# COLORING TREES IN REVERSE MATHEMATICS

DAMIR D. DZHAFAROV AND LUDOVIC PATEY

Abstract. The tree theorem for pairs ($\mathsf{TT}^2_2$), first introduced by Chubb, Hirst, and McNicholl, asserts that given a finite coloring of pairs of comparable nodes in the full binary tree $2^{<\omega}$, there is a set of nodes isomorphic to $2^{<\omega}$ which is homogeneous for the coloring. This is a generalization of the more familiar Ramsey's theorem for pairs ($\mathsf{RT}^2_2$), which has been studied extensively in computability theory and reverse mathematics. We answer a longstanding open question about the strength of $\mathsf{TT}^2_2$, by showing that this principle does not imply the arithmetic comprehension axiom ($\mathsf{ACA}_0$) over the base system, recursive comprehension axiom ($\mathsf{RCA}_0$), of second-order arithmetic. Combined with a recent result of Patey's that $\mathsf{TT}^2_2$ is strictly stronger than $\mathsf{RT}^2_2$, this establishes $\mathsf{TT}^2_2$ as the first known example of a natural combinatorial principle to occupy the interval strictly between $\mathsf{ACA}_0$ and $\mathsf{RT}^2_2$. The proof of this fact uses an extension of the bushy tree forcing method, and develops new techniques for dealing with combinatorial statements formulated on trees, rather than on $\omega$.

## 1. Introduction

Reverse mathematics is an area of mathematical logic devoted to classifying mathematical theorems according to their logical strength. The setting for this endeavor is *second-order arithmetic* which is a formal system strong enough to encompass (countable analogues of) most results of classical mathematics. It consists of the usual Peano axioms for the natural numbers, together with the *comprehension scheme*, consisting of axioms asserting that the set of all $x \in \mathbb{N}$ satisfying a given formula $\varphi$ exists. Fragments of this system obtained by weakening the comprehension scheme are called *subsystems of second-order arithmetic*. The logical strength of a theorem is then measured according to the weakest such subsystem in which that theorem can be proved. This is a two-step process: the first consists in actually finding such a subsystem, and the second in showing that the theorem "reverses", i.e., is in fact equivalent to this subsystem, over a fixed weak base system. One way to think about such a reversal is that it precisely captures the techniques needed to prove the given theorem. By extension, two theorems that turn out to be equivalent to the same subsystem (and hence to each other) can thus be regarded as requiring the same basic techniques to prove. The observation mentioned above, that most theorems can be classified into just a few categories, refers to the fact that most theorems are either provable in the weak base system, or are equivalent over it to one of four other subsystems.

The base system here is the *recursive comprehension axiom* ($\mathsf{RCA}_0$), which restricts the comprehension scheme to $\Delta^0_1$-definable sets. This system corresponds

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roughly to constructive mathematics, sufficing to prove the existence of all the computable sets, but not of any noncomputable ones. A considerably stronger system is the *arithmetical comprehension axiom* ($\mathsf{ACA}_0$), which adds comprehension for sets definable by arithmetical formulas, i.e., formulas whose quantifiers only range over variables for numbers (as opposed to variables for sets of numbers). This system suffices to solve the halting problem, i.e., the problem of determining whether a given computer program halts on a given input, and as such is considerably stronger than $\mathsf{RCA}_0$. Three other important systems are *weak König's lemma* ($\mathsf{WKL}_0$), *arithmetical transfinite recursion* ($\mathsf{ATR}_0$), and the $\Pi^1_1$*-comprehension axiom* ($\Pi^1_1$-$\mathsf{CA}_0$). In order of increasing strength, these are arranged thus:

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0.$$

We refer the reader to Simpson [33] for a complete treatise on reverse mathematics, and to Soare [34] for general background on computability theory.

A striking observation, repeatedly demonstrated in the literature, is that most theorems investigated in this framework are either provable in the base system $\mathsf{RCA}_0$, or else equivalent over $\mathsf{RCA}_0$ to one of the other four subsystems listed above. It is from this empirical fact that these systems derive their commonly-used moniker, "the big five". The initial focus of the subject was almost exclusively on a kind of zoological classification of theorems according to which of the five categories they belong to. In the interval between $\mathsf{RCA}_0$ and $\mathsf{ACA}_0$, the study of which has received the overwhelming majority of attention in the literature, an early exception to this classification project was Ramsey's theorem for pairs. We recall its statement.

**Definition 1.1.** Fix $X \subseteq \omega$ and $n, k \geq 1$.

(1) $[X]^n$ is the set of all tuples $\langle x_0, \ldots, x_{n-1}\rangle \in X^n$ with $x_0 < \cdots < x_{n-1}$.
(2) A *$k$-coloring of* $[X]^n$ is a function $f : [X]^n \to \{0, \ldots, k-1\}$.
(3) A set $Y \subseteq X$ is *homogeneous for* $f$, or *$f$-homogeneous*, if there is a color $c < k$ such that $f([Y]^n) = \{c\}$.

We identify $\{0, \ldots, k-1\}$ with $k$, and so write a coloring simply as $f : [X]^n \to k$. We also write $f(x_0, \ldots, x_{n-1})$ in place of $f(\langle x_0, \ldots, x_{n-1}\rangle)$ for $\langle x_0, \ldots, x_{n-1}\rangle \in [X]^n$.

**Statement 1.2** (Ramsey's theorem for $n$-tuples and $k$ colors ($\mathsf{RT}^n_k$))**.** For every coloring $f : [\omega]^n \to k$, there is an infinite $f$-homogeneous set.

It is easy to see that over $\mathsf{RCA}_0$, $\mathsf{RT}^n_2$ is equivalent to $\mathsf{RT}^n_k$ for any $k \geq 2$, so in practice, we will usually restrict $k$ to 2.[1] The effective analysis of Ramsey's theorem was initiated by Jockusch [17] in the 1970s. Recasting some of his results in the parlance of reverse mathematics, we see that $\mathsf{RCA}_0$ proves $\mathsf{RT}^1_2$, that $\mathsf{ACA}_0$ proves $\mathsf{RT}^n_2$ for every $n$, and that for $n \geq 3$, $\mathsf{RT}^n_k$ and $\mathsf{ACA}_0$ are in fact equivalent (see [32, Theorem III.7.6]). The situation for $n = 2$ is different. Hirst [16, Corollary 6.12] showed that $\mathsf{RT}^2_2$ is not provable in $\mathsf{WKL}_0$ (see also [14, Corollary 6.12]), while much later, answering what had by then become a major question, Seetapun showed

[1] The situation is quite different in the closely related investigation of Ramsey's theorem under reducibilities stronger than provability in $\mathsf{RCA}_0$, such as Weihrauch reducibility and computable reducibility. This analysis has gained much prominence recently as an extension of the traditional framework of reverse mathematics. (See Dorais et al. [8] for an introduction, and Brattka [1] for an updated bibliography.) In this setting, the number of colors does indeed make a difference. In particular, Patey [29, Theorem 3.14] has shown that if $k > j \geq 2$ then $\mathsf{RT}^n_k$ is not computably reducible to $\mathsf{RT}^n_j$.

that $\mathsf{RT}^2_2$ does not imply $\mathsf{ACA}_0$ (see [30, Theorems 2.1 and 3.1]). Thus, $\mathsf{RT}^2_2$ does not obey the "big five" phenomenon.

