k}| = |S_b|$. By Lemma 4.5, $X \in |\operatorname{Decorate}(E_{a^*}, h)|$ if and only if $X \in |E_b|$. Because X meets each D_n and $U \cup V$, by the definition of a Baire code, we have $X \in |\operatorname{Decorate}(E_{a^*}, h)|$ if and only if $X \in U$. To establish the contradiction, it suffices to show that $X \in |E_b|$ if and only if $X \in V$.

Observe $X \in |E_b|$, if and only if, for some q extending p, we have $0^e 1^\frown q \prec X$ and $q \in W_e^{H_b}$. But this happens if and only if for some such q, we have $0^e 1^\frown q \in V$. $\square$

6. DPB implies HYP generics exist in ω-models

The next theorem shows that DPB implies the existence of hyperarithmetic generics in ω-models. In short, if $\mathcal{M}$ has Z but no $\Delta^1_1(Z)$-generics, there is a pseudo-ordinal which $\mathcal{M}$ thinks is well-founded. This pseudo-ordinal can be used to construct a code for the following subset of M:

$$\bigcup_b E_b^Z \cap \{X : b \text{ is least s.t. } X \text{ is not generic relative to } H_b^Z\}$$

After decorating this code, it becomes determined for every non-$\Delta^1_1(Z)$-generic. If this code has a Baire decomposition, meeting the associated dense sets creates a $\Delta^1_1(Z)$-generic.

Theorem 6.1. *If $\mathcal{M}$ is an ω-model which satisfies* DPB*, then for every $Z \in \mathcal{M}$, there is a $G \in \mathcal{M}$ such that G is Δ^1_1-generic relative to Z.*

Proof. Let M be the second-order part of an ω-model which satisfies DPB. Then by Proposition 2.6, whenever $Z \in M$, we also have that $H_b^Z \in M$ for every $b \in \mathcal{O}^Z$.

Case 1: Suppose $\mathcal{M}$ is a β-model (that is, for every tree $T \in M$, if $\mathcal{M} \models$ "T is well-founded", then T is truly well-founded.) Let $Z \in M$. Because $\{G : G \text{ is } \Delta^1_1(Z)\text{-generic}\}$ is a $\Sigma^1_1(Z)$ set, the Z-computable tree corresponding to the $\Sigma^1_1(Z)$ statement "there is a $\Delta^1_1(Z)$-generic" has a path in M, and that path computes a $\Delta^1_1(Z)$-generic G. Therefore, the theorem holds when $\mathcal{M}$ is a β-model.

Case 2: Suppose that there is some $T \in M$ which $\mathcal{M}$ believes is well-founded, but in reality is ill-founded. Let $Z \in M$, and without loss of generality assume that $Z \geq_T T$ (without this assumption we find a $\Delta^1_1(Z \oplus T)$-generic G, but such G is also $\Delta^1_1(Z)$-generic.) There is a Z-computable function which, given the index of a truly well-founded Z-computable tree, outputs an element of $\mathcal{O}^Z$ which bounds the rank of the tree. Applying that function to T produces a pseudo-notation a^* such that $W^Z_{p(a^*)}$ is not truly well-founded, but it has no descending sequence in M.

Relativize the definitions of $<_*$, ranked trees, Decorate, and Lemmas 4.4 and 4.5 to Z.

As in the previous theorem, there are a Z-computable procedures which map any $b \in \mathcal{O}^Z$ to a code E_b of rank $b + O(1)$ such that

$$|E_b| = \bigcup_{e,r:r\in W_e^{H_b^Z}} [0^e 1^\frown r],$$

and map each $b \in \mathcal{O}^Z$ to a code S_b of rank $b + O(1)$ such that

$$|S_b| = \{X \in 2^\omega : X \text{ is not 1-generic relative to } H^Z_b, \\ \text{but for all } c <^Z_* b,\ X \text{ is 1-generic relative to } H^Z_c\},$$

and alternating codes P_b and N_b of rank $b + O(1)$ such that

$$|P_{b+O(1)}| = |S_b| \cap |E_b|, \qquad |N_{b+O(1)}| = |S_b| \cap |E^c_b|.$$

By overflow, these functions are total for any $b <^Z_* a^*$. And fixing a large enough finite number k, we may assume the resulting trees are all alternating and $(b+k)$-ranked relative to Z.

