STRONG REDUCTIONS BETWEEN COMBINATORIAL PRINCIPLES

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Dedicated to Robert I. Soare, on the occasion of his retirement.

Abstract. This paper is a contribution to the growing investigation of strong reducibilities between Π^1_2 statements of second-order arithmetic, viewed as an extension of the traditional analysis of reverse mathematics. We answer several questions of Hirschfeldt and Jockusch [13] about Weihrauch (uniform) and strong computable reductions between various combinatorial principles related to Ramsey's theorem for pairs. Among other results, we establish that the principle SRT^2_2 is not Weihrauch or strongly computably reducible to $\mathsf{D}^2_{<\infty}$, and that COH is not Weihrauch reducible to $\mathsf{SRT}^2_{<\infty}$, or strongly computably reducible to SRT^2_2. The last result also extends a prior result of Dzhafarov [9]. We introduce a number of new techniques for controlling the combinatorial and computability-theoretic properties of the problems and solutions we construct in our arguments.

1. Introduction

The traditional approach in reverse mathematics has been to compare a given theorem with several benchmark subsystems of second-order arithmetic, and isolate the weakest that it can be proved in, and the strongest that can in turn be proved from it, all over the base theory RCA_0. Often, these two subsystems coincide, meaning that the given theorem is actually equivalent over RCA_0 to one of the benchmark subsystems. The early tendency in the subject was to focus almost exclusively on theorems of this kind, giving rise to what is often called the "big five phenomenon" (see, e.g., Montalbán [19, Section 1]), after the five principal subsystems that arise most commonly in this endeavor. The past decade, however, has seen a growing shift away from this tendency, and towards looking at principles that do not admit equivalences to any of the benchmark subsystems. Spurred on primarily by interest in the strength of Ramsey's theorem for pairs, this new direction has since generated a zoo (see [10]) of inequivalent principles from various branches of mathematics with an intricate and fascinating structure of relationships. We refer the reader to Hirschfeldt's wonderful new text [12] for an introduction and summary. For a comprehensive general background on reverse mathematics, we refer to Simpson [25].

1.1. **Strong reductions.** We take up an even newer direction in reverse mathematics, in which we replace provability over RCA_0 by stronger relations, as a means

The author was partially supported by NSF grant DMS-1400267, and is grateful to Denis Hirschfeldt, David Nichols, Ludovic Patey, and the anonymous referee for numerous helpful comments.

of getting at finer distinctions between members of the above-mentioned zoo. The majority of principles studied in reverse mathematics are Π^1_2 statements of the form

$$\forall X \Theta(X) \to \exists Y \Psi(X, Y),$$

where Θ and Ψ are arithmetical properties of sets. The sets X satisfying Θ are then called the *instances* of the principle, and the sets Y satisfying $\Psi(X, Y)$ are the *solutions* to X. To be precise, the above representation need not be unique, but for each principle we typically have a particular one in mind, and so with it, a particular set of instances and solutions. For example, in the statement of König's lemma, that every finitely-branching tree of infinite height contains an (infinite) path through it, the instances are the finitely-branching trees of infinite height, and the solutions the paths. There is a well-understood correspondence between the strength of a theorem, in the sense of reverse mathematics, and the relative complexities, in the sense of computability theory, of instances and solutions of that theorem (see [15, Section 1]). For instance, if for every set X, principle P has an X-computable instance all of whose solutions compute X', while principle Q satisfies that every X-computable instance of it has an X'-computable solution, then every ω-model of P will also be a model of Q. Thus, modulo issues of induction, a computability-theoretic fact yields an implication over RCA_0.

More generally, consider the following reducibilities between Π^1_2 statements.

Definition 1.1. Let P and Q be Π^1_2 principles.

(1) Q is *computably reducible* to P, written $\mathsf{Q} \leq_{\mathrm{c}} \mathsf{P}$, if every instance X of Q computes an instance $\widehat{X}$ of P, such that for every solution $\widehat{Y}$ to $\widehat{X}$, we have that $X \oplus \widehat{Y}$ computes a solution Y to X.
(2) Q is *strongly computably reducible* to P, written $\mathsf{Q} \leq_{\mathrm{sc}} \mathsf{P}$, if every instance X of Q computes an instance $\widehat{X}$ of P, such that every solution $\widehat{Y}$ to $\widehat{X}$ computes a solution Y to X.
(3) Q is *Weihrauch reducible* to P, written $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}$, if there exist Turing functionals Φ and Ψ such that for every instance X of Q, we have that Φ^X is an instance of P, and for every solution $\widehat{Y}$ to Φ^X we have that $\Psi^{X \oplus \widehat{Y}}$ is a solution to X.
(4) Q is *strongly Weihrauch reducible* to P, written $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$, if there exist Turing functionals Φ and Ψ such that for every instance X of Q, we have that Φ^X is an instance of P, and for every solution $\widehat{Y}$ to Φ^X we have that $\Psi^{\widehat{Y}}$ is a solution to X.

The notions of $\leq_{\mathrm{sc}}$ and $\leq_{\mathrm{c}}$ arise frequently in the proofs of implications over RCA_0, but were first formally defined and studied by Dzhafarov [9]. The notions of $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ (also sometimes called *uniform reducibility* and *strong uniform reducibility*, respectively) were introduced by Weihrauch [27], under a different formulation than given above, and have been widely applied in the study of computable analysis. Later, these were independently re-discovered by Dorais et. al. [7], and shown to be the uniform versions of computable reducibility and strong computable reducibility, respectively (see [7, Appendix A]). The investigation of these notions as an extension of the traditional framework of reverse mathematics has seen a recent surge of interest. (An updated bibliography is maintained by Brattka [1].)

It is easy to see that all of these reducibilities are transitive, and Figure 1 summarizes the relationships hold between them. Furthermore, if Q is reducible to P

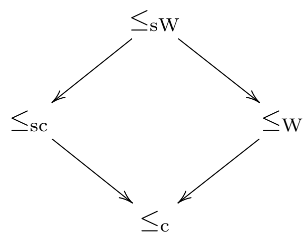

FIGURE 1. Relations between notions of reduction. An arrow from one reducibility to another means that whenever Q is reducible to P according to the first then it is also reducible according to the second. In general, no relations hold other than the ones shown.

according to any one of these notions, then, as in the example above, every ω-model of P is an ω-model of Q. Not every implication over RCA_0, or even over ω-models, must come from some such reduction, but in practice, most do. (Hirschfeldt and Jockusch [13, Section 4.1] introduced a related reduction notion that does capture implication over ω-models.) Consequently, our motivation for studying principles under these reducibilities, as opposed to the traditional framework of reverse mathematics, is twofold. First, we can tease apart subtle differences between principles that are impossible to detect with provability in RCA_0 alone. And second, where we do not know if there is an implication over RCA_0 between two principles, we can lend credence to a negative answer by showing that some of the stronger reducibilities fail. As a way of extending the scope of reverse mathematics, this program has been seeing increasing interest, as we describe further below.

In the present article, we apply this analysis, for both of the above-mentioned motivations, to several combinatorial principles related to Ramsey's theorem for pairs. Our work is inspired by Questions 1.6 (1)–(4) below, posed by Hirschfeldt and Jockusch in a draft of [13], which we provide answers to here.

1.2. **Ramsey's theorem.** For a set X, let $[X]^n$ denote the set of all subsets F of X of size n. We define a *k-coloring of* $[X]^n$ to be a map $c : [X]^n \to C$, where C is a non-empty finite set. Most commonly, X is ω and C is the set $k = \{0, \ldots, k-1\}$ for some $k \geq 1$. As is customary, if $F \in [X]^n$ and the members of F are $x_0 < \cdots < x_{n-1}$, we write $c(x_0, \ldots, x_{n-1})$ in place of $c(F)$. A set $H \subseteq X$ is *homogeneous* for $c : [X]^n \to C$ if there is a $j \in C$ such that c is constant on $[H]^n$ and equal to j, in which case we also say H is *homogeneous with color j*. Ramsey's theorem is the following Π^1_2 principle, for fixed $n, k \geq 1$.

Statement 1.2 (RT^n_k)**.** Every coloring $c : [\omega]^n \to k$ has an infinite homogeneous set.

For any (standard) k and j, it is easy to see that RT^n_k is equivalent over RCA_0 to RT^n_j. However, this result turns out to be sensitive to the distinctions drawn by Definition 1.1. While $\mathsf{RT}^n_j \leq_{\mathrm{sW}} \mathsf{RT}^n_k$ for all $j < k$, it was shown by Dorais et al. [7, Theorem 3.1] that $\mathsf{RT}^n_k \not\leq_{\mathrm{sW}} \mathsf{RT}^n_j$. This was subsequently improved by Hirschfeldt and Jockusch [13, Theorem 3.3] and Rakotoniaina [21] (see [3, Theorem 4.21]) to show that also $\mathsf{RT}^n_k \not\leq_{\mathrm{W}} \mathsf{RT}^n_j$, and more recently by Patey [20, Corollary 3.15] to show that even $\mathsf{RT}^n_k \not\leq_{\mathrm{c}} \mathsf{RT}^n_j$.