The quest to better understand $\mathsf{RT}^2_2$, and in particular, of why its strength differs from that of most other theorems, has developed into a highly active and fruitful line of research, as a result of which, a "zoo" of mathematical principles has emerged, with a complex system of relationships between them (see [10]). We refer the reader to Hirschfeldt [14, Section 6] for a survey. Figure 1 gives a snapshot of this zoo. A

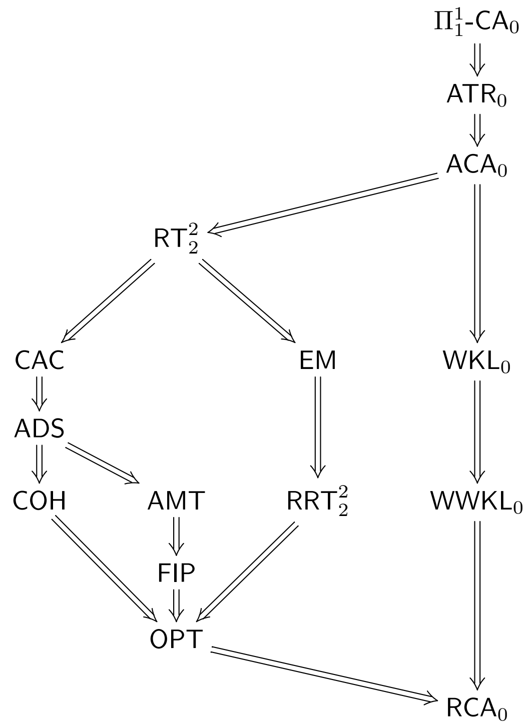


FIGURE 1. Relationships among selected principles between $\mathsf{RCA}_0$ and $\mathsf{ACA}_0$. Arrows denote implications over $\mathsf{RCA}_0$, all of which are strict in this diagram.

conspicuous feature of this diagram is that $\mathsf{RT}^2_2$ lies above every other principle, with the exception of $\Pi^1_1$-$\mathsf{CA}_0$, $\mathsf{ATR}_0$, $\mathsf{ACA}_0$, $\mathsf{WKL}_0$, and $\mathsf{WWKL}_0$. (Whether or not $\mathsf{RT}^2_2$ also implies the latter two was a longstanding problem, which was resolved only recently, by Liu [22, 23], who showed that $\mathsf{RT}^2_2$ does not imply even $\mathsf{WWKL}_0$.) While some of these principles are weaker forms of Ramsey's theorem that were introduced explicitly as a way of obtaining partial results about $\mathsf{RT}^2_2$, a large number of others were studied for their own sake and with independent motivations, and come from a variety of mathematical areas (including from outside of combinatorics, such as model theory, set theory, and algorithmic randomness).

The above suggests that $\mathsf{RT}^2_2$ is a naturally strong theorem in relation to principles lying below $\mathsf{ACA}_0$. Notably, there have been no examples of a natural principle[2] lying strictly below $\mathsf{ACA}_0$ and strictly above $\mathsf{RT}^2_2$. The only candidate to be such a principle that has been previously looked at is the so-called *tree theorem for pairs*, introduced by Chubb, Hirst, and McNicholl [5], and defined below. (Recently, Patey and Frittaion [13] have shown this to be closely related to a version of the Erdős-Rado theorem.)

**Definition 1.3** (Chubb, Hirst, and McNicholl [5]). Fix $T \subseteq 2^{<\omega}$ and $n, k \geq 1$.

(1) $[T]^n$ is the set of all tuples $\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in T^n$ with $\sigma_0 \prec \cdots \prec \sigma_{n-1}$.
(2) A *$k$-coloring of* $[T]^n$ is a function $f : [T]^n \to k$.
(3) A set $Y \subseteq T$ is *homogeneous for $f$*, or *$f$-homogeneous*, if there is a color $c < k$ such that $f([Y]^n) = \{c\}$.
(4) $T$ is *isomorphic to* $2^{<\omega}$, written $T \cong 2^{<\omega}$, if there is a bijection $i : 2^{<\omega} \to T$ such that $\sigma \preceq \tau$ if and only if $i(\sigma) \preceq i(\tau)$ for all $\sigma, \tau \in 2^{<\omega}$.

As for colorings of subsets of $\omega$, we write $f(\sigma_0, \ldots, \sigma_{n-1})$ in place of $f(\langle \sigma_0, \ldots, \sigma_{n-1} \rangle)$ for $\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in [T]^n$.

**Statement 1.4** (Tree theorem for $n$-tuples and $k$ colors ($\mathsf{TT}^n_k$)). For every coloring $f : [2^{<\omega}]^n \to k$, there is an $f$-homogeneous set $H \subseteq 2^{<\omega}$ such that $H \cong 2^{<\omega}$.

Chubb, Hirst, and McNicholl [5, Section 1] showed that basic results about $\mathsf{TT}^n_k$ parallel the situation for Ramsey's theorem. As before, we may safely assume $k = 2$. The base system $\mathsf{RCA}_0$ suffices to prove $\mathsf{TT}^1_2$, and $\mathsf{ACA}_0$ suffices to prove $\mathsf{TT}^n_2$ for each $n$. It is also easy to see that $\mathsf{TT}^n_2$ implies $\mathsf{RT}^n_2$ over $\mathsf{RCA}_0$ for all $n$, so for $n \geq 3$, $\mathsf{TT}^n_2$ is equivalent to $\mathsf{ACA}_0$, and hence to $\mathsf{RT}^n_2$. By contrast, Patey [28, Corollary 4.12] has shown that $\mathsf{TT}^2_2$ is strictly stronger than $\mathsf{RT}^2_2$, leaving open the possibility that $\mathsf{TT}^2_2$ might imply $\mathsf{ACA}_0$. Whether this is the case has been an open question for some time, originally appearing in [5, Question 2], subsequently also asked about by Montalbán [25, Section 2.2.4], and variously explored by Corduan, Groszek, and Mileti [7], Dzhafarov, Hirst, and Lakins [11], Chong, Li, and Yang [4], and Corduan and Groszek [6]. Our main theorem in this paper is a solution to this question.

**Main Theorem.** $\mathsf{RCA}_0 \nvdash \mathsf{TT}^2_2 \to \mathsf{ACA}_0$.

Thus, we establish $\mathsf{TT}^2_2$ as the first (and only known) natural principle to lie strictly in the interval between $\mathsf{ACA}_0$ and $\mathsf{RT}^2_2$, and so dethrone $\mathsf{RT}^2_2$ as the "strongest of the weak principles". Our proof of this fact develops entirely new techniques for dealing with Ramsey-like principles in the tree setting. The key difficulty here is that standard arguments for building a homogeneous set for a coloring (e.g., Mathias forcing constructions, as in the proof of Seetapun's theorem) seem, on a close inspection, very dependent on the linearity of the partial order ($\omega$) on which the coloring is defined. Thus, when dealing instead with colorings on $2^{<\omega}$, many of the combinatorial methods from the linear setting are no longer applicable or obviously adaptable. The proof of our main theoem thus offers a way to get around these problems.

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[2] Here, we mean *combinatorially* natural, and in particular, representing a single combinatorial problem. For example, $\mathsf{RT}^2_2 + \mathsf{WKL}_0$ lies strictly below $\mathsf{ACA}_0$ (see [30, Theorem 3.1]), and since $\mathsf{RT}^2_2$ does not imply $\mathsf{WKL}_0$, this is also strictly above $\mathsf{RT}^2_2$. But $\mathsf{RT}^2_2 + \mathsf{WKL}_0$ is a conjunction of two principles, and hence not natural in our sense.

This outline of the rest of the paper is as follows. In Section 2, we lay down some of the background notions and notations that we will use in the sequel. The proof of our main theorem is organized into a stable and cohesive part, in the manner first employed by Cholak, Jockusch, and Slaman [3], though stability and cohesiveness are far less obvious notions in the tree setting than in the linear. In Section 3, we consider $\mathsf{TT}^1_2$, and prove that it admits the so-called strong cone avoidance property, which then allows us to conclude that the stable tree theorem for pairs does not imply $\mathsf{ACA}_0$. In Section 4, we show that the cohesive tree theorem for pairs admits cone avoidance, and hence also does not imply $\mathsf{ACA}_0$. Combining these results then yields our main theorem.

## 2. Background and definitions

Our terminology and notation in this paper is standard, e.g., as in Downey and Hirschfeldt [9]. Throughout, we reserve the term *tree* for downward-closed subsets of $2^{<\omega}$, and refer to other subsets of $2^{<\omega}$ (including those with a tree structure on them) simply as *sets*.

**Definition 2.1.** Let $T \subseteq 2^{<\omega}$ be non-empty.

(1) A node $\tau \in T$ is a *successor* of $\sigma \in T$ if $\tau \succ \sigma$ and there is no $\xi \in T$ such that $\sigma \prec \xi \prec \tau$.
(2) A *leaf* is a node without any successor. We denote by $\mathrm{lvs}(T)$ the set of leaves of $T$.
(3) A node $\tau \in T$ is *at level $n$ in $T$* if there exist precisely $n$ proper initial segments of $\tau \in T$.
(4) A *root* of $T$ is a node at level 0 in $T$.
(5) The set $T$ is *at level $n$* if every leaf is at level $n$.
(6) We let $T \restriction n$ be the set $\{\sigma \in T : \sigma \text{ is at level at most } n\}$.