Letting h be the name of the nice decorating function mapping $b+k$ to (P_b, N_b), consider the code $T := \text{Decorate}^Z(E_{a^*}, h)$. Observe that since λ in E_{a^*} is a $\cup$, we know that λ in T is a $\cup$.

If T is not determined, let $G \in M$ be such that G is not determined. We claim that G is $\Delta^1_1(Z)$-generic. If G is not $\Delta^1_1(Z)$-generic, then there is some least $b \in \mathcal{O}^Z$ with $b <^Z_* a^*$ such that G is not 1-generic relative to H^Z_b. Then we would have $G \in |S_b|$, and therefore by Lemma 4.5, G would be determined in T.

If T is determined, then since $\mathcal{M}$ models DPB, let $(U_\sigma, V_\sigma)_{\sigma\in T} \in M$ be a Baire decomposition for T. Let $\{D_i\}_{i<\omega} \in M$ be the associated sequence of dense sets as in Proposition 2.9. For any $p \in 2^{<\omega}$, define $D_{i,p} = \{q : p^\frown q \in D_i\}$. We claim that any $G \in \cap_{i,p} D_{i,p}$ is $\Delta^1_1(Z)$-generic. For this we argue that every dense open $B \in \Delta^1_1(Z)$ actually contains $D_{i,p}$ for some i, p. Let $b \in \mathcal{O}^Z$ and e be such that $B = W_e^{H^Z_b}$. Then $T_{\langle 2(b+k)+1\rangle} = \text{Decorate}(P_{b+k}, h)$, where $|P_{b+k}| = |S_b| \cap |E_b|$. Therefore, there is some $\sigma \in T$ such that $T_\sigma = \text{Decorate}(E_b, h)$. Since E_b has a union at the root, this σ is a union. Let $p = 0^e 1$. We claim that $D_{\ell,p} \subseteq B$, where $D_\ell = \cup_m U_{\sigma^\frown m} \cup V_\sigma$. Let q be such that $p^\frown q \in D_\ell$. To finish the proof, we need to show that $[q] \subseteq B$.

For the remainder of this proof, any X which meets the following conditions will be called *sufficiently generic*:

- $X \in \cap_i D_i$, and
- X is 1-generic relative to H^Z_{b+k}

Observe that for every $r \in 2^\omega$, there is a sufficiently generic $X \in M$ with $r \prec X$. Also, observe that for all such X and all codes R which are c-ranked for some $c \leq_* b + k$, the second condition implies that c, X and R satisfy the conditions of Lemma 4.4, and so $X \in |\text{Decorate}(R, h)|$ if and only if $X \in |R|$. Finally, by

Proposition 2.9, for all sufficiently generic X and all $\tau \in T$, we have $X \in |T_\tau|$ if and only if $X \in U_\tau$.

If X is sufficiently generic and $p^\frown q \prec X$, then $X \in p^\frown B$, and so $X \in |E_b|$, and so by Lemma 4.4, $X \in |\operatorname{Decorate}(E_b, h)| = |T_\sigma|$. Therefore, it is impossible that $X \in V_\sigma$, so we conclude $p^\frown q \in U_{\sigma^\frown m}$ for some m. Therefore, for sufficiently generic X with $p^\frown q \prec X$, we have $X \in |T_{\sigma^\frown m}|$.

If $m = 2c+1$ for some $c \leq_* b+k$, then $T_{\sigma^\frown m} = \operatorname{Decorate}(P_c, h)$. But for any sufficiently generic X, we have $X \notin |P_c|$, so this case is impossible. Therefore, $m = 2n$ for some $\langle n\rangle \in E_b$. It follows from the definition of Decorate that $T_{\sigma^\frown m} = \operatorname{Decorate}((E_b)_{\langle n\rangle}, h)$. So for sufficiently generic X with $p^\frown q \prec X$, we have $X \in |(E_b)_{\langle n\rangle}|$.