Note that RT^1_k for standard k is just the pigeonhole principle, and as such is provable in RCA_0. It follows by a well-known result of Jockusch [16, Lemma 5.9] that if $n \geq 3$ then RT^n_k is equivalent to ACA_0 over RCA_0. By contrast, Seetapun (see [22, Theorem 2.1]) showed that this is false for the $n = 2$ case: RT^2_k is strictly weaker than ACA_0. Seetapun's theorem set off an industry of research over the past twenty years into pinning down the precise strength of RT^2_k. A prominent approach to this problem is that of Cholak, Jockusch, and Slaman [4, Section 7], who introduced two closely related principles that RT^2_2 can be decomposed into. The first is a restriction to a special class of colorings: say $c : [X]^2 \to C$ is *stable* if for each $x \in X$ the limit $\lim_{y \in X} c(x, y)$ exists, meaning there is $j \in C$ and a z such that $c(x, y) = j$ for all $y \in X$ with $y \geq z$. (When $X = \omega$ or there is no possibility of confusion, we write simply $\lim_y c(x, y)$ in place of $\lim_{y \in X} c(x, y)$.)

Statement 1.3 (SRT^2_k)**.** Every stable coloring $c : [\omega]^2 \to k$ has an infinite homogeneous set.

We let $\mathsf{SRT}^2_{<\infty}$ denote the statement $\forall k\, \mathsf{SRT}^2_k$ (and reserve the notation SRT^2 for when we do not wish to specify a subscript). Typically, one works not with SRT^2_k directly, but with D^2_k, defined as follows. Call a set $L \subseteq X$ *limit-homogeneous* for a stable coloring $c : [X]^2 \to C$ if there is a $j \in C$ such that $\lim_y c(x, y) = j$ for all $x \in L$. Thus, every homogeneous set is obviously limit-homogeneous. Conversely, if L is limit-homogeneous, then $c \oplus L$ can uniformly compute a homogeneous set by thinning.

Statement 1.4 (D^2_k)**.** Every stable coloring $c : [\omega]^2 \to k$ has an infinite limit-homogeneous set.

We let $\mathsf{D}^2_{<\infty}$ denote the statement $\forall k\, \mathsf{D}^2_k$. Using the limit lemma, it is easy to see that SRT^2_k and D^2_k are computably equivalent, as are $\mathsf{SRT}^2_{<\infty}$ and $\mathsf{D}^2_{<\infty}$ (see [4, Lemma 3.5] for a proof.) That this can be formalized in RCA_0 is nontrivial, and was shown by Chong, Lempp, and Yang [5, Theorem 1.4]. As for RT, there has been interest in SRT^2 and D^2 for different numbers of colors: Patey [20, Theorem 3.14] has shown that $\mathsf{D}^2_k \not\leq_{\mathrm{c}} \mathsf{RT}^2_j$ whenever $j < k$.

The second related principle introduced by Cholak, Jockusch, and Slaman [4] is a sequential version of RT^1_2 with finite errors. Given a family of sets $\vec{X} = \langle X_n : n \in \omega \rangle$, a set Y is called $\vec{X}$*-cohesive* if for each n, either $Y \cap X_n$ or $Y \cap \overline{X_n}$ is finite.

Statement 1.5 (COH)**.** Every family of sets $\vec{X} = \langle X_n : n \in \omega \rangle$ has an infinite cohesive set Y.

Over RCA_0, the principles RT^2_k and $\mathsf{SRT}^2_k + \mathsf{COH}$ are equivalent (see Mileti [18, Corollary A.1.4]), and for many years it was a major open question in reverse mathematics whether this split is proper. Hirschfeldt et al. [14, Theorems 2.3 and 3.7] constructed an ω-model of $\mathsf{COH} + \neg\mathsf{SRT}^2_2$, thereby showing that COH does not imply RT^2_2, but the question of whether SRT^2_2 implies COH was only answered considerably later, by Chong, Slaman, and Yang [6]. The model for the the latter separation, however, has nonstandard first-order part, and so the question of whether every ω-model of SRT^2_2 is also a model of COH remains open.

In light of the discussion above, one way of moving towards a negative answer to this question is to compare the strengths of COH and SRT^2_k under the stronger reducibilities of Definition 1.1. As a first step in this direction, Dzhafarov [9, Corollary 1.10] showed that $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{D}^2_{<\infty}$. In an earlier draft of their own paper on stronger reducibilities between variants of Ramsey's theorem (since updated to reflect our results below), Hirschfeldt and Jockusch [13] asked about extensions of this fact.

Question 1.6 (Hirschfeldt and Jockusch)**.**

(1) Is it the case that $\mathsf{SRT}^2_2 \leq_{\rm W} \mathsf{D}^2_2$?
(2) Is it the case that $\mathsf{SRT}^2_2 \leq_{\rm sc} \mathsf{D}^2_2$?
(3) Is it the case that $\mathsf{COH} \leq_{\rm W} \mathsf{SRT}^2_2$?
(4) Is it the case that $\mathsf{COH} \leq_{\rm sc} \mathsf{SRT}^2_2$?

Note that the fourth question is only of interest if the second has a negative answer, since otherwise its answer is no by the corresponding result for D^2_2 mentioned above. These questions, in turn, motivate the other two. But there is an additional reason the first question is interesting. Namely, Hirschfeldt and Jockusch [13, Proposition 4.8] showed that the answer to it is almost yes, in that SRT^2_2 is reducible to two iterated applications of D^2_2. (This is made precise using the notion of *generalized uniform reducibility*; see [13, Section 4.2]. In the parlance of Weihrauch reducibility, it can also be stated as $\mathsf{SRT}^2_2 \leq_{\rm W} \mathsf{D}^2_2 * \mathsf{D}^2_2$, where $*$ denotes the *compositional product*; see [2, Proposition 2.4] for the definition.)

We give negative answers to all four of the above questions, and along the way obtain other results about these, and related, principles with regards to Weihrauch and strong computable reducibility. In the case of each of the first three questions, our proofs will use an ad hoc combinatorial argument to diagonalize solutions to instances of the principle on the left, plus an intricate generalization of the proof of Seetapun's theorem from [22] to build solutions to instances of the principle on the right. Consequently, we dedicate Section 2 to developing this generalization. By contrast, to obtain a negative answer to the fourth question we introduce a new method for directly constructing homogeneous sets for colorings. While the arguments based on our extension of Seetapun's argument produce relatively effective constructions, the same is not true of this new method, which uses countably many iterates of the hyperjump.

Our computability-theoretic notation and conventions throughout will be standard, following Soare [26] or Downey and Hirschfeldt [8, Chapter 2].

2. A generalization of Seetapun's argument

For a tree $T \subseteq \omega^{<\omega}$, we write $\operatorname{ran}(T)$ for $\bigcup_{\alpha \in T} \operatorname{ran}(\alpha)$, and $|T|$ for $\sup_{\alpha \in T} |\alpha|$. For an infinite set X, let $Inc(X)$ be the set of all increasing sequences of elements of X, i.e., all $\sigma \in \omega^{<\omega}$ with $\sigma(n) \in X$ for all $n < |\sigma|$ and $\sigma(n) < \sigma(n')$ for all $n < n' < |\sigma|$. So if T is a subtree of $Inc(X)$, then any path through T has infinite range. Additionally, if such a T has bounded width, then it is finite as a set of strings if and only $\operatorname{ran}(T)$ is finite, if and only if $|T|$ is a finite number, if and only if T is well-founded.

Throughout, if X and Y are sets, we will write $X < Y$ to mean that X is finite and $\max X < \min Y$. If $A = \{x_0, \ldots, x_{n-1}\}$, we will also sometimes write $x_0, \ldots, x_{n-1} < Y$. If T and U are subtrees of $Inc(\omega)$, we will write $T < U$ to mean that $\operatorname{ran}(T) < \operatorname{ran}(U)$. Note that this implies that $\operatorname{ran}(T)$ is finite, hence in particular that T is well-founded.

Definition 2.1. Let φ and ψ be properties of finite sets.

(1) A *φ-tree* is a finite subtree T of $Inc(\omega)$ of bounded width such that if $\alpha \in T$ is a terminal node, then $\varphi(F)$ holds for some finite set $F \subseteq \mathrm{ran}(\alpha)$.
(2) A *φ-sequence* is a (finite or infinite) sequence $T_0 < T_1 < \cdots$ of φ-trees.
(3) The *ψ-generated subtree* of a φ-sequence $T_0 < T_1 < \cdots$ is the tree of all $\alpha \in \omega^{<\omega}$ such that $\alpha(n) \in \mathrm{ran}(T_n)$ for all $n < |\alpha|$, and $\neg\psi(F)$ holds for all finite sets $F \subseteq \mathrm{ran}(\alpha \upharpoonright |\alpha| - 1)$.

We highlight a few basic but important observations about the above definition.