**Definition 2.2.** Let $T \subseteq 2^{<\omega}$ be non-empty.

(1) $T$ is *$h$-branching* for a function $h : \omega \to \omega$ if it has a unique root and every node at level $n$ in $T$ which is not a leaf has exactly $h(n)$ immediate successors.
(2) $T$ is *2-branching* if it is $h$-branching for the constant function $h(n) = 2$.
(3) $T$ is *perfect* if each node has at least 2 successors.
(4) $T$ is *isomorphic to* $2^{<\omega}$, written $T \cong 2^{<\omega}$, if $T$ is perfect and 2-branching.

Note that if $T$ is perfect then it has a $T$-computable subset isomorphic to $2^{<\omega}$. The definition of being isomorphic to $2^{<\omega}$ here is different than that given in Section 1, but the two are readily seen to be equivalent.

Cholak, Jockusch, and Slaman [3, Section 7] developed a prominent framework for studying Ramsey's theorem for pairs, namely, by introducting the so-called stable Ramsey's theorem for pairs ($\mathsf{SRT}^2_2$) and the cohesive principle ($\mathsf{COH}$), into which they showed $\mathsf{RT}^2_2$ can be decomposed. We recall the definitions.

**Definition 2.3.** Fix $f : [\omega]^2 \to k$ and an infinite $X \subseteq \omega$.

(1) $f$ is *stable over $X$* if for every $x$, there is an $s > x$ such that $f(x, y) = f(x, s)$ for all $y \geq s$.
(2) $f$ is *stable* if it is stable over $\omega$.

**Statement 2.4** (Stable Ramsey's theorem for pairs and $k$ colors ($\mathsf{SRT}^2_k$))**.** For every stable coloring $f : [\omega]^2 \to k$, there is an infinite $f$-homogeneous set.

**Definition 2.5.** Let $\vec{R} = \langle R_0, R_1, \ldots\rangle$ be a sequence of subsets of $\omega$. An infinite set $C \subseteq \omega$ is *$\vec{R}$-cohesive* if for each $i \in \omega$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

**Statement 2.6** (Cohesive principle ($\mathsf{COH}$))**.** For every uniform sequence of sets $\vec{R}$, there is a $\vec{R}$-cohesive set.

**Proposition 2.7** (Cholak, Jockusch, and Slaman [3, Lemma 7.11]; Mileti [24, Claim A.1.3])**.** $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 + \mathsf{COH}$.

The utility of this decomposition fact is that each of $\mathsf{SRT}^2_2$ and $\mathsf{COH}$ can be regarded as a form of $\mathsf{RT}^1_2$ (the infinitary pigeonhole principle), and hence is in many ways easier to work with than $\mathsf{RT}^2_2$. (See [14, Section 6.4] for an insightful discussion.)

For the tree theorem, notions of stability and cohesiveness were first considered by Dzhafarov, Hirst, and Lakins [11]. As it turns out, both these notions admit not just one but several reasonable adaptations from the linear to the tree setting; in [11], the authors identified and studied five distinct such notions. For our purposes here, we will deal only with what was in [11, Definition 3.2] called *1-stability*. Since no confusion can arise here, we will refer to this simply as *stability* below.

**Definition 2.8.** Fix $f : [2^{<\omega}]^2 \to k$ and $T \cong 2^{<\omega}$.

(1) $f$ is *stable over* $T \cong 2^{<\omega}$ if for every $\sigma \in T$, there is a color $c < k$ and a level $n \in \omega$ such that $f(\sigma, \tau) = c$ for every $\tau \in T$ such that $\tau \succ \sigma$ and $|\tau| > n$.
(2) $f$ is *stable* if it is stable over $2^{<\omega}$.

We have the following restrictions of $\mathsf{TT}^2_k$, and an associated decomposition fact.

**Statement 2.9** ($\mathsf{STT}^2_k$)**.** For every stable coloring $f : [2^{<\omega}]^2 \to k$, there is an $f$-homogeneous set $H \subseteq 2^{<\omega}$ such that $H \cong 2^{<\omega}$.

**Statement 2.10** ($\mathsf{CTT}^2_k$)**.** For every coloring $f : [2^{<\omega}]^2 \to k$, there is a set $H \subseteq 2^{<\omega}$ such that $H \cong 2^{<\omega}$ and over which $f$ is stable.

**Proposition 2.11** (Dzhafarov, Hirst, and Lakins [11, Proposition 3.18])**.** $\mathsf{RCA}_0 \vdash \mathsf{TT}^2_k \leftrightarrow \mathsf{STT}^2_k + \mathsf{CTT}^2_k$.

The analogy between $\mathsf{SRT}^2_k$ and $\mathsf{STT}^2_k$ is clear. The analogy between $\mathsf{COH}$ and $\mathsf{CTT}^2_k$ stems from the fact, due to Hirschfeldt and Shore [15, Proposition 4.4 and 4.8], that over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ (and hence over $\omega$-models), $\mathsf{COH}$ is equivalent to the principle $\mathsf{CRT}^2_k$, which asserts that for every coloring $f : [\omega]^2 \to k$ there is an infinite set over which $f$ is stable.

All the principles of the kind we are discussing here have the same syntactic form as statements in the language of second-order arithmetic, namely

$$(\forall X)[\phi(X) \to (\exists Y)[\theta(X, Y)]], \tag{1}$$

where $\phi$ and $\theta$ are arithmetical formulas. The sets $X$ satisfying $\phi(X)$ are commonly called the *instances* of the principle, and for each such $X$, the sets $Y$ satisfying $\theta(X, Y)$ are called the *solutions to* $X$. (While the presentation in (1) is not unique, in practice there is always a fixed one we have in mind, and hence an implicitly understood class of instances and solutions for it.) For example, the instances of $\mathsf{STT}^n_k$ are the stable colorings $f : [2^{<\omega}]^2 \to k$, and the solutions to any such $f$ are the $f$-homogeneous sets $H \cong 2^{<\omega}$. (See also [8, Section 1] and [14, Section 1.4] for further examples.) The terminology is not so important for us in this paper, but it does allow us to state in a general form the following properties.

**Definition 2.12.** Let $\mathsf{P}$ be a principle as in (1) .

(1) $\mathsf{P}$ admits *cone avoidance* if for every set $Z$, every set $C \not\leq_T Z$, and every $Z$-computable instance $I$ of $\mathsf{P}$, there is a solution $S$ to $I$ such that $C \not\leq_T S \oplus Z$.
(2) $\mathsf{P}$ admits *strong cone avoidance* if for every set $Z$, every set $C \not\leq_T Z$, and every instance $I$ of $\mathsf{P}$, there is a solution $S$ to $I$ such that $C \not\leq_T S \oplus Z$.

The term "cone avoidance" refers to the fact that the solution $S$ above avoids the upper cone $\{X : X \geq_T C\}$. The adjective "strong" in part 2 of the definition refers to the fact that the instance $I$ there is arbitrary, and in particular, not necessarily computable from the set $Z$, as in part 1. The following lemma is standard.

**Lemma 2.13.** *If* $\mathsf{P}$ *admits cone avoidance, then* $\mathsf{RCA}_0 \nvdash \mathsf{P} \to \mathsf{ACA}_0$.

*Proof.* We define a chain of sets $Z_0 \leq_T Z_1 \leq_T \cdots$ as follows. Let $Z_0 = \emptyset$, and suppose inductively that we have defined $Z_n$ and that $\emptyset' \not\leq_T Z_n$. Say $n = \langle m, e \rangle$, so that $m \leq n$. If $\Phi_e^{Z_m}$ does not define an instance of $\mathsf{P}$, we set $Z_{n+1} = Z_n$. Otherwise, call this instance $I$, and regard it as a $Z_n$-computable set. By cone avoidance for $\mathsf{P}$, there exists a solution $S$ to $I$ such that $\emptyset' \not\leq_T S \oplus Z_n$. We set $Z_{n+1} = S \oplus Z_n$. This completes the definition of the chain. Now let $\mathcal{M}$ be the $\omega$-model with second order part $\{X : (\exists n)[X \leq_T Z_n]\}$. Then by construction, $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{P}$, but $\emptyset' \notin \mathcal{M}$ and so $\mathcal{M} \not\models \mathsf{ACA}_0$. $\square$

Finally, we recall the definition of Mathias forcing, which is commonly employed in the construction of homogeneous sets. In the sequel, for subsets $A$ and $B$ of $\omega$, we write $A < B$ if $A$ is finite and $\max A < \min B$.