Now we will use a property of the codes E_b which follows from how they are defined at the beginning of the proof of Theorem 5.1. The code E_b was obtained as the union of many codes $F_{b,e,r}$, at whose leaves the only options are $[0^e 1^\frown r]$ or $\emptyset$. The code E_b was also post-processed so that it would be alternating, but while this process can break up the codes $F_{b,e,r}$, it can never combine them together. Therefore, for every $\langle n\rangle \in E_b$, there is an r such that whenever $\langle n\rangle^\frown\tau \in E_b$ is a leaf, its attached clopen set is either $[0^e 1^\frown r]$ or $\emptyset$. Fixing r associated to $n = m/2$ for the m found above, we observe that an evaluation map that works for one $Y \in [0^e 1^\frown r]$ works for all such Y, and we conclude that $|(E_b)_{\langle n\rangle}|$ is equal to either $\emptyset$ or $[0^e 1^\frown r]$. It must be the latter because $X \in |(E_b)_{\langle n\rangle}|$ for all sufficiently generic X with $p^\frown q \prec X$. It follows that $[r] \subseteq B$. Furthermore, any sufficiently generic X that does not extend $p^\frown r$ must be out of $|(E_b)_{\langle n\rangle}|$, so it must be that $[q] \subseteq [r]$. Therefore, $[q] \subseteq B$, as desired. $\square$

7. Application to the Borel dual Ramsey theorem

As an application of Theorem 3.6, we identify a natural formulation of the Borel dual Ramsey theorem for 3 partitions and ℓ colors ($\mathsf{Borel\text{-}DRT}^3_\ell$) as a principle which lies strictly below ATR_0, but all of whose ω-models are closed under hyperarithmetic reduction.

Theorem 7.1 (Borel dual Ramsey theorem, [CS84])**.** *For every Borel ℓ-coloring of the set of partitions of ω into exactly k pieces, there is an infinite partition p of ω and a color $i < \ell$ such that every way of coarsening p down to exactly k pieces is given color i.*

Since the set of partitions of ω into exactly k pieces can be coded naturally as a Borel subset of k^ω, a natural way to formulate the hypotheses of the above theorem is roughly "Whenever there are Borel codes $T_1, \dots T_\ell$ such that for every $X \in k^\omega$, we have $X \in |\cup_{i<\ell} T_i|$, ..." (See below for a precise formalization).

Therefore, the Borel dual Ramsey theorem has a natural formulation in terms of determined Borel sets. In [PV85, DFSW17], it was shown that a solution to $\mathsf{Borel\text{-}DRT}^k_\ell$ can in general be obtained by a two-step process:

1. Use the fact that every Borel set has the property of Baire to come up with a Baire approximation to each color in the given coloring.
2. Apply a purely combinatorial principle CDRT^k_ℓ to a coloring of $(k-1)^{<\omega}$ obtainable from the Baire approximation from (1).

If we represent the coloring in the natural way described below, then DPB can be used to carry out (1). It was known to Simpson (see [DFSW17]) that CDRT^3_ℓ

follows from Hindman's Theorem (HT), which follows from ACA_0^+ by [BHS87]. Therefore, the following natural formalization of $\mathsf{Borel\text{-}DRT}^3_\ell$ follows from $\mathsf{DPB} + \mathsf{ACA}_0^+$. We first give the formalization of the space of k-partitions of ω, and then the formalization of $\mathsf{Borel\text{-}DRT}^3_\ell$.

Definition 7.2 (Partitions of ω, [DFSW17])**.** *In RCA_0, a partition of ω into exactly k pieces is a function $p \in k^\omega$ such that p is surjective, and for each $i < k-1$,*

$$\min\{n : p(n) = i\} < \min\{n : p(n) = i+1\}.$$

A partition of ω into infinitely many pieces is a surjective function $p \in \omega^\omega$ which satisfies the above condition for each $i \in \omega$.

The set of partitions described above is an open subset of k^ω representable in RCA_0 by a determined Borel code, as the reader can verify. (For the case $k = 3$, the set in question is the union of the sets $O_{a,b}$ introduced at the start of the proof of Theorem 7.4.) Let P_3 denote this determined Borel code in the case $k = 3$.