- The ψ-generated subtree is a finitely-branching subtree of $Inc(\omega)$.
- If the ψ-generated subtree of an infinite φ-sequence is finite, it is a ψ-tree.
- If the ψ-generated subtree U of some φ-sequence $T_0 < T_1 < \cdots$ is a ψ-tree, then U is also the ψ-generated subtree of $T_0 < \cdots < T_{|U|-1}$.
- If φ is Σ^0_1 and there exists an infinite φ-sequence, then there also exists a computable infinite φ-sequence.
- If ψ is Σ^0_1, then the ψ-generated subtree of any computable infinite φ-sequence is computable and computably-bounded.

We add to this list the following two general results.

Lemma 2.2. *If there exists no infinite φ-sequence, then there is a number z such that $\neg\varphi(F)$ holds for all finite sets $F > z$.*

Proof. If there is no such z, we can define a sequence of finite sets $F_0 < F_1 < \cdots$ such that $\varphi(F_n)$ holds for all n. For each n, let T_n be the tree of all initial segments of the principal function of F_n. Then $T_0 < T_1 < \cdots$ is an infinite φ-sequence. □

Lemma 2.3. *If there exists an infinite φ-sequence, and the ψ-generated subtree of every infinite φ-sequence is finite, then there exists an infinite ψ-sequence.*

Proof. Let $T_0 < T_1 < \cdots$ be any φ-sequence, and assume that for some $n \in \omega$, we have defined ψ-trees U_m for all $m < n$. Since all the U_m are finite, there is an s such that $U_m < T_s$ for all $m < n$. Now $T_s < T_{s+1} < \cdots$ is a φ-sequence, so if we let U_n be its ψ-generated subtree, then U_n is finite and hence a ψ-tree. By choice of s, we have $U_m < U_n$ for all $m < n$, so by induction, $U_0 < U_1 < \cdots$ is a ψ-sequence. □

In the above construction, notice that if, for each n, we choose the least s such that $U_m < T_s$ for all $m < n$, then $s = \sum_{m<n} |U_m|$.

Definition 2.4. Fix $k \geq 1$, and let $\varphi_0, \ldots, \varphi_{k-1}$ be properties of finite sets. A *$\langle\varphi_0, \ldots, \varphi_{k-1}\rangle$-forest* is a collection of sequences of trees

$$\{T_{j,0} < \cdots < T_{j,s_j} : j < k\}$$

with the following properties. For each $j < k$, the sequence $T_{j,0} < \cdots < T_{j,s_j}$ is a φ_j-sequence, and for each $j < k-1$ and each $n < s_{j+1}$, the tree $T_{j+1,n}$ is the ψ_{j+1}-generated subtree of

$$T_{j,s} < \cdots < T_{j,s+|T_{j+1,n}|-1},$$

where $s = \sum_{m<n} |T_{j+1,m}|$.

So for instance, in a $\langle\varphi_0, \varphi_1, \varphi_2\rangle$-forest $\{T_{0,j} < \cdots < T_{j,s_j} : j < 3\}$, we have:

- the tree $T_{1,0}$ is the φ_1-generated subtree of $T_{0,0} < \cdots < T_{0,|T_{1,0}|-1}$;
- the tree $T_{1,1}$ is the φ_1-generated subtree of $T_{0,|T_{1,0}|} < \cdots < T_{0,|T_{1,0}|+|T_{1,1}|-1}$;
- the tree $T_{2,0}$ is the φ_2-generated subtree of $T_{1,0} < \cdots < T_{1,|T_{2,0}|-1}$;

and so on.

Lemma 2.5. *Fix $k \geq 1$, let $\varphi_0, \ldots, \varphi_{k-1}$ be properties of finite sets. Suppose there exists an infinite φ_0-sequence, and that for each $j < k-1$, every infinite φ_j-sequence has finite φ_{j+1}-generated subtree. Then there exists a $\langle \varphi_0, \ldots, \varphi_{k-1} \rangle$-forest, and moreover, if $\varphi_0, \ldots, \varphi_{k-1}$ are Σ^0_1, then it can be found uniformly computably.*

Proof. By Lemma 2.3, there exists an infinite φ_j-sequence $T_{j,0} < T_{j,1} < \cdots$ for each $j < k$. By the proof of that lemma and the subsequent remark, we may choose these sequences so that for each $j < k-1$ and every n, the tree $T_{j+1,n}$ is the φ_{j+1}-generated subtree of the sequence $T_{j,s} < T_{j,s+1} < \cdots$, where $s = \sum_{m<n} |T_{j+1,m}|$. Let $s_{k-1} = 0$, and for $j < k-1$, let $s_j = (\sum_{n \leq s_{j+1}} |T_{j+1,n}|) - 1$. Then $\{T_{0,j} < \cdots < T_{j,s_j} : j < k\}$ is a $\langle \varphi_0, \ldots, \varphi_{k-1} \rangle$-forest. □

The next result is the main motivation behind Definition 2.1, and appears in less general form in [9, Lemma 2.3]. It is an elaboration on the combinatorial core of Seetapun's argument [22, Lemma 2.14], which is essentially the special case when $k = 2$ below.

Lemma 2.6. *Fix $k \geq 1$, let $\varphi_0, \ldots, \varphi_{k-1}$ be properties of finite sets, and suppose $\{T_{0,j} < \cdots < T_{j,s_j} : j < k\}$ is a $\langle \varphi_0, \ldots, \varphi_{k-1} \rangle$-forest. Given $l : \omega \to k$, there is a $j < k$ and a terminal α in some $T_{j,n}$ such that $l(x) = j$ for all $x \in \operatorname{ran}(\alpha)$.*

Proof. Suppose the conclusion of the lemma fails for all $j < k-1$. We claim that for each such j, every $T_{j,n}$ contains some terminal α having an x in its range with $l(x) > j$. This is true of $j = 0$ by assumption, so assume it for an arbitrary $j < k-1$. Fix any $n \leq s_{j+1}$, and let $s = \sum_{m<n} |T_{j+1,m}|$ and $t = s + |T_{j+1,n}| - 1$, so that $T_{j+1,n}$ is the φ_{j+1}-generated subtree of $T_{j,s} < \cdots < T_{j,t}$. For each m with $s \leq m \leq t$, choose an x_m in some $\alpha \in T_{j,m}$ with $l(x_m) > j$. Then $\alpha = \langle x_s, \ldots, x_t \rangle$ is a terminal node in $T_{j+1,n}$, and we have $l(x) \geq j+1$ for all $x \in \operatorname{ran}(\alpha)$. If $j+1 < k-1$, we must have that $l(x) > j+1$ for some $x \in \operatorname{ran}(\alpha)$, by assumption, which completes the induction. If, on the other hand, $j+1 = k-1$, then we have $l(x) = k-1$ for all $x \in \operatorname{ran}(\alpha)$. Thus, the conclusion of the lemma holds for $k-1$. □

We close by noting that Definitions 2.1 and 2.4 can both be formulated for restrictions to any infinite set X, by replacing $Inc(\omega)$ with $Inc(X)$ throughout. (For instance, we define a *φ-tree inside* X to be a finite subtree T of $Inc(X)$ of bounded width such that if $\alpha \in T$ is terminal then $\varphi(F)$ holds for some finite $F \subseteq \operatorname{ran}(\alpha)$.) All of the above lemmas extend in the obvious way to this more general setting.

3. Comparing SRT^2 and D^2

In this section, we prove that SRT^2_2 is not Weihrauch or strongly computably reducible to $\mathsf{D}^2_{<\infty}$. These results are comparatively straightforward, and thus serve as a good illustration of how the setup of the preceding section will be applied in the considerably more complicated arguments of Section 4.

We begin with the following definition.

Definition 3.1. Let $\mathbb{P}$ be the following notion of forcing. A condition is a triple $p = \langle n^p, c^p, \ell^p \rangle$, where n^p is a number, c^p is a function $[n^p]^2 \to 2$, and ℓ^p is a function $n^p \to 2 \times \omega$ such that if $\ell^p(x) = \langle v, u \rangle$ for some $x < n^p$ then $c^p(x, y) = v$ for all y with $u \leq y < n^p$. A condition q extends p if $n^q \geq n^p$, $c^q \supseteq c^p$, and $\ell^q \supseteq \ell^p$.

Thus, c^p is a finite approximation to a coloring of pairs, and we think of ℓ^p as a commitment about what color the numbers smaller than n^p should limit to under this coloring, and by what point their colors should stabilize. It is clear that if G is a sufficiently generic filter on $\mathbb{P}$, then $\bigcup_{p \in G} c^p$ is a stable coloring of pairs. We say a stable coloring $c : [\omega]^2 \to 2$ *extends* a condition p, and write $c \succeq p$, if c extends c^p as a function and respects ℓ^p, in the sense that if $\ell^p(x) = \langle v, u\rangle$ then $c(x, y) = v$ for all $y \geq u$.

Our results below, while driven by an interest in $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sc}}$, actually prove considerably more. Namely, each constructs an instance c of SRT^2_2 such that *every* instance of $\mathsf{D}^2_{<\infty}$, computable from c or not, has a solution not computing (uniformly or strongly, respectively) any solution to c. Such stronger versions of $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sc}}$ were first asked about by Montalbán (unpublished), in connection with the results of [9], and have subsequently also emerged in [13] and [20]. The "forward" reductions in the definitions of these reductions, in other words, seem less essential in many cases.