**Definition 2.14.**

(1) A *Mathias condition* is a pair $(F, X)$, where $F$ is a finite subset of $\omega$, $X$ is an infinite subset of $\omega$ called the *reservoir*, and $F < X$.
(2) A Mathias condition $(E, Y)$ *extends* $(F, X)$, written $(E, Y) \leq (F, X)$, if $F \subseteq E \subseteq F \cup X$ and $Y \subseteq X$.

A set $S$ is said to *satisfy* a condition $(F, X)$ if $S$ is infinite and $F \subseteq S \subseteq F \cup X$.

We refer the reader to Cholak, Jockusch, and Slaman [3, Sections 4 and 5] for some prominent examples of the use of Mathias forcing in reverse mathematics, and to Cholak, Dzhafarov, Hirst, and Slaman [2] for some general computability-theoretic facts about this forcing notion. We assume familiarity with the basics of forcing in arithmetic, as described, e.g., in Shore [31]. Below, we will work with a number of forcing notions that are defined as combinatorial elaborations of Mathias forcing, each giving rise to a forcing language and forcing relation in the standard manner (see [31, Section 3.2]).

## 3. Partitions of trees and strong cone avoidance

Our starting point is to prove a tree analogue of the following result about the infinitary pigeonhole principle.

**Theorem 3.1** (Dzhafarov and Jockusch [12, Lemma 3.2])**.** $\mathsf{RT}^1_2$ *admits strong cone avoidance.*

We prove our analogue as Theorem 3.7 below, from which we will also obtain cone avoidance for $\mathsf{STT}^2_2$.

We begin with a slightly weaker result, which we preface with a definition. Given a finite set $F \subseteq 2^{<\omega}$, we denote by $[F]^{\preceq}$ the set of $\tau \in 2^{<\omega}$ extending some $\sigma \in F$.

We write $[F]^{\prec}$ for the set of $\tau \in 2^{<\omega}$ properly extending some $\sigma \in F$. Note that if the elements of $F$ are pairwise incomparable then $[F]^{\prec} = [F]^{\preceq} \setminus F$.

**Definition 3.2.** Given $T \subseteq 2^{<\omega}$ be non-empty, let $\mathbb{P}_T$ consist, for all $n \in \omega$, of all $n$-tuples $\langle \sigma_0, \dots, \sigma_{n-1} \rangle$ of pairwise incomparable nodes of $T$. If $\langle \sigma_0, \dots, \sigma_{n-1} \rangle$ and $\langle \tau_0, \dots, \tau_{m-1} \rangle$ belong to $\mathbb{P}_T$ for some $n, m \in \omega$, we write $\langle \tau_0, \dots, \tau_{m-1} \rangle \succeq_T \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ if $m = n$ and $\tau_i \succeq \sigma_i$ for each $i < n$.

Thus, for every finite set $F \subseteq T$, we have that $\mathrm{lvs}(F) \in \mathbb{P}_T$. Going forward, we notate elements of $\mathbb{P}_T$ as tuples $\vec{\sigma}$, and we write $\sigma \in \vec{\sigma}$ if $\vec{\sigma} = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ and $\sigma = \sigma_i$ for some $i < n$.

**Definition 3.3.** Fix a perfect set $T \subseteq 2^{<\omega}$.

(1) A formula $\varphi(U)$ (where $U$ is a finite coded set) is *essential below* $\vec{\sigma} \in \mathbb{P}_T$ if for every $\vec{\tau} \succeq_T \vec{\sigma}$, there is a finite set $R \subseteq T \cap [\vec{\tau}]^{\preceq}$ such that $\varphi(R)$ holds.
(2) A set $A$ is *$T$-densely $Z$-hyperimmune* if for every $\vec{\sigma} \in \mathbb{P}_T$ and every $\Sigma^{0,Z}_1$ formula $\varphi(U)$ essential below $\vec{\sigma}$, there is a finite set $R \subseteq T \cap [\vec{\sigma}]^{\preceq} \cap \overline{A}$ such that $\varphi(R)$ holds.

**Theorem 3.4.** *Fix a set $Z$, a set $C \not\leq_T Z$ and a $Z$-computable perfect set $T \subseteq 2^{<\omega}$. For every set $A \subseteq T$ which is $T$-densely $Z$-hyperimmune, there is a set $G \subseteq T \cap \overline{A}$ such that $G \cong 2^{<\omega}$ and $C \not\leq_T G \oplus Z$.*

*Proof.* Fix $Z$, $C$, $T$ and $A \subseteq T$. We will build the set $G$ by forcing. Our forcing conditions are pairs $(F, \vec{\sigma})$, where $F \subseteq T \cap \overline{A}$ is a finite 2-branching set and $\vec{\sigma} \succeq_T \mathrm{lvs}(F)$. One can see the condition $c = (F, \vec{\sigma})$ as the Mathias condition $\tilde{c} = (F, T \cap [\vec{\sigma}]^{\preceq})$. A condition $d = (E, \vec{\tau})$ *extends* a condition $c = (F, \vec{\sigma})$ (written $d \leq c$) if the Mathias condition $\tilde{d}$ extends the Mathias condition $\tilde{c}$, that is, $F \subseteq E \subseteq F \cup (T \cap [\vec{\sigma}]^{\preceq})$ and $T \cap [\vec{\tau}]^{\preceq} \subseteq T \cap [\vec{\sigma}]^{\preceq}$. The following lemma shows that every sufficiently generic filter yields a set $G \cong 2^{<\omega}$.

**Lemma 3.5.** *For every condition $c = (F, \vec{\sigma})$ and for every $\sigma \in \vec{\sigma}$, there is some extension $d = (E, \vec{\tau})$ of $c$ such that $E \cap [\sigma]^{\prec} \neq \emptyset$.*

*Proof.* Fix $c$ and $\vec{\sigma}$. We add a split to $F$ above $\sigma$, and then prune $T \cap [\vec{\sigma}]^{\preceq}$ accordingly to obtain $\vec{\tau}$. Let $\varphi(U)$ be the $\Sigma^{0,Z}_1$ formula which holds if $U$ contains at least 2 incomparable nodes in $[\sigma]^{\preceq}$. The formula $\varphi(U)$ is essential below $\sigma$, so by $T$-dense $Z$-hyperimmunity of $A$, there is some finite set $R \subseteq T \cap [\vec{\sigma}]^{\preceq} \cap \overline{A}$ such that $\varphi(R)$ holds. Unfolding the definition of $\varphi(R)$, there are two incomparable nodes $\xi_0, \xi_1 \in T \cap [\sigma]^{\preceq} \cap \overline{A}$. Let $\vec{\tau} \succeq_T \mathrm{lvs}(F \cup \{\xi_0, \xi_1\})$ be such that $[\vec{\tau}]^{\preceq} \subseteq [\vec{\sigma}]^{\preceq}$. The condition $d = (F \cup \{\xi_0, \xi_1\}, \vec{\tau})$ is the desired extension of $c$. $\square$

A set $G$ *satisfies* $c = (F, \vec{\sigma})$ if $G \cong 2^{<\omega}$, $G \subseteq T \cap \overline{A}$, and $G$ satisfies the Mathias condition $\tilde{c}$, that is, $F \subseteq G \subseteq F \cup (T \cap [\vec{\sigma}]^{\preceq})$.

**Lemma 3.6.** *For every condition $c$ and every Turing functional $\Gamma$, there is an extension $d$ of $c$ forcing $\Gamma^{G \oplus Z} \neq C$.*

*Proof.* Fix $c = (F, \vec{\sigma})$ and $\Gamma$. Let $\varphi(U)$ be the $\Sigma^{0,Z}_1$ formula which holds if there is some $n \in \omega$ and two sets $E_0, E_1 \subseteq U$ such that $F \cup E_0$ and $F \cup E_1$ are both 2-branching, and $\Gamma^{(F \cup E_0) \oplus Z}(n) \neq \Gamma^{(F \cup E_1) \oplus Z}(n)$. We have two cases.