Definition 7.3 (Formal Borel dual Ramsey theorem for 3 partitions and ℓ colors)**.** *In RCA_0, $\mathsf{Borel\text{-}DRT}^3_\ell$ is the principle which states: Whenever $T_0, \ldots T_{\ell-1}$ are Borel codes such that for all $X \in |P_3|$, we have $X \in |\bigcup_{i<\ell} T_i|$, then there is an infinite partition p of ω and a color $i < \ell$ such that whenever $X \in |P_3|$, $X \circ p \in |T_i|$.*

It follows from the hypotheses of this theorem that the codes $\{T_i\}_{i<\ell}$ are all determined. Therefore, the discussion preceding the formal definitions proves that $\mathsf{DPB} + \mathsf{ACA}_0^+ \vdash \mathsf{Borel\text{-}DRT}^3_2$ over RCA_0.

The ω-model which was constructed to prove Theorem 3.6 is closed under hyperarithmetic reduction, and therefore satisfies ACA_0^+ as well as DPB. Therefore, $\mathsf{Borel\text{-}DRT}^3_\ell$ holds in this model, while ATR_0 does not. This shows that the formulation of $\mathsf{Borel\text{-}DRT}^3_\ell$ discussed here is strictly weaker than ATR_0.

On the other hand, we have the following, which essentially follows from a more detailed version of the analysis in Section 4 of [DFSW17].

Theorem 7.4. *Let $\ell \in \omega$ with $\ell \geq 2$. Every ω-model of $\mathsf{Borel\text{-}DRT}^3_\ell$ is closed under hyperarithmetic reduction.*

Proof. It suffices to consider the case $\ell = 2$. We will first define some important subsets of 3^ω. For each a, b with $0 < a < b$, let $O_{a,b}$ be the clopen set given by the finite collection of strings

$$O_{a,b} = \{\sigma \in 3^{b+1} : a = \min\{n : \sigma(n) = 1\} \text{ and } b = \min\{n : \sigma(n) = 2\}\}$$

Then the set of partitions of ω into exactly 3 pieces is given by $P_3 = \bigcup_{0<a<b} O_{a,b}$.

Let M be the second-order part of an ω-model $\mathcal{M}$ of $\mathsf{Borel\text{-}DRT}^3_2$. We first show that $\mathcal{M}$ satisfies ACA_0. Let $A \in M$. Let R be the following labeled Borel code.[4]

$$R = \bigcup_{0<a<b} \bigcap_{s>b} C_{a,b,s} \text{ where } C_{a,b,s} = \begin{cases} O_{a,b} & \text{if } A'_b \upharpoonright a = A'_s \upharpoonright a \\ \emptyset & \text{otherwise.} \end{cases}$$

Then R is determined. For any $X \in 3^\omega$, there is at most one pair a, b such that $X \in O_{a,b}$, so an evaluation map for X in R may safely put zeros at every node of

[4]We use standard computability-theoretic notation: for any $s \in \mathbb{N}$, let A'_s denote $\{x < s : \phi^A_{x,s}(x)\downarrow\}$, and for any X let $X \upharpoonright s$ denote the string σ of length s describing the characteristic function of X on $\{0, \ldots, s-1\}$.

R except for the root and the nodes of the distinguished subtree $\cap_{s>b} C_{a,b,s}$. The leaves of that subtree can be $X \oplus A$-computably filled out. Then the root of R and the root of the subtree $\cap_{s>b} C_{a,b,s}$ may be non-uniformly supplied with their unique correct values.

Exactly as in the proof of [DFSW17, Theorem 4.5], for any infinite partition p of ω which is homogeneous for the coloring defined by $|R|, |R^c|$, we have $p \geq_T A'$. (The coloring defined here is the same as the one defined in the proof of that theorem; the function needing to be dominated is the least modulus function for A'_s. Here we have just been more precise about the code R in order to argue that it is determined.) Therefore, $\mathcal{M} \models \mathsf{ACA}_0$.