Theorem 3.2. *Let Ψ be a Turing functional. There is a computable coloring $c : [\omega]^2 \to 2$ such that for every $k \geq 1$, every stable coloring $d : [\omega]^2 \to k$ has an infinite limit-homogeneous set L with the property that Ψ^L is not equal to any infinite homogeneous set for c.*

Proof. We may assume Ψ is $\{0, 1\}$-valued, so that if it is total, it defines a set. Also, if there is an infinite set X such that Ψ^X does not define an infinite set for any $Y \subseteq X$, we may let c be arbitrary, and for each $d : [\omega]^2 \to k$ simply choose an infinite limit-homogeneous set $L \subseteq X$. Thus, assume this is not the case. We build a computable sequence of conditions $p_0 \geq p_1 \geq \cdots$, and let $c = \bigcup_{t\in\omega} c^{p_t}$.

Let $p_0 = \langle 0, \emptyset, \emptyset\rangle$, and assume inductively that we have defined p_t for some $t = \langle k, m\rangle \in \omega$. Let $\varphi(F)$ be the formula asserting that $F > m$ and there exist two numbers $x_1 > x_0 \geq n^{p_t}$ with $\Psi^F(x_0)\downarrow = \Psi^F(x_1)\downarrow = 1$. We consider the following three cases.

Case 1: there exists no infinite φ-sequence. By Lemma 2.2, there is a number z such that $\neg\varphi(F)$ holds for all finite sets $F > z$. In this case, if Y is any subset of $X = \{x \in \omega : x > \max\{m, z\}\}$ and Ψ^Y is total, then the set it defines contains at most one number $x \geq n^{p_t}$, and so is finite. This contradicts our assumption above about Ψ.

Case 2: there exists an infinite φ-sequence with infinite φ-generated subtree. Let P be any path through this tree, recalling that it has infinite range. By definition, $\neg\varphi(F)$ holds for every finite set $F \subseteq \mathrm{ran}(P)$. Hence, if Y is any subset of $X = \{x \in \mathrm{ran}(P) : x > m\}$ and Ψ^Y is total, the set it defines is finite, which is again a contradiction.

Case 3: otherwise. Define $\varphi_0 = \cdots = \varphi_{k-1} = \varphi$, and observe that by the failure of Cases 1 and 2, we are exactly in the hypotheses of Lemma 2.5. Since φ is Σ^0_1, we may consequently effectively search for, and find, a $\langle\varphi_0, \ldots, \varphi_{k-1}\rangle$-forest $\langle T_{j,0} < \cdots < T_{j,s_j} : j < k\rangle$. Since each $T_{j,i}$ is a φ_j-tree, for every terminal $\alpha \in T_{j,i}$ there is a finite set $F > m$ and two numbers $x_1 > x_0 \geq n^{p_t}$ with $\Psi^F(x_0)\downarrow = \Psi^F(x_1)\downarrow = 1$. Let u be the maximum of all these computations, across all α and F, which by usual conventions also means that u is bigger than all the numbers x_0 and x_1. Let p_{t+1} be the extension of p_t with $n^{p_{t+1}} = u$, and with $c^{p_{t+1}}(x, y) = 0$ and $\ell^{p_{t+1}}(x) = \langle 1, u\rangle$ for all x, y with $n^{p_t} \leq x < y < u$.

Suppose $d : [\omega]^2 \to k$ is given. We may assume that for each $j \in C$ there are infinitely many x with $\lim_y d(x, y) = j$. Otherwise, we would let $C \subseteq k$ be maximal such that this is the case for all $j \in C$, and let m be such that $\lim_y d(x, y) \in C$ for all $x > m$. We would then work not with d but with the coloring $d' : [\omega]^2 \to |C|$ defined as follows. Writing $C = \{c_0, \ldots, c_{|C|-1}\}$ then for all $x < y$, if $d(x, y) = c_j$ for some j then $d'(x, y) = j$, and otherwise $d'(x, y) = 0$. Note that this d' would have the property that for each $j < |C|$ there are infinitely many x with $\lim_y d'(x, y) = j$, and that d and d' have the same infinite limit-homogeneous sets.

Now let $\langle T_{j,0} < \cdots < T_{j,s_j} : j < k\rangle$ be the $\langle \varphi_0, \ldots, \varphi_{k-1}\rangle$-forest from the construction of p_{t+1} with $t = \langle k, u\rangle$. By Lemma 2.6, there is a $j < k$, an $i \leq s_j$, and a terminal $\alpha \in T_{j,i}$, such that $\lim_y d(x, y) = j$ for all $x \in \mathrm{ran}(\alpha)$. Fix $F > u$ inside $\mathrm{ran}(\alpha)$ and $x_1 > x_0 \geq n^{p_t}$ with $\Psi^F(x_0)\downarrow = \Psi^F(x_1)\downarrow = 1$. Since infinitely many numbers limit to j under d, there is an infinite limit-homogeneous set L for d extending F above the use of these computations. In particular, Ψ^L agrees with Ψ^F on x_0 and x_1, so if Ψ^L is total the set it defines contains x_0 and x_1. But by construction, $c(x_0, x_1) = 0$ and $\lim_y c(x_0) = 1$, so this set cannot be homogeneous for c. Since L is also homogeneous for the original d, this completes the proof. □

Corollary 3.3. $\mathsf{SRT}^2_2 \not\leq_{\mathrm{W}} \mathsf{D}^2_{<\infty}$.

For our next result, recall that a set X is said to have *PA degree* relative to a set Y, written $X \gg Y$, if every Y-computable infinite finitely-branching tree has an X-computable path through it. By a result of Simpson [24, Theorem 6.5], if $X \gg Y$ then there is a Z with $X \gg Z \gg Y$. Below, we say a coloring $c : [\omega]^2 \to 2$ is 1-X-*generic* if for each $\Sigma^0_1(X)$-definable class of $\mathbb{P}$-conditions W there is a $p \preceq c$ that meets W (meaning $p \in W$) or avoids it (meaning $q \notin W$ for all $q \leq p$).

Theorem 3.4. *Let X be a set of PA degree, and let $c : [\omega]^2 \to 2$ be 1-X-generic for $\mathbb{P}$. Given $k \geq 1$, a finite set C of size k, and an infinite set $Y \ll X$, every stable coloring $d : [Y]^2 \to C$ has an infinite limit-homogeneous set $L \subseteq Y$ that computes no infinite homogeneous set for c.*

Proof. Fix k, C, Y, and $d : [Y]^2 \to C$. We may assume that for every infinite $Z \subseteq Y$ satisfying $Z \ll X$, and for each $j < k$, there are infinitely many $x \in Z$ with $\lim_y d(x, y) = j$. If not, then by repeatedly passing to subsets we could find a $Z \subseteq Y$ with $Z \ll X$ and a set $C' \subseteq C$ such that $d \upharpoonright [Z]^2$ has this property for all $j \in C'$. We would then work instead with the coloring $d' : [Z]^2 \to C'$ such that for all $x < y$, if $d(x, y) \in C$ then $d'(x, y) = d(x, y)$, and otherwise $d'(x, y)$ is the least $j \in C'$. In particular, every infinite limit-homogeneous set for d' (contained in Z) would then be an infinite limit-homogeneous for d.

For ease of notation, we may also assume $C = k$. We build infinite limit-homogeneous sets $L_0, \ldots, L_{k-1}$ for d, with colors $0, \ldots, k-1$, respectively, such that for every functional Ψ there is a $j < k$ with the property that Ψ^{L_j} is not equal to any infinite homogeneous set for c. From here it follows that one of the L_j computes no such homogeneous set, as desired.

We build $L_0, \ldots, L_{k-1}$ by forcing with conditions $(F_0, \ldots, F_{k-1}, I)$, where each F_j is a finite set satisfying $\lim_y d(x, y) = j$ for all $x \in F_j$, and $I > F_0, \ldots, F_{k-1}$ is an infinite subset of Y satisfying $I \ll X$. A condition $(F'_0, \ldots, F'_{k-1}, I')$ extends $(F_0, \ldots, F_{k-1}, I)$ if $F_j \subseteq F'_j \subseteq F_j \cup I$ for each $j < k$, and $I' \subseteq I$. Thus, this is just forcing with k many Mathias conditions sharing a common reservoir. We define

$$(F_{0,0}, \ldots, F_{0,k-1}, I_0) \geq (F_{1,0}, \ldots, F_{1,k-1}, I_1) \geq \cdots$$

with $\lim_s |F_{s,j}| = \infty$ for all $j < k$, and take $L_j = \bigcup_s F_{s,j}$. Let $(F_{0,0}, \ldots, F_{0,k-1}, I_0) = (\emptyset, \ldots, \emptyset, \omega)$, and suppose that we have defined $(F_{s,0}, \ldots, F_{s,k-1}, I_s)$ for some s.