*Case 1.* $\varphi(U)$ is essential below $\vec{\sigma}$. Since $A$ is $T$-densely $Z$-hyperimmune, there is a finite set $R \subseteq T \cap [\vec{\sigma}]^{\preceq} \cap \overline{A}$ such that $\varphi(R)$ holds. Let $E \subseteq R$ be such that $F \cup E$

is 2-branching and $\Gamma^{(F \cup E) \oplus Z}(n) \neq C(n)$. Since $E \subseteq [\vec{\sigma}]^{\preceq} \succeq_T \text{lvs}(F)$, we can define some $\vec{\tau} \in \mathbb{P}_T$ such that $\vec{\tau} \succeq_T \text{lvs}(F \cup E)$ and $T \cap [\vec{\tau}]^{\preceq} \subseteq T \cap [\vec{\sigma}]^{\preceq}$. Since $E \subseteq T \cap \overline{A}$, the pair $(F \cup E, \vec{\tau})$ is a valid condition extending $c$ and forcing $\Gamma^{G \oplus Z} \neq C$.

*Case 2.* $\varphi(U)$ is not essential below $\vec{\sigma}$. Let $\vec{\tau} \succeq_T \vec{\sigma}$ be such that $\varphi(R)$ does not hold for every finite set $R \subseteq T \cap [\vec{\tau}]^{\preceq}$. The condition $d = (F, \vec{\tau})$ extends $c$ and forces $\Gamma^{G \oplus Z}$ to be either partial, or $Z$-computable. □

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter, where $c_s = (F_s, \vec{\sigma}_s)$, and let $G = \bigcup_s F_s$. By the definition of a condition, $G \subseteq \overline{A}$. By an iterated use of Lemma 3.5, $G \cong 2^{<\omega}$, and by Lemma 3.6, $C \not\leq_T G \oplus Z$. This completes the proof of Theorem 3.4. □

The proof of the next theorem is in a sense dual to the proof of Theorem 3.4. We could use basically the same forcing notion here as there, though we need less control in the conditions here and so use a slightly simpler forcing.

**Theorem 3.7.** $\mathsf{TT}^1_2$ *admits strong cone avoidance.*

*Proof.* Fix a set $Z$, a set $C \not\leq_T Z$, and a set $A \subseteq 2^{<\omega}$. By the cone avoidance basis theorem of Jockusch and Soare [18, Theorem 2.5], there is a Turing ideal $\mathcal{M} \models \mathsf{WKL}$ such that $Z \in \mathcal{M}$ and $C \notin \mathcal{M}$. First, suppose there is a perfect set $X \in \mathcal{M}$ such that $A$ is $X$-densely $X \oplus Z$-hyperimmune. Since $X, Z \in \mathcal{M}$ we have $X \oplus Z \in \mathcal{M}$, hence $C \not\leq_T X \oplus Z$. So we can apply Theorem 3.4 (with $X \oplus Z$ instead of $Z$, and $X$ instead of $T$) to get a $G \subseteq X \cap \overline{A}$ such that $G \cong 2^{<\omega}$ and $C \not\leq_T G \oplus X \oplus Z$, so also $C \not\leq_T G \oplus Z$, in which case we are done.

So suppose now that the above situation does not apply. We will now build a set $G \subseteq A$ by forcing. Our forcing conditions are Mathias conditions $(F, X)$, where $F \subseteq A$ is a finite 2-branching set, $X \in \mathcal{M}$, and $F \cup X$ is a perfect set. The extension is the usual Mathias extension.

**Lemma 3.8.** *For every condition $c = (F, X)$ and for every leaf $\sigma$ of $F$, there is some extension $d = (E, Y)$ of $c$ such that $E \cap [\sigma]^{\prec} \neq \emptyset$.*

*Proof.* Fix $c$ and $\sigma$. Since $F \cup X$ is perfect, so is $X_\sigma = X \cap [\sigma]^{\preceq}$. Moreover, $X_\sigma \in \mathcal{M}$, so by assumption, $A$ is not $X_\sigma$-densely $X_\sigma \oplus Z$-hyperimmune. Unfolding the definition, there is a $\Sigma^{0, X_\sigma \oplus Z}_1$ formula $\varphi(U)$ essential below some $\vec{\tau} \in \mathbb{P}_{X_\sigma}$, such that $R \cap A \neq \emptyset$ whenever $\varphi(R)$ holds and $R \subseteq X \cap [\vec{\tau}]^{\preceq}$. Let $\vec{\xi_0} \succeq_{X_\sigma} \vec{\tau}$ and $\vec{\xi_1} \succeq_{X_\sigma} \vec{\tau}$ be such that $[\vec{\xi_0}]^{\preceq} \cap [\vec{\xi_1}]^{\preceq} = \emptyset$. By essentiality of $\varphi(U)$, there are two finite sets $R_0 \subseteq X \cap [\vec{\xi_0}]^{\preceq}$ and $R_1 \subseteq X \cap [\vec{\xi_1}]^{\preceq}$ such that $\varphi(R_0)$ and $\varphi(R_1)$ hold. In particular, we can pick some $\rho_0 \in R_0 \cap A$ and $\rho_1 \in R_1 \cap A$. Note that $\rho_0, \rho_1 \in X \cap [\vec{\tau}]^{\preceq} \subseteq X \cap [\sigma]^{\preceq}$, and therefore $F \cup \{\rho_0, \rho_1\}$ is 2-branching. Let $Y \subseteq X$ be obtained by removing finitely many elements, so that $(F \cup \{\rho_1, \rho_1\}, Y)$ is a Mathias condition. Since $\rho_0, \rho_1 \in X$ and $F \cup X$ is perfect, so is $F \cup \{\rho_0, \rho_1\} \cup Y$. Therefore, the condition $d = (F \cup \{\rho_0, \rho_1\}, Y)$ is the desired extension of $c$. □

We say a set $G$ *satisfies* $c = (F, X)$ if $G \cong 2^{<\omega}$, $G \subseteq A$, and $G$ satisfies $c$ as a Mathias condition.

**Lemma 3.9.** *For every condition $c$ and every Turing functional $\Gamma$, there is an extension $d$ of $c$ forcing $\Gamma^{G \oplus Z} \neq C$.*

*Proof.* Fix $c = (F, X)$ and $\Gamma$. For every $\sigma \in \mathrm{lvs}(F)$, the set $X_\sigma = X \cap [\sigma]^{\preceq}$ is perfect and belongs to $\mathcal{M}$, so by assumption, $A$ is not $X_\sigma$-densely $X_\sigma \oplus Z$-hyperimmune. Unfolding the definition, there is a $\Sigma^{0,X_\sigma \oplus Z}_1$ formula $\varphi_\sigma(U)$ essential below some $\vec{\tau}_\sigma \in \mathbb{P}_{X_\sigma}$, such that $R \cap A \neq \emptyset$ whenever $\varphi_\sigma(R)$ holds and $R \subseteq X_\sigma \cap [\vec{\tau}_\sigma]^{\preceq}$. Fix a $X \oplus Z$-computable enumeration $\vec{\xi}_0, \vec{\xi}_1, \dots$ of all $\vec{\xi} \in \mathbb{P}_{X_\sigma}$ such that $\vec{\xi} \succeq_{X_\sigma} \vec{\tau}_\sigma$. For each $i \in \omega$, let $R_i \subseteq X \cap [\vec{\xi}_i]^{\preceq}$ be a finite set such that $\varphi_\sigma(R_i)$ holds. Note that $R_i \cap A \neq \emptyset$ by assumption. Define $\mathcal{C}_\sigma$ to be the class of all $P \in X^\omega$ such that $(\forall i \in \omega) P(i) \in R_i$. Since the $R_i$ are finite, $\mathcal{C}_\sigma$ is a $\Pi^{0,X \oplus Z}_1$ class. In particular, by choice of the $R_i$, there is some $P \in \mathcal{C}_\sigma$ such that $\mathrm{ran}(P) \subseteq A$. Moreover, by the usual pairing argument, for every $P \in \mathcal{C}_\sigma$, there is some $\tau \in \vec{\tau}_\sigma$ such that $\mathrm{ran}(P)$ is dense in $X \cap [\tau]^{\preceq}$.