Now suppose that $A \in M$ and $3 \cdot 5^e \in \mathcal{O}^A$. Suppose that for all $d \leq_{\mathcal{O}} 3 \cdot 5^3$, we have $H^A_d \in M$. Then we claim that $H^A_{3 \cdot 5^e} \in M$. By a result of Jockusch [Joc68] discussed in more detail below, the hyperarithmetic sets are exactly those that can be computed from sufficiently fast-growing functions. As in [DFSW17, Theorem 4.7], we construct a Borel coloring which forces any solution to $\mathsf{Borel\text{-}DRT}^3_2$ to compute a sufficiently fast-growing function. To prove associated Borel code is determined, we need a more detailed analysis than what was given in [DFSW17].

More specifically, Jockusch's result has plenty of uniformity: there are computable functions h and k such that for all $d \in \mathcal{O}^A$, whenever $g : \omega \to \omega$ dominates the increasing function

$$f_d(n) := \phi^{H^A_d}_{h(d)}(n),$$

we have

$$\phi_{k(d)}(A \oplus g) = H^A_d.$$

(To get this from the proof of [Joc68, Theorem 6.8], apply [Rog87, Exercise 16-98] to conclude that the sets H^A_d are in fact uniformly Turing equivalent to implicitly $\Pi^0_1(A)$-definable functions f_d.)

Uniformly in $d \in \mathcal{O}^A$ and $a, b, \in \omega$, and A, there are Borel codes $C_{a,b,d}$ of well-founded rank $d + O(1)$ such that

$$C_{a,b,d} = \begin{cases} O_{a,b} & \text{if } b \geq f_d(a) \\ \emptyset & \text{otherwise.} \end{cases}$$

The uniformity follows from the existence of h above and the A-uniformity of producing a formula of $L_{\omega_1,\omega}$ to assess facts about H^A_d.

For each $n < \omega$, let $d_n = \phi_e(n)$. Now let R be the labeled Borel code

$$R = \bigcup_{0<a<b} \bigcap_{i \leq a} C_{a,b,d_i}.$$

For any $X \in 3^\omega$, there is at most one pair of a, b such that $X \in O_{a,b}$, so as above, any evaluation map for X in R can safely fill in zeros everywhere except for the root of R and the distinguished subtree rooted at $\cap_{i \leq a} C_{a,b,d_i}$. This subtree has well-founded rank $d_a + O(1)$, so the unique evaluation map on it is $H^A_{d_a+O(1)}$-computable. Because $H^A_d \in M$ for all $d \leq_{\mathcal{O}} 3 \cdot 5^e$, this evaluation map exists in M. Therefore, R is determined in M.

Now let $p \in M$ be any infinite partition of ω which is a solution to $BorelDRT^3_2$ for the coloring $|R|, |R^c|$. Define, for each i,

$$p_i = \min\{n : p(n) = i\}.$$

As in the proof of [DFSW17, Theorem 4.5], for every $0 < s < t$, there is a coarsening $X_{s,t}$ of p to exactly three blocks such that $p_s = \min\{n : X_{s,t}(n) = 1\}$ and $p_t = \min\{n : X_{s,t}(n) = 2\}$. Since t can be chosen arbitrarily large, for every s there is a t such that

$$X_{s,t} \in \bigcap_{i \le p_s} C_{p_s, p_t, d_i}$$

and therefore $P_3 \circ p$ is monochromatic for color R, and $s < t$ implies that for all $i \le p_s$, we have $p_t \ge f_{d_i}(p_s)$. Therefore, p computes a sequence of functions $\{g_i : i \in \omega\}$ such that for all i and n, $g_i(n) \ge f_{d_i}(n)$. (Given i and n, let s be large enough that $i, n \le p_s$, and output p_{s+1}.) Therefore, $A \oplus p$ computes

$$\bigoplus_i \phi_{k(d_i)}(A \oplus g_i) = \bigoplus_i H^A_{d_i} = H^A_{3 \cdot 5^e},$$

as was needed. □

We end this section with a question about robustness. The formalization of Borel-DRT^3_2 given above is one we find quite natural. However, another possible way to state the hypothesis of this theorem would be "Whenever there are Borel codes $T_1, \ldots T_\ell$ such that for every $X \in k^\omega$, there is an i such that $X \in |T_i|$, ..."