If s is even, we wish to add one more element to each of the eventual sets L_j. By our assumption above, for each $j < k$, there must be infinitely many $x \in I_s$ with $\lim_y d(x,y) = j$. Hence, we may define $(F_{s+1,0}, \ldots, F_{s+1,k-1}, I_{s+1})$ so that $|F_{s+1,j}| = |F_{s,j}| + 1$ for all j, as desired.

Now let s be odd, say $s = 2e+1$. Let Ψ be the eth member of some fixed listing of all Turing functionals. For each $m \geq \min I_s$ and each $j < k$, let $\varphi^m_j(F)$ be the formula asserting that $F \geq m$ and there exist two numbers $x_1 > x_0 \geq m$ with $\Psi^{F_{s,j} \cup F}(x_0) \downarrow = \Psi^{F_{s,j} \cup F}(x_1) \downarrow = 1$. Let D be the set of all $\mathbb{P}$-conditions p such that for some $m \geq n^p$, the following are true.

- There exists a $\langle \varphi^m_0, \ldots, \varphi^m_{k-1} \rangle$-forest $\langle T_{j,0} < \cdots < T_{j,s_j} : j < k \rangle$ inside I_s.
- For each terminal α in each $T_{j,i}$, there is a finite set $F \geq m$ contained in $\mathrm{ran}(\alpha)$ and two numbers $x_1 > x_0 \geq m$ with $\Psi^{F_{s,j} \cup F}(x_0) \downarrow = \Psi^{F_{s,j} \cup F}(x_1) \downarrow = 1$, such that $x_0, x_1 < n^p$ and $c^p(x_0, x_1) = 0$ and $\ell^p(x_0) = \langle 1, u \rangle$ for some u.

Then D is Σ^0_1-definable over I_s, and hence over X, so since c is 1-X-generic, there must be a condition $p \preceq c$ that meets or avoids D.

Suppose first that p meets D. By Lemma 2.6, there is a $j < k$, an $i \leq s_j$, and a terminal $\alpha \in T_{j,i}$, such that $\lim_y d(x,y) = j$ for all $x \in \mathrm{ran}(\alpha)$. Since $m \geq \min I_s$, by definition of D there is an $F \geq \min I_s$ contained in $\mathrm{ran}(\alpha)$ and $x_1 > x_0 \geq \min I_s$ with $\Psi^{F_{s,j} \cup F}(x_0) \downarrow = \Psi^{F_{s,j} \cup F}(x_1) \downarrow = 1$, such that $c^p(x_0, x_1) = 0$ and $\ell^p(x_0) = \langle 1, u \rangle$ for some u. In particular, $\lim_y d(x,y) = j$ for all $x \in F$, and since $T_{j,i}$ is a φ^m_j-tree inside I_s, also $F \subseteq I_s$. Let $F_{s+1,j} = F_{s,j} \cup F$, and for all $j' \neq j$ let $F_{s+1,j'} = F_{s,j'}$. Let I_{s+1} be the set of all elements in I_s greater than F and the uses of the computations $\Psi^{F_{s,j} \cup F}(x_0)$ and $\Psi^{F_{s,j} \cup F}(x_1)$. Then $(F_{s+1,0}, \ldots, F_{s+1,k-1}, I_{s+1})$ is an extension of $(F_{s,0}, \ldots, F_{s,k-1}, I_s)$ guaranteeing that if Ψ^{L_j} is total, it contains x_0 and x_1. But as $c(x_0, x_1) = 0$ and $\lim_y c(x_0, y) = 1$, this set is not homogeneous for c.

So suppose instead that p avoids D, and let m be the larger of $\min I_s$ and n^p. We consider the following three cases.

Case 1: for some $j < k$, there exists no infinite φ^m_j-sequence inside I_s. Fix n_0 such that $\neg\varphi(F)$ holds for all finite sets subsets $F > n_0$ of I. Let $I_{s+1} = \{x \in I_s : x > n_0\}$ and let $F_{s+1,0}, \ldots, F_{s+1,k-1}$ be $F_{s,0}, \ldots, F_{s,k-1}$, respectively. Then $(F_{s+1,0}, \ldots, F_{s+1,k-1}, I_{s+1})$ extends $(F_{s,0}, \ldots, F_{s,k-1}, I_s)$, and if B is any set with $F_{s+1,j} \subseteq B \subseteq F_{s+1,j} \cup I_{s+1}$ and Ψ^B is total, the set it defines can contain at most one numbers $x \geq m$. In particular, Ψ^{L_j} cannot be an infinite set.

Case 2: for some $j < k-1$, there exists an I_s-computable infinite φ^m_j-sequence inside I_s with infinite φ^m_{j+1}-generated subtree. Notice that every path through this subtree is an infinite subset of I_s. Moreover, this subtree is I_s-computable and I_s-computably bounded, so since $I_s \ll X$, there is a path P satisfying $P \ll X$. By definition, $\neg\varphi(F)$ holds for every finite set $F \subseteq \mathrm{ran}(P)$. Let $I_{s+1} = \mathrm{ran}(P)$ and let $F_{s+1,0}, \ldots, F_{s+1,k-1}$ be $F_{s,0}, \ldots, F_{s,k-1}$, respectively. As in the previous case, this ensures that $\Psi^{L_{j+1}}$ cannot be an infinite set.

Case 3: otherwise. We are in the hypotheses of Lemma 2.5, so we may fix a $\langle \varphi^m_0, \ldots, \varphi^m_{k-1} \rangle$-forest $\langle T_{j,0} < \cdots < T_{j,s_j} : j < k \rangle$ inside I_s. For every terminal $\alpha \in T_{j,i}$ there is a finite set F and two numbers $x_1 > x_0 \geq m$ with $\Psi^{F_{s,j} \cup F}(x_0) \downarrow = \Psi^{F_{s,j} \cup F}(x_1) \downarrow = 1$. Let u be the maximum of all these computations, across all α and F, which means u is also bigger than all the numbers x_0 and x_1. Let q be the extension of p with $n^q = u$, and with $c^q(x, y) = 0$ and $\ell^q(x) = \langle 1, u\rangle$ for all x, y with $n^p \leq x < y < u$. In particular, since $m \geq n^p$ and $u > x_1 > x_0 \geq m$, we have that $c^q(x_0, x_1) = 0$ and $\ell^q(x_0) = \langle 1, u\rangle$. But this makes q an extension of p in D, a contradiction. Hence, this case cannot obtain. $\square$

It is easy to show that there is a 1-generic for $\mathbb{P}$ computable in $\emptyset'$ and that every such generic is low. The proof is the same as for Cohen forcing; see [17, Lemma 2.6].

Corollary 3.5. *There is a low stable coloring $c : [\omega]^2 \to 2$ such that for every $k \geq 2$, every stable coloring $d : [\omega]^2 \to k$ has an infinite limit-homogeneous set L that computes no infinite homogeneous set for c.*

Proof. Take X in the theorem to be a set of low PA degree, and let c be any low 1-X-generic for $\mathbb{P}$. $\square$

Note that since SRT^2_2 and D^2_2 are computably equivalent, we cannot hope to replace "low" above with "computable". The corollary can thus be seen as saying that this is a sharp division with respect to the jump hierarchy.

Corollary 3.6. $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_{<\infty}$.

4. COH and uniform reducibility

In this section, we look at Weihrauch reducibility and the principle COH, and among other things, show that COH is not Weihrauch reducible to $\mathsf{SRT}^2_{<\infty}$. The main technical ingredient will be Lemma 4.2 below. To state it, we first need some definitions. Say $\sigma' \in \omega^{<\omega}$ is a 1-*extension* of $\sigma \in \omega^{<\omega}$ if $\sigma' \succeq \sigma$ and $|\sigma'| = |\sigma| + 1$.

Definition 4.1.

(1) A *tree enumeration* is a partial computable function U from ω to the finite subsets of $Inc(\omega)$ such that $U(0) \simeq \{\emptyset\}$, and if $U(x+1)\downarrow$ then $U(x)\downarrow$, and every string in $U(x + 1)$ is a 1-extension of some string in $U(x)$.

(2) An *uniform sequence of tree enumerations* is a computable (set of indices for a) sequence of tree enumerations $\langle U_l : l \in \omega\rangle$ such that if $U_l(x)\downarrow$ and $U_{l'}(x')\downarrow$ for some $l' < l$ and numbers x, x', then there is a stage s such that $U_l(x)[s]\uparrow$ and $U_{l'}(x)[s]\downarrow$.

Thus, all the members of $U(x)$ are strings of length x, and we think of U as enumerating a subtree of $\omega^{<\omega}$, which we also identify with U. If U is total as a function, then U as a tree is actually computable, computably bounded, and infinite. And in this case, the the non-extendible nodes of U are precisely those σ that belong to some $U(x)$ but have no extension in $U(x + 1)$. We say σ *looks extendible* at s if it has an extension in the largest x such that $U(x)[s]\downarrow$.