Let $\mathcal{C}$ be the $\Pi^{0,X \oplus Z}_1$ class of all $\langle P_\sigma : \sigma \in \mathrm{lvs}(F)\rangle$, where $P_\sigma \in \mathcal{C}_\sigma$ such that, for every pair of finite sets $E_0, E_1 \subseteq \bigcup_{\sigma \in \mathrm{lvs}(F)} \mathrm{ran}(P_\sigma)$ with $F \cup E_0$ and $F \cup E_1$ both 2-branching, it is not the case that $\Gamma^{(F \cup E_0) \oplus Z} \downarrow \neq \Gamma^{(F \cup E_1) \oplus Z} \downarrow$. We have two cases.

*Case 1.* $\mathcal{C}$ is empty. For each $\sigma \in \mathrm{lvs}(F)$, let $P_\sigma \in \mathcal{C}_\sigma$ be such that $\mathrm{ran}(P_\sigma) \subseteq A$. In particular, $\langle P_\sigma : \sigma \in \mathrm{lvs}(F)\rangle \notin \mathcal{C}$, so by definition of $\mathcal{C}$, there is a finite set $E \subseteq \bigcup_{\sigma \in \mathrm{lvs}(F)} \mathrm{ran}(P_\sigma)$ such that $F \cup E$ is 2-branching, and some $n \in \omega$ such that $\Gamma^{(F \cup E) \oplus Z}(n) \neq C(n)$. Let $Y \subseteq X$ be obtained by removing finitely elements, so that $(F \cup E, Y)$ is a valid Mathias condition. Since $E \subseteq X$ and $F \cup X$ is perfect, so is $F \cup E \cup Y$. The condition $d = (F \cup E, Y)$ is an extension of $c$ forcing $\Gamma^{G \oplus Z} \neq C$.

*Case 2.* $\mathcal{C}$ is non-empty. Since $\mathcal{M} \models \mathsf{WKL}$, there is some $\langle P_\sigma : \sigma \in \mathrm{lvs}(F)\rangle \in \mathcal{C} \cap \mathcal{M}$. For each $\sigma \in \mathrm{lvs}(F)$, let $\tau_\sigma \in \vec{\tau}_\sigma$ be such that $\mathrm{ran}(P_\sigma)$ is dense in $X \cap [\tau_\sigma]^{\preceq}$. Let $Y$ be obtained by $\vec{P} \oplus Z$-computably thinning out the set $X$ so that $X \cap [\tau_\sigma]^{\preceq}$ is perfect for each $\sigma \in \mathrm{lvs}(F)$. Note that $Y \in \mathcal{M}$. The condition $d = (F, Y)$ is an extension of $c$ forcing $\Gamma^{G \oplus Z}$ to be either partial, or $X \oplus Z$-computable. □

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter, where $c_s = (F_s, X_s)$, and let $G = \bigcup_s F_s$. By the definition of a condition, $G \subseteq A$. By an iterated use of Lemma 3.8, $G \cong 2^{<\omega}$, and by Lemma 3.9, $C \not\leq_T G \oplus Z$. This completes the proof of Theorem 3.7. □

For general interest, we note the following immediate consequence of Theorem 3.7, which may be considered a first step in the direction of proving our main theorem. However, in the proof of the main theorem, we will actually need the full version of Theorem 3.7 rather than merely this corollary.

**Corollary 3.10.** $\mathsf{STT}^2_2$ *admits cone avoidance. Hence,* $\mathsf{RCA}_0 \nvdash \mathsf{STT}^2_2 \to \mathsf{ACA}_0$.

*Proof.* Consider any set $Z$, any $Z$-computable stable $f : [2^{<\omega}]^2 \to 2$, and any $C \not\leq_T Z$. For each $c < 2$, let $A_c$ be the set of all $\sigma \in 2^{<\omega}$ such that $(\exists n)(\forall \tau)[(\tau \succeq \sigma \wedge |\tau| \geq n) \to f(\sigma, \tau) = c$. By stability of $f$, $A_0 = \overline{A_1}$. By Theorem 3.7, there exists a $c < 2$ and a $G \subseteq A_c$ such that $G \cong 2^{<\omega}$ and $C \not\leq_T G \oplus Z$. Now $G$ can be $(G \oplus Z)$-computably pruned to $H \subseteq G$ such that $H \cong 2^{<\omega}$ and $f(\sigma, \tau) = c$ for all $\langle \sigma, \tau \rangle \in [H]^2$. And we have $H \oplus Z \leq_T G \oplus Z$, hence $C \not\leq_T H \oplus Z$. The rest of the corollary now follows by Lemma 2.13. □

## 4. The tree theorem for pairs and cone avoidance

Our goal in this section is to prove our main theorem, which we will do in the following more specific form.

**Theorem 4.1.** $\mathsf{TT}^2_2$ *admits cone avoidance. Hence,* $\mathsf{RCA}_0 \nvdash \mathsf{TT}^2_2 \to \mathsf{ACA}_0$.

In order to prove the theorem, we need to introduce an adaptation of the bushy tree forcing framework. Bushy tree forcing was developed by Kumabe [20] and Kumabe and Lewis [21] and has been employed to prove a number of results in algorithmic randomness and classical computability theory, particularly to do with diagonally-noncomputable functions. We refer the reader to Khan and Miller [19] for a survey on some of these results, along with a primer on bushy tree forcing as it is used to prove them.

The use of this forcing for the purposes of studying combinatorial principles like Ramsey's theorem is more recent, with some early examples by Patey [26, 27, 29]. Our treatment here will be self-contained.

Recall from Definition 2.1 that $\tau \in T \subseteq 2^{<\omega}$ is *at level $n$ in $T$* if there exist precisely $n$ proper initial segments of $\tau \in T$, and the set $T$ is *at level $n$* if every leaf is at level $n$. Recall from Definition 2.2 that a set $T \subseteq 2^{\omega}$ is *$h$-branching* for a function $h$ if it has a unique root and every node at level $n$ in $T$ which is not a leaf has exactly $h(n)$ immediate successors.

**Definition 4.2.** Given two sets $T, S \subseteq 2^{<\omega}$, we write $S \lhd T$ if $S \subseteq T$ and whenever $\tau \in S$ and $\sigma$ is a proper initial segment of $\tau$ in $T$, then $\sigma \in S$. We say that a set $B \subseteq T$ is *$h$-big in $T$* for some function $h$ if there is an $h$-branching set $S \lhd T$ such that $\operatorname{lvs}(S) \subseteq B$.

In particular, if $S \lhd T$, then any node at level $n$ in $S$ is at level $n$ in $T$. Also note that relation $\lhd$ is transitive. The following lemma is a standard combinatorial fact about bushy tree forcing (see [19], Lemma 2.4). As our framework is slightly different from the general one, we provide a proof for the sake of completeness.

**Lemma 4.3.** *Fix $T \subseteq 2^{<\omega}$. If $B \cup C$ is $(h+g-1)$-big in $T$, then either $B$ is $h$-big in $T$ or $C$ is $g$-big in $T$.*

*Proof.* Fix an $(h+g-1)$-branching set $S \lhd T$ such that $\operatorname{lvs}(S) \subseteq B \cup C$. We label each $\sigma \in S$ by either $B$ or $C$, as follows. Label each $\sigma \in \operatorname{lvs}(S)$ by $B$ if $\sigma \in B$, and by $C$ if $\sigma \notin B$ (in which case, of course, $\sigma \in C$). Now fix $\sigma \in S \smallsetminus \operatorname{lvs} S$, and assume by induction that every successor of $\sigma$ in $S$ has already been labeled. Say $\sigma$ is at level $n$. If at least $h(n)$ successors of $\sigma$ are labeled by $B$, then label $\sigma$ by $B$ as well. Otherwise, label $\sigma$ by $C$, and notice that as $S$ is $(h+g-1)$-branching, this means at least $g(n)$ successors of $\sigma$ in $S$ are labeled by $C$. This completes the labeling. We now define $S' \lhd T$ as follows. Add the root of $S$ to $S'$. Having added $\sigma$ to $S'$, add to $S'$ either the least $h(n)$ successors of $\sigma$ labeled by $B$ or the least $g(n)$ successors labeled by $C$, depending as $\sigma$ is itself labeled $B$ or $C$, respectively. Then $S'$ witnesses that either $B$ is $h$-big in $T$ or $C$ is $g$-big in $T$, as desired. $\square$

**Definition 4.4.** Fix $k \geq 2$ and $p \geq 1$. Define $h_{k,p} : \omega \to \omega$ inductively as follows:

(1) $h_{k,p}(0) = kp - 1$;
(2) $h_{k,p}(n+1) = h_{k,kp-1}(n)$ for all $n$.