The subtle difference lies in the fact that if $X \in |\cup_{i<\ell} T_i|$, the evaluation map for X in that code must also prove that $X \in |T_i|$ or $X \in |T_i^c|$ for each $i < \ell$. In the slight variant just mentioned, it is enough to know that for some i, $X \in T_i$ (and possibly have no information about X in the codes T_j for $j \neq i$.) This variant does not, at least on its face, lead to any conclusion about whether, or in what sense, any of the T_i must be determined.

Question 7.5. *How robust is the given formalization of* Borel-DRT^3_2*? In particular, is it equivalent to the variant described above?*

8. Questions

Several directions of further questions immediately suggest themselves. Most results here concern ω-models. It is not immediately clear how to formalize the statement "for every Z, there is a $\Delta^1_1(Z)$-generic" in reverse math. To our knowledge no one has even yet examined the reverse math strength of "$\Delta^1_1 = HYP$". Once a reasonable reverse math way of formalizing these principles exists, it would be natural to ask how these principles are related to principles about (determined) Borel sets.

In the context of ω-models, there are some gaps remaining. For example, we have seen that every ω-model of DPB models $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$ and the existence of Δ^1_1 generics.

Question 8.1. *Suppose $M \subseteq 2^\omega$ is closed under join, satisfies $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$, and for every $Z \in M$, there is a $G \in M$ that is $\Delta^1_1(Z)$-generic. Does it follow that $\mathcal{M} \models$* DPB*?*

One way that the above question could have a negative answer would be if DPB implied some theory of hyperarithmetic analysis strictly stronger than $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}$.

Question 8.2. *Which theorems of hyperarithmetic analysis are implied by* DPB*, and which are incomparable with it?*

We built an ω-model of DPB by adjoining many mutually Σ^1_1-generics.

Question 8.3. *Does every ω-model of* DPB *contain a* Σ^1_1*-generic?*

Whether in ω-models or full reverse math, many other theorems involving Borel sets may now have interesting reverse math content when considering their determined versions. A forthcoming paper will address the strength of "Every determined Borel set is measurable." We mention that the statement "Every determined Borel set has the perfect set property" is equivalent to ATR_0, because "Every closed set has the perfect set property" already implies ATR_0 by by [Sim09, V.5.5], so here the way of defining a Borel set does not add additional strength.

Turning now to $\mathsf{Borel\text{-}DRT}^3_\ell$, we have seen that any ω-model of it is closed under hyperarithmetic reduction.

Question 8.4. *Is* $\mathsf{Borel\text{-}DRT}^3_\ell$ *a theory of hyperarithmetic analysis?*

Finally, there is the issue of robustness. There are some possible variations on what could be considered as an evaluation map. For example, a weaker version of an evaluation map would be a partial function $f :\subseteq T \to \{0,1\}$ such that $f(\lambda)$ is defined; and whenever $\sigma \in T$ is a $\cup$, and $f(\sigma) = 1$, there is an n such that $f(\sigma^\frown n) = 1$; and whenever $\sigma \in T$ is a $\cap$ and $f(\sigma) = 1$, for all n, $\sigma^\frown n \in T$ implies $f(\sigma^\frown n) = 1$; and similarly for when $f(\sigma) = 0$. We did not investigate, but it would be interesting to know, the extent to which the results of this paper are robust under variations of this definition.

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Department of Mathematics, University of Connecticut, Storrs, Connecticut U.S.A.
E-mail address: `eric.astor@uconn.edu`

Department of Mathematics, University of Connecticut, Storrs, Connecticut U.S.A.
E-mail address: `damir.dzhafarov@uconn.edu`

Department of Mathematics, University California-Berkeley, Berkeley, California U.S.A.
E-mail address: `antonio@math.berkeley.edu`

Department of Mathematics, University of Connecticut, Storrs, Connecticut U.S.A.
E-mail address: `solomon@math.uconn.edu`

Department of Mathematics, Penn State University, State College, Pennsylvania U.S.A.
E-mail address: `lzw299@psu.edu`