We can now think of a uniform sequence of tree enumerations as a sequence of attempts at enumerating an infinite tree, such that each attempt stops before the next attempt begins. In particular, at most one of these attempts can succeed. Observe also that there is a computable listing of all (indices of) uniform sequences of tree enumerations. We assume that if $U_l(x)[s]\downarrow$ for some x then $l < s$, and say U_l *looks infinite* at s if there is an x such that $U_l(x)[s]\downarrow$ and $U_{l'}(0)[s]\uparrow$ for all $l' > l$. Note that if U is infinite as a tree, then the range of every path through U is infinite.

The way this notion will come up is as follows. Suppose $\varphi_0, \ldots, \varphi_{k-1}$ are Σ^0_1 formulas of finite sets. We can then search for a computable $\langle \varphi_0, \ldots, \varphi_{k-1} \rangle$-forest using the simple inductive method in the proof of Lemma 2.5, which we call the *canonical search*. Recall how this goes. For each j, we try to build a φ_j-sequence. As this sequence gets longer, we build more of its φ_{j+1}-generated subtree: each new member of the φ_j-sequence (i.e., each new φ_j-tree) allows us to build one more level of this subtree. If the subtree eventually becomes a φ_{j+1}-tree (i.e., every terminal node has a finite subset of its range satisfying φ_{j+1}), we can add it to a φ_{j+1}-sequence, and start the construction of a new φ_{j+1}-generated subtree. This gives rise to a uniform sequence of tree enumerations: we let U_l enumerate the levels of the lth φ_{j+1}-generated subtree, until, if ever, it becomes a φ_{j+1}-tree. (For φ_0, we look at longer and longer segments of ω, or whatever infinite set we are working inside, to try to build the lth φ_0-tree, so we let U_l enumerate these initial segments.)

We can now state the lemma. Define the following fast-growing computable function, $\# : \omega \to \omega$. Let $\#(0) = 1$, and for $k \geq 1$ let

$$\#(k) = (k+1) \cdot \#(k-1)^k.$$

Thus, $\#(1) = 2$, $\#(2) = 3 \cdot 2^2 = 12$, $\#(3) = 4 \cdot 12^3 = 6912$, and so on.

Lemma 4.2. *Let Ψ be a Turing functional. Fix $k \geq 1$ and finite sets $C_0 \subseteq C$ with $|C - C_0| = k$. Let d be a partial computable coloring $[\omega]^2 \to C$, and let $\langle U_l : l \in \omega \rangle$ be an ordered sequence of tree enumerations. There exists a computable coloring $c : \omega \to k+1$, and for each $j < k$, a computable coloring $c_j : \omega \to \#(k-1)$, with the following properties.*

1. *If d is total and stable, some U_l is infinite, and $\lim_y d(x, y) \notin C_0$ for almost all x in the range of each path through U_l, then d has an infinite homogeneous set H contained in the range of a path through U_l such that Ψ^H is not an infinite almost-homogeneous set for c or one of the c_j.*
2. *Indices for c and the c_j as computable functions can be found uniformly from k, canonical indices for C_0 and C, an index for d as a partial computable function, an index for $\langle U_l : l \in \omega \rangle$, and an index for Ψ.*

We prove Lemma 4.2 in the next two subsections, using induction on k.

4.1. **The $k = 1$ case.** Note that since $\#(k-1) = \#(0) = 1$ in this case, and since every set is homogeneous for the trivial coloring $\omega \to 1$, we build only the coloring $c : \omega \to k + 1 = 2$ here.

Proof of Lemma 4.2 for $k = 1$. Assume for simplicity that $C - C_0 = \{0\}$. We build c, and for each $\sigma \in \omega^{<\omega}$, try to define a finite set $H_\sigma \subseteq \mathrm{ran}(\sigma)$ homogeneous for d with color 0 such that $H_\sigma > H'_\sigma$ for all $\sigma \succ \sigma'$. At each stage, all but finitely many of the H_σ will be undefined, and each of those that are defined may subsequently become undefined and redefined any number of times. However, if d satisfies the hypotheses of the lemma and U_l is infinite, we ensure there is a path P through U_l satisfying one of the following two outcomes.

- d has an infinite homogeneous set $H \subseteq \mathrm{ran}(P)$ such that Ψ^H is not infinite.
- There exists $\sigma_0 \prec \sigma_1 \prec \cdots \prec P$ such that each of the sets H_{σ_n} stabilizes to a finite set H_{P_n}, and $H = \bigcup_n H_{P_n} \subseteq \mathrm{ran}(P)$ is infinite and Ψ^H is not almost-homogeneous for c.

Certainly this suffices to prove the lemma.

We write $d(x,y)[s]\downarrow$ to mean that the coloring d converges on (x,y) in s or fewer steps, and follow the convention that if $d(x,y)[s]\downarrow$ then also $d(x,y')[s]\downarrow$ for all y' with $x<y'\leq y$. Given x, we computably guess at $\lim_y d(x,y)$ as follows: at stage s, choose the largest y with $x<y\leq s$ such that $d(x,y)[s]\downarrow$, and guess $d(x,y)$ to be the limit. Thus, if d is not actually a stable coloring, our guess about the limit may change infinitely often, but otherwise it will eventually be correct.

Construction. Initially, let all the H_σ be undefined. At the start of stage s, assume we have defined c on $\omega\upharpoonright s$ and let l be such that U_l looks infinite. We only define H_σ at this stage if σ is a terminal node in U_l that looks extendible at this stage. Alongside this definition, we also define a number $u_\sigma\geq H_\sigma$. By induction, we assume that each such σ has the same number, say n, of (not necessarily proper) initial segments σ' for which $H_{\sigma'}$ is defined already, and that if $H_{\sigma'}$ is defined then σ' is such an initial segment. Given any σ on U_l (not necessarily terminal), we denote its $(m+1)$st initial segment σ' such that $H_{\sigma'}$ is currently defined by σ_m, so that necessarily $n<m$ by assumption. Thus, if $\sigma'\prec\sigma$ and σ'_m is defined then so is σ_m and $\sigma'_m=\sigma_m$. We will also maintain that if σ and σ' are any two strings with σ_m and σ'_m defined then $|\sigma_m|=|\sigma'_m|$.

Now for all stages $t\geq s$, we start coloring $c(t)=n \bmod 2$, not defining or undefining any sets, until one or more of the following conditions applies at t.

(1) Each terminal σ in U_l that looks extendible has H_σ undefined (so $\sigma\neq\sigma_{n-1}$), and there is a finite set F such that:
  - $\emptyset\neq F\subseteq\operatorname{ran}(\sigma)$;
  - $\max_{m<n} u_{\sigma_m}<F$;
  - there is an $x\geq\max_{m<n} u_{\sigma_m}$ such that $\Psi^{\bigcup_{m<n}H_{\sigma_m}\cup F}(x)\downarrow=1$;
  - $d(x,y)\downarrow=0$ for all $x<y$ with $x\in\bigcup_{m<n}H_{\sigma_m}\cup F$ and $y\in F$;
  - all the $x\in F$ look like they limit to 0 under d.

(2) Some σ_m no longer looks extendible.

(3) Some x in some H_{σ_m} looks like it limits to a color other than 0.

(4) U_l no longer looks infinite.

By usual conventions, the first condition means that t is larger than $\max F$, the number x, and the use of the computation $\Psi^{\bigcup_{m<n}H_{\sigma_m}\cup F}(x)\downarrow=1$. In particular, $c(x)=n \bmod 2$. If this condition applies, fix the least such F in each σ, define $H_\sigma=F$, and define $u_\sigma=t$. If the second condition applies, then for any such σ_m we undefine the set H_{σ_m} and number u_{σ_m}. If the third condition applies, we choose the least m for which there is some such H_{σ_m}, and then undefine $H_{\sigma'_{m'}}$ and $u_{\sigma'_{m'}}$ for all σ' and all $m'\geq m$. Note that each terminal σ on U_l that looks extendible still has the same number of initial segments σ' for which $H_{\sigma'}$ is defined, and in the first case it is $n+1$ many. If the fourth condition applies, undefine all H_σ and u_σ, and start over with U_{l+1} instead of U_l. This completes the construction.

Verification. Clearly, c is total and computable. So suppose d is is total and stable, U_l is infinite, and almost all x in the range of each path through U_l limit to 0 under d. If there is a path P through U_l and an infinite homogeneous set $H\subseteq\operatorname{ran}(P)$ for d such that Ψ^H is not an infinite set, then we are done. So suppose not. Let s_0 be the least stage after which U_l looks infinite and none of the finitely many x that do not limit to 0 look like they do. Note that if H_σ is defined after stage s_0, it is consequently to a set all of whose members limit to 0 under d.