Note that for all $k, p$ as above we have $kp - 1 \geq 1$, so $h_{k,p}$ is well-defined. This function was designed so that it satisfies the following combinatorial lemma.

**Lemma 4.5.** *Fix $k \geq 2$ and $p \geq 1$. Let $T \subseteq 2^{<\omega}$ be an $h_{k,p}$-branching set at level $n$, and $g : [T]^2 \to k$ be a coloring. There is a $p$-branching set $S \lhd T$ at level $n$*

*such that*

$$(\forall \sigma \in S \setminus \mathrm{lvs}(S))(\exists c < k)(\forall \tau \in \mathrm{lvs}(S))[\sigma \prec \tau \to g(\sigma, \tau) = c].$$

*Proof.* By induction over $n$. The case $n = 0$ is vacuously true. Let $T \subseteq 2^{<\omega}$ be an $h_{k,p}$-branching set at level $n+1$ and $g : [T]^2 \to k$ be a coloring. For each node $\xi$ at level 1 in $T$, let $T_\xi = T \cap [\xi]^{\preceq}$ and $g_\xi : [T_\xi]^2 \to k$ be the restriction of $g$ to $T_\xi$. Note that every node at level $m$ in $T_\xi$ is at level $m+1$ in $T$. In particular, if it is not a leaf in $T_\xi$, then it has $h_{k,p}(m+1) = h_{k,kp-1}(m)$ immediate successors in $T_\xi$. Therefore, $T_\xi$ is an $h_{k,kp-1}$-branching set at level $n$. By induction hypothesis, there is a $(kp-1)$-branching set $S_\xi \lhd T_\xi$ at level $n$ such that

$$(\forall \sigma \in S_\xi \setminus \mathrm{lvs}(S_\xi))(\exists c < k)(\forall \rho \in \mathrm{lvs}(S_\xi))[\sigma \prec \rho \to g(\sigma, \rho) = c].$$

Note that since $T$ is $h_{k,p}$-branching, there are $h_{k,p}(0) = kp-1$ nodes at level 1 in $T$, so $T_1 = \{\epsilon\} \cup \bigcup_\xi S_\xi$ is a $(kp-1)$-branching set at level $n+1$, where $\epsilon$ is the root of $T$. Moreover, $T_1 \lhd T$. For each $i < k$, let $B_i = \{\tau \in \mathrm{lvs}(T_1) : g(\epsilon, \tau) = i\}$. The set $B_0 \cup \dots \cup B_{k-1}$ is $(kp-1)$-big in $T_1$, so by Lemma 4.3, there is some $i < k$ such that $B_i$ is $p$-big in $T_1$. Then by definition of $p$-bigness of $B$, there is a $p$-branching set $S \lhd T_1 \lhd T$ such that $\mathrm{lvs}(S) \subseteq B_i$. We claim that $S$ satisfies the desired property. Fix some $\sigma \in S \setminus \mathrm{lvs}(S)$. We have two cases.

In the first case, $\sigma = \epsilon$. Let $c = i$. Since $\mathrm{lvs}(S) \subseteq B_i$, then for each $\tau \in \mathrm{lvs}(S)$ such that $\sigma \prec \tau$, $g(\epsilon, \tau) = i = c$.

In the second case, $\sigma \in T_\xi$ for some $\xi$ at level 1 in $T$. Let $c < k$ be such that $(\forall \rho \in \mathrm{lvs}(S_\xi))[\sigma \prec \rho \to g(\sigma, \rho) = c]$. This color exists by definition of $S_\xi$. For every $\rho \in \mathrm{lvs}(S)$ such that $\sigma \prec \rho$, $\rho \in S_\xi$, so $f(\sigma, \rho) = c$. This completes the proof of Lemma 4.5. ☐

**Lemma 4.6.** *Fix some $k \geq 2$, a set $Z$, a $Z$-computable $h_{k,2}$-branching perfect set $T \subseteq 2^{<\omega}$ and a $Z$-computable coloring $f : [2^{<\omega}]^2 \to k$. For every $n$, there is a 2-branching set $S \lhd T \upharpoonright n$ at level $n$ and a $Z$-computable set $X \subseteq T$ such that $S \cup X$ is perfect and*

$$(\forall \sigma \in S)(\exists c < k)(\forall \rho \in X)[\sigma \prec \rho \to f(\sigma, \rho) = c].$$

*Proof.* Fix $n$. For each $\tau \in \mathrm{lvs}(T \upharpoonright n)$, let $\sigma_0, \sigma_1, \dots, \sigma_n$ be the initial segments of $\tau$ in $T$, with $\sigma_n = \tau$. Apply $\mathsf{TT}^1_{k^{n+1}}$ over $T \cap [\tau]^{\prec}$ with the coloring $\rho \mapsto \langle f(\sigma_0, \rho), \dots, f(\sigma_n, \rho)\rangle$ to obtain a $Z$-computable set $X_\tau \subseteq T \cap [\tau]^{\prec}$ such that $\{\tau\} \cup X_\tau$ is perfect and $X_\tau$ is homogeneous for some color $\langle c^\tau_{\sigma_0}, \dots, c^\tau_{\sigma_n}\rangle$. Let $g : [T \upharpoonright n]^2 \to k$ be the coloring defined by

$$g(\sigma, \tau) = \begin{cases} c^\tau_\sigma & \text{if } \tau \in \mathrm{lvs}(T \upharpoonright n) \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 4.5 applied to $T \upharpoonright n$ and $g$, there is a 2-branching set $S \lhd T \upharpoonright n$ at level $n$ such that

$$(\forall \sigma \in S \setminus \mathrm{lvs}(S))(\exists c < k)(\forall \tau \in \mathrm{lvs}(S))[\sigma \prec \tau \to g(\sigma, \tau) = c]$$

Let $X = \bigcup_{\tau \in \mathrm{lvs}(S)} X_\tau$. Since $X_\tau \subseteq [\tau]^{\prec}$ and $\{\tau\} \cup X_\tau$ is perfect for every $\tau \in \mathrm{lvs}(S)$, then $S \cup X$ is perfect. We claim that $S$ and $X$ satisfy the desired property. Fix some $\sigma \in S$. We have two cases.

In the first case, $\sigma \in \mathrm{lvs}(S)$. Set $c = c^\sigma_\sigma$. For every $\rho \in X$ such that $\sigma \prec \rho$, $\rho \in X_\sigma$. By definition of $X_\sigma$, $f(\sigma, \rho) = c^\sigma_\sigma = c$.

In the second case, $\sigma$ is a proper initial segment of some $\tau \in \mathrm{lvs}(S)$. Set $c = c^{\tau}_{\sigma}$. For every $\rho \in X$ such that $\sigma \prec \rho$, there is some $\tau_1 \in \mathrm{lvs}(S)$ such that $\rho \in X_{\tau_1}$. By definition of $S$, there is some color $c_1$ such that $(\forall \tau_2 \in \mathrm{lvs}(S))[\sigma \prec \tau_2 \to g(\sigma, \tau_2)] = c_1$. By letting $\tau_2 = \tau$, we obtain $c_1 = c^{\tau}_{\sigma}$, and by letting $\tau_2 = \tau_1$, we obtain $c_1 = c^{\tau_1}_{\sigma}$. It follows that $c = c^{\tau_1}_{\sigma}$, and by definition of $X_{\tau_1}$, $f(\sigma, \rho) = c^{\tau_1}_{\sigma} = c$. This completes the proof of Lemma 4.6. $\square$

Our next result is the weaker form of Theorem 4.1 for $\mathsf{CTT}^2_2$. Notice that together with Corollary 3.10, this gives us cone avoidance for both $\mathsf{STT}^2_2$ and $\mathsf{CTT}^2_2$. However, this is by itself not enough to yield Theorem 4.1, as it does not imply cone avoidance for the join, $\mathsf{STT}^2_2 + \mathsf{CTT}^2_2$. We will prove that this is the case at the end of the section.