For every n, we claim there is a stage $s \geq 0$ such that if P is a path through U_l and $\sigma \prec P$ is a terminal node of U_l at stage s (which necessarily looks extendible), then each H_{σ_m} with $m < n$ is defined and has stabilized to some finite set H_{P_m}. This means, in particular, that every element of H_{P_m} actually limits to 0 under d. We prove the claim by induction. Fix n, and let s witness that the claim holds for all $m < n$. We may assume that s enumerates enough of U_l so that no terminal σ on U_l equals σ_{n-1}, so the latter is a proper initial segment of σ. For each path P through U_l and $\sigma \prec P$, write u_{P_m} for u_{σ_m} if H_{σ_m} has stabilized to H_{P_m}. If from some stage after s onwards, some (and hence every) terminal σ on U_l that looks extendible has H_σ undefined, then it must be that the first condition of the construction does not apply. By choice of s_0 and s, and the fact that U_l is a finitely-branching tree, this can only happen if there is a number $z \geq \max_{m<n} u_{P_m}$ and a path P through U_l whose range has no finite subset $F > z$ homogeneous for d with color 0 such that $\Psi^{\bigcup_{m<n} H_{Q_m} \cup F}(x) \downarrow= 1$ for some $x \geq \max_{m<n} u_{P_m}$. But then let H be any infinite homogeneous set for d containing $\bigcup_{m<n} H_{P_m}$ and otherwise only elements of $\mathrm{ran}(P)$ bigger than z, and observe that Ψ^H is bounded by $\max_{m<n} u_{P_m}$ and is thus finite, which is a contradiction. We conclude that there are infinitely many stages after s at which the H_σ for σ terminal in U_l and looking extendible are defined. And since, for any path P through U_l and any $\sigma \prec P$, such an H_σ can later only be undefined if some element of it starts looking like it does not limit to 0 (since it will always look extendible), it is now easy to see that one of these definitions must become permanent.

To conclude the proof, fix any path P through U_l. Note that since each H_{P_n} is non-empty, $H = \bigcup_n H_{P_n}$ is infinite. Furthermore, for each n there is an $x \geq \max_{m<n} u_{P_m} \geq \bigcup_{m<n} H_{P_m}$ such that $c(x) = n \bmod 2$ and $\Psi^{\bigcup_{m \leq n} H_{P_m}}(x) \downarrow= 1$. Since $\min F_{m+1}$ is larger than the use of this computation, this also means that $\Psi^H(x) \downarrow= 1$. Thus, Ψ^H contains infinitely many numbers colored 0 by c, and infinitely many colored 1, and so is not almost-homogeneous for c. □

4.2. **The $k > 1$ case.** In the $k = 1$ case of Lemma 4.2, the coloring d is essentially a 1-coloring, so finding a homogeneous set for it is straightforward. In the general case, we must now instead use the technology of Section 2 to build homogeneous sets, which complicates the argument considerably.

Proof of Lemma 4.2 for $k > 1$. We focus on the differences with the $k = 1$ case. Assume the lemma for k, and for simplicity, assume also that $C - C_0 = \{0, \ldots, k-1\}$. Along with c, we try to define k many finite sets $H_{\sigma,0}, \ldots, H_{\sigma,k-1}$ for each $\sigma \in \omega^{<\omega}$, with $H_{\sigma,j}$ homogeneous for d with color j. To build the c_j, we appeal to the inductive hypothesis. If d and U_l satisfy the hypotheses of the lemma, we ensure there is a path P through U_l satisfying one of the following outcomes.

- d has an infinite homogeneous set $H \subseteq \mathrm{ran}(P)$ such that Ψ^H is not infinite.
- For some $j < k$, d has an infinite homogeneous set $H \subseteq \mathrm{ran}(P)$ such that Ψ^H is not almost-homogeneous for c_j.
- There exists $\sigma_0 \prec \sigma_1 \prec \cdots \prec P$ such that for each $j < k$, each of the sets $H_{\sigma_n,j}$ stabilizes to a finite set $H_{P_n,j}$, and for some such j, $H = \bigcup_n H_{P_n,j} \subseteq \mathrm{ran}(P)$ is infinite and Ψ^H is not almost-homogeneous for c.

Construction of c. Whenever we define one of $H_{\sigma,0}, \ldots, H_{\sigma,k-1}$ for some σ then we define all of them, and exactly one of these sets is non-empty. As before, we also define a number u_σ, and we follow the same conventions and notations. Initially, let all the $H_{\sigma,j}$ be undefined. At the start of stage s, we assume that c is defined on $\omega \upharpoonright s$, and that every terminal σ on U_l that looks extendible has the same number, n, of initial segments σ' for which $H_{\sigma',0}, \ldots, H_{\sigma',k-1}$ are defined. For each such σ and each $j < k$, let $\varphi_{\sigma,j}(F)$ be the Σ^0_1 formula asserting:

- $\emptyset \neq F \subseteq \operatorname{ran}(\sigma)$;
- $\max_{m<n} u_{\sigma_m} < F$;
- there is an $x \geq \max_{m<n} u_{\sigma_m}$ such that $\Psi^{\bigcup_{m<n} H_{\sigma_m,j} \cup F}(x) \downarrow= 1$;
- $d(x,y) \downarrow= j$ for all $x < y$ with $x \in \bigcup_{m<n} H_{\sigma_m,j} \cup F$ and $y \in F$.

Note that if σ is terminal in U_l at s and σ' is terminal at some $t \geq s$, then $\varphi_{\sigma,j}$ and $\varphi_{\sigma',j}$ are the same formula so long as we did not define or undefine any sets between stages s and t.

For all stages $t \geq s$, we start coloring $c(t) = n \bmod (k+1)$ until one of the following conditions applies at t.

(1) Each terminal σ in U_l that looks extendible has H_σ undefined and the canonical search has found a $\langle \varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1} \rangle$-forest.
(2) Some σ_m no longer looks extendible.
(3) Some x in some $H_{\sigma_m,j}$ looks like it limits to a color other than j.
(4) U_l no longer looks infinite.

If the first condition applies, consider any terminal σ on U_l that looks extendible, and let $\langle T_{j,0} < \cdots < T_{j,s_j} : j < k \rangle$ be the $\langle \varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1} \rangle$-forest inside it. Thus, for each j, every terminal α in every $T_{j,i}$ has a finite subset F of its range satisfying $\varphi_{\sigma,j}$. Now if for each σ there is a j and an F as above all of whose elements look like they limit to j under d, then define $H_{\sigma,j} = F$ for the least such j and F, and define $H_{\sigma,j'} = \emptyset$ for all $j' \neq j$ and $u_\sigma = t$. Otherwise, do nothing. If the second condition applies, then for any such σ_m we undefine the sets $H_{\sigma_m,0}, \ldots, H_{\sigma_m,k-1}$ and number u_{σ_m}. If the third condition applies, fix the least m for which there is some such $H_{\sigma_m,j}$, and then undefine $H_{\sigma'_{m'},0}, \ldots, H_{\sigma'_{m'},k-1}$ and $u_{\sigma'_{m'}}$ for all σ' and all $m' \geq m$. Finally, if the fourth condition applies, undefine all $H_{\sigma,j}$. Naturally, if we undefine some H_σ we terminate our canonical search in any extensions of σ. This completes the construction of c.

Construction of c_j. We first define an ordered sequence of tree enumerations, $\langle \widehat{U}_m : m \in \omega \rangle$. At the beginning of s, fix the largest m such that we have defined $\widehat{U}_m(0)$ (necessarily to be $\{\emptyset\}$), and let $x_0 > 0$ be least with $\widehat{U}_m(x_0)$ still undefined. Let l be such that U_l looks infinite at stage s. In the construction of c at this stage, we either begin, or are in the midst of, the canonical search for a $\langle \varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1} \rangle$-forest for each terminal σ on U_l that looks extendible. We assume the search has not found a new φ_j-tree inside the range of each such σ since we started defining $\widehat{U}_m$. We then let $\widehat{U}_m(x_0)$ be undefined until one of the following conditions applies at some stage $t \geq s$. We divide the first condition in two depending as $j = 0$ or $j > 0$.

(1) ($j = 0$) U_l has enumerated new terminal nodes that look extendible, and for at least one such node σ the canonical search has found no new $\varphi_{\sigma,j}$-tree.
(1) ($j > 0$) U_l has enumerated new terminal nodes that look extendible, for each such node σ the canonical search has found a new $\varphi_{\sigma,j-1}$-tree, and for at least one such node σ the canonical search has found no new $\varphi_{\sigma,j}$-tree.

(2) The canonical search has found a new φ_j-tree in the range of each terminal node in U_l that looks extendible.
(3) The canonical search terminates (successfully or unsuccessfully) for each terminal node in U_l that looks extendible.

If the first condition applies with $j = 0$, fix any such σ for which the canonical search has found no new $\varphi_{\sigma,j}$-tree. If $x_0 = 1$, let $\widehat{U}_m(x_0)$ enumerate $\sigma(|\sigma| - 1)$ as a string of length 1. Otherwise, assume inductively that $\widehat{U}_m(x_0 - 1)$ enumerated the string $\sigma(z) \cdots \sigma(|\sigma| - 2)$ for some number z, and let $\widehat{U}_m(x_0)$ enumerate $\sigma(z) \cdots \sigma(|\sigma| - 2)\sigma(|\sigma| - 1)$. By induction, if $\widehat{U}_m$ turns out to be infinite, then every path through $\widehat{U}_m$ will be a co-initial segment of some path through U_l. In particular, the range of any path through $\widehat{U}_m$ is contained in the range of some path through U_l.