**Theorem 4.7.** *$\mathsf{CTT}^2_2$ admits cone avoidance.*

*Proof.* Fix a set $Z$, a set $C \not\leq_T Z$, and a $Z$-computable coloring $f : [2^{<\omega}]^2 \to 2$. We will construct a set $G \subseteq 2^{<\omega}$ over which $f$ is stable, such that $G \cong 2^{<\omega}$ and $C \not\leq_T G \oplus Z$. The set $G$ will be constructed by a forcing whose conditions are Mathias conditions $(F, X)$, where $F$ is a finite 2-branching set, $C \not\leq_T X \oplus Z$, and $F \cup X$ is a perfect set. Moreover, we require that

$$(\forall \sigma \in F)(\exists c < 2)(\forall \tau \in X)[\sigma \prec \tau \to f(\sigma, \tau) = c]$$

The extension is the usual Mathias extension. The following lemmas shows that every sufficiently generic filter for this notion of forcing yields a set $G \cong 2^{<\omega}$.

**Lemma 4.8.** *For every condition $c = (F, X)$ and for every leaf $\sigma$ of $F$, there is some extension $d = (E, Y)$ of $c$ such that $E \cap [\sigma]^{\prec} \neq \emptyset$.*

*Proof.* Fix some leaf $\sigma \in F$. Since $F \cup X$ is perfect, we can pick three pairwise-incomparable nodes $\xi_0, \xi_1, \xi_2 \in X \cap [\sigma]^{\prec}$. In particular, $X \cap [\xi_i]^{\prec}$ is perfect for each $i < 3$, so by applying $\mathsf{TT}^1_2$ to $X \cap [\xi_i]^{\prec}$ with $\rho \mapsto f(\xi_i, \rho)$, we obtain an $X \oplus Z$-computable set $X_i \subseteq X \cap [\xi_i]^{\prec}$ and a color $c_i < 2$ such that $\{\xi_i\} \cup X_i$ is perfect, and $f(\xi_i, \rho) = c_i$ for each $\rho \in X_i$. By the pigeonhole principle, there is some $c < 2$ and some $i_0 < i_1 < 3$ such that $c = c_{i_0} = c_{i_1}$. By removing finitely many elements of $(X \setminus [\sigma]^{\prec}) \cup X_{i_0} \cup X_{i_1}$, we obtain a $Z$-computable set $Y$ such that the condition $d = (F \cup \{\xi_{i_0}, \xi_{i_1}\}, Y)$ is a valid extension of $c$. $\square$

We say a set $S$ *satisfies* $c = (F, X)$ if $S \cong 2^{<\omega}$ and $S$ satisfies $c$ as a Mathias condition.

**Lemma 4.9.** *For every condition $c$ and every Turing functional $\Gamma$, there is an extension $d$ of $c$ forcing $\Gamma^{G \oplus Z} \neq C$.*

*Proof.* Fix $c = (F, X)$ and $\Gamma$. For every $\xi \in \mathrm{lvs}(F)$, let $X_\xi$ be an $X \oplus Z$-computable $h_{2,2}$-branching perfect subtree of $X \cap [\xi]^{\prec}$. Let $\mathcal{C}$ be the $\Pi^{0, X \oplus Z}_1$ class of all $\langle S_\xi : \xi \in \mathrm{lvs}(F)\rangle$ such that $S_\xi \lhd X_\xi$ is a perfect 2-branching set for each $\xi \in \mathrm{lvs}(F)$. Let $\mathcal{D}$ be the $\Pi^{0, X \oplus Z}_1$ class of all $\langle S_\xi : \xi \in \mathrm{lvs}(F)\rangle \in \mathcal{C}$ such that for every pair $E_0, E_1 \subseteq \bigcup_\xi S_\xi$ of finite sets with $F \cup E_0$ and $F \cup E_1$ both 2-branching, it is not the case that $\Gamma^{(F \cup E_0) \oplus Z} \downarrow = \Gamma^{(F \cup E_1) \oplus Z} \downarrow$. We have two cases.

*Case 1.* $\mathcal{D}$ is empty. By compactness, there is some $n$ such that for every $\langle S_\xi : \xi \in \mathrm{lvs}(F)\rangle$ where $S_\xi \lhd X_\xi \upharpoonright n$ is a 2-branching set at level $n$, there is a set $E \subseteq \bigcup_\xi S_\xi$ and some $m$ such that $F \cup E$ are is 2-branching, and $\Gamma^{(F \cup E) \oplus Z}(m) \neq C(m)$. By

Lemma 4.6 applied to each $X_\xi$, there are some $\langle S_\xi : \xi \in \mathrm{lvs}(F)\rangle$ where $S_\xi \lhd X_\xi \upharpoonright n$ is a 2-branching set at level $n$, and some $Z$-computable sets $\langle Y_\xi \subseteq X_\xi : \xi \in \mathrm{lvs}(F)\rangle$ such that $S_\xi \cup Y_\xi$ is perfect and

$$(\forall \sigma \in S_\xi)(\exists c < 2)(\forall \rho \in Y_\xi)[\sigma \prec \rho \rightarrow f(\sigma, \rho) = c].$$

Let $Y \subseteq \bigcup_\xi Y_\xi$ be obtained by removing finitely elements, so that $(F \cup E, Y)$ is a valid Mathias condition and $F \cup E \cup Y$ is perfect. The condition $d = (F \cup E, Y)$ is an extension of $c$ forcing $\Gamma^{G \oplus Z} \neq C$.

*Case 2.* $\mathcal{D}$ is non-empty. By the cone avoidance basis theorem, there is some $\vec{S} = \langle S_\xi : \xi \in \mathrm{lvs}(F)\rangle \in \mathcal{D}$ such that $C \not\leq_T \vec{S} \oplus Z$. Let $Y$ be obtained by removing finitely many elements of $\bigcup_\xi S_\xi$ so that $F \cup Y$ is perfect. The condition $d = (F, Y)$ is an extension of $c$ forcing $\Gamma^{G \oplus Z}$ to be either partial, or $X \oplus Z$-computable. □

Let $\mathcal{F} = \{c_0, c_1, \dots\}$ be a sufficiently generic filter, where $c_s = (F_s, X_s)$, and let $G = \bigcup_s F_s$. By definition of a condition, $f$ is stable over $G$. By Lemma 4.8, $G \cong 2^{<\omega}$, and by Lemma 4.9, $C \not\leq_T G \oplus Z$. This completes the proof of Theorem 4.7. □

We are now ready to prove that $\mathsf{TT}^2_2$ admits cone avoidance.

*Proof of Theorem 4.1.* Fix a set $Z$, a set $C \not\leq_T Z$, and a $Z$-computable coloring $f : [2^{<\omega}]^2 \to 2$. By cone avoidance of $\mathsf{CTT}^2_2$ (Theorem 4.7), there is a set $H_0 \subseteq 2^{<\omega}$ such that $H_0 \cong 2^{<\omega}$, $C \not\leq_T H_0 \oplus Z$, and over which $f$ is stable. Define the $\Delta^{0, H_0 \oplus Z}_2$ set $A \subseteq H_0$ by

$$A = \{\sigma \in H_0 : (\exists n)(\forall \tau \in H_0)[(\tau \succ \sigma \wedge |\tau| \geq n) \rightarrow f(\sigma, \tau) = 1\}.$$

By strong cone avoidance of $\mathsf{TT}^1_2$ (Theorem 3.7), there is a subset $H_1$ of $A$ or $H_0 \setminus A$ such that $H_1 \cong 2^{<\omega}$ and $C \not\leq_T H_1 \oplus H_0 \oplus Z$. Note $H_1 \oplus H_0 \oplus Z$ computes an $f$-homogeneous set $H \subseteq H_1$ such that $H \cong 2^{<\omega}$. In particular, $C \not\leq_T H \oplus Z$. □

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Department of Mathematics, University of Connecticut, Storrs, Connecticut U.S.A.
*E-mail address*: damir@math.uconn.edu

Department of Mathematics, University of California, Berkeley, Berkeley, California U.S.A.
*E-mail address*: ludovic.patey@computability.fr