If the first condition applies with $j > 0$, consider any σ for which the canonical search has found no new $\varphi_{\sigma,j}$-tree, and let T be the least new $\varphi_{\sigma,j-1}$-tree that has been found. If $x_0 = 1$, let $\widehat{U}_m$ enumerate every element of $\mathrm{ran}(T)$. If $x_0 > 1$, assume inductively that there is a $\sigma' \prec \sigma$ such that $\widehat{U}_m(x_0 - 1)$ enumerated some string τ with $\tau(|\tau| - 1)$ in a $\varphi_{\sigma',j-1}$-tree $T' < T$. For each such τ and each $x \in \mathrm{ran}(T)$, let $\widehat{U}_m(x_0)$ enumerate the string τx. In other words, for the longest initial segment of σ that $\widehat{U}_m$ already enumerated some string τ for, it now enumerates all 1-extensions of τ with last bit from $\mathrm{ran}(T)$. By induction, it follows that each string τ that $\widehat{U}_m$ enumerates on behalf of some σ has $\mathrm{ran}(\tau)$ contained in a $\varphi_{\sigma,j-1}$-tree and hence in $\mathrm{ran}(\sigma)$. Thus, if $\widehat{U}_m$ is infinite, every path through it has range contained in the range of some path through U_l.

If the second or third condition applies, leave $\widehat{U}_m(x_0)$ undefined and define instead $\widehat{U}_{m+1}(0) = \{\emptyset\}$. So in particular, we never define $\widehat{U}_m$ on x_0 or any larger numbers.

It is easy to see that the $\widehat{U}_m$ indeed form an ordered sequence of tree enumerations. Next, let $C_0' = C_0 \cup \{j\}$, so that $|C - C_0'| = k - 1$. Apply the inductive hypothesis to the coloring d, the sets C and C_0', and $\langle \widehat{U}_m : m \in \omega \rangle$ to obtain a coloring $c' : \omega \to k$, and for each $j' < k - 1$, a coloring $c_{j'}' : \omega \to \#(k - 2)$. Finally, let $c_j : \omega \to \#(k - 1)$ be defined by

$$c_j(x) = \langle c'(x), c_0'(x), \ldots, c_{k-2}'(x) \rangle.$$

Note that every infinite almost-homogeneous set for c_j is also almost-homogeneous for c' and each of the $c_{j'}'$. This completes the construction.

Verification. Suppose d is total and stable, U_l is infinite, and almost all x in the range of each path through U_l limit to a color in $C - C_0$ under d. As in the $k = 1$ case, we may assume there is no path P through U_l whose range contains an infinite homogeneous set H for d such that Ψ^H is not infinite.

Now fix any $j < k$, and suppose one of the tree enumerations $\widehat{U}_m$ defined in the construction of c_j is infinite as a tree. Then at each stage s after we start defining $\widehat{U}_m$ there must be a terminal σ that looks extendible in U_l such that the canonical search for a $\langle \varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1} \rangle$-forest in the construction of c has not yet terminated. Hence, there is a path P through U_l such that this is true of every $\sigma \prec P$ that is terminal in U_l (and necessarily looks extendible) at such a stage s. As noted in the construction, the formulas $\varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1}$ do not change between such initial segments σ of P while the canonical search is ongoing, so we can write simply $\varphi_{P,j}$ in place of $\varphi_{\sigma,j}$. Also, the number of $\sigma' \preceq \sigma$ for which $H_{\sigma'}$ is defined cannot change for any such σ, since doing so terminates the canonical search along P. So if this number is n, then for any sufficiently long initial segment σ of P we have that $H_{\sigma_m,j}$ is defined for each $m < n$, and we can write simply H_{P_m} in place of H_{σ_m}, and u_{P_m} in place of u_{σ_m}. In particular, every element of $H_{P_m,j}$ actually limits to j under d (possibly trivially so, if the set is empty), because otherwise this set would be eventually undefined and the search terminated.

Fix any path Q through $\widehat{U}_m$, and recall the construction of c_j. If $j = 0$, then Q is a co-initial segment of some path P through U_l, inside whose range we find no $\varphi_{P,j}$-trees. If $j > 0$, then for some path P through U_l, we have that Q is a path through an infinite $\varphi_{P,j}$-generated subtree of an infinite $\varphi_{P,j-1}$-sequence. Either way, this means that no finite $F \subseteq \operatorname{ran}(Q)$ satisfies $\varphi_{P,j}$. Either way, this means that if $F \subseteq \operatorname{ran}(Q)$ is homogeneous for d with color j then there is no $x \geq \max_{m<n} u_{P_m}$ with $\Psi^{\bigcup_{m<n} H_{P_m,0} \cup F}(x) \downarrow = 1$. Now if infinitely many elements of $\operatorname{ran}(Q)$ limit to j under d, then let H be any infinite homogeneous set for d with color j that contains $\bigcup_{m<n} H_{P_m,j}$ and otherwise only contains elements of $\operatorname{ran}(Q)$. Then H is a subset of $\operatorname{ran}(P)$ and Ψ^H is bounded by $\max_{m<n} H_{P_m}$ and so is finite, contradicting our assumption above.

We conclude that if Q is any path through $\widehat{U}_m$, then almost all x in $\operatorname{ran}(Q)$ limit to a color other than j under d. Since $\operatorname{ran}(Q)$ is a subset of $\operatorname{ran}(P)$ for some path P through U_l, this means that almost all x in $\operatorname{ran}(Q)$ limit to one of $1, \ldots, k-1$, or equivalently, to a member of $C - C'_0$, as defined in the construction of c_j. But then we are precisely in the hypothesis of the lemma for $k-1$, and d consequently has an infinite homogeneous set H, contained in the range of some infinite path through $\widehat{U}_m$ and hence some infinite path through U_l, such that Ψ^H is not almost-homogeneous for c_j.

Going forward, we may thus assume that no $\widehat{U}_m$ defined in the construction of any c_j is infinite. Let s_0 be the least stage after which U_l looks infinite and none of the x that do not limit to a color in $C - C_0$ under d look like they do. For every n, we claim there is a stage $s \geq s_0$ such that if P is a path through U_l and $\sigma \prec P$ is a terminal node of U_l at stage s, then each $H_{\sigma_m,j}$ with $m < n$ has stabilized to some finite set $H_{P_m,j}$. Fix n and assume the claim for all $m < n$, as witnessed by some $s \geq s_0$. We may assume s is large enough so that no terminal σ on U_l that looks extendible is equal to σ_{n-1}.

Suppose first that at all sufficiently large stages $t \geq s$, some (and hence every) terminal σ on U_l that looks extendible has $H_{\sigma,0}, \ldots, H_{\sigma,k-1}$ undefined. By choice of s_0 and l, conditions 3 and 4 of the construction cannot apply at any such t, and whenever condition 2 applies it does not change how many initial segments σ' with $H_{\sigma'}$ defined the terminal nodes that still look extendible have. Thus, as above, the formulas $\varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1}$ do not change between compatible strings σ at these stages t. This means that if the canonical search ever finds a $\langle \varphi_{\sigma,0}, \ldots, \varphi_{\sigma,k-1} \rangle$-forest at any of these stages but we do not define $H_{\sigma,0}, \ldots, H_{\sigma,k-1}$, then the same forest is found again at the next stage, and also for any newly enumerated extensions of σ. But the only reason we might fail to define new sets when a forest is found is if for some σ that looks extendible there is no finite F in the range of any terminal α in any $\varphi_{\sigma,j}$-tree such that F satisfies $\varphi_{\sigma,j}$ and every element of F looks like it limits to j under d. By Lemma 2.6 and the fact that U_l is finitely-branching, this is impossible.

It follows that at any of these stages t, there is a terminal σ in U_l that looks extendible such that the canonical search has not found a $\langle \varphi_{\sigma,0}, \dots, \varphi_{\sigma,k-1}\rangle$-forest. (That is, σ witnesses that condition 1 of the construction does not apply at t.) There are two cases that can cause this situation.

Case 1: at any stage t as above, there is a terminal σ in U_l that looks extendible such that the canonical search does not find a new $\varphi_{\sigma,0}$-tree. Let m be such that at the least stage t as above, we are defining $\widehat{U}_m$ in the construction of c_0. Then it is easy to see that $\widehat{U}_m$ is infinite.

Case 2: for some $j > 0$, there are infinitely many stages t as above such that the canonical search finds a new $\varphi_{\sigma,j-1}$-tree for each terminal σ in U_l that looks extendible, but at each such t there is at least one σ such that this search does not find a new $\varphi_{\sigma,j}$-tree. Let m be such that at the least stage t as above, we are defining $\widehat{U}_m$ in the construction of c_j. Then $\widehat{U}_m$ is infinite.

Since both cases result in contradictions, we conclude that there are infinitely many stages after s at which the H_σ for σ terminal in U_l and looking extendible are defined. For any path P through U_l and any $\sigma \prec P$, such an $H_{\sigma,j}$ can later only be undefined if some element of it starts looking like it does not limit to j under d. Since the $H_{\sigma,j}$ are always chosen from a certain $\langle \varphi_{\sigma,0}, \dots, \varphi_{\sigma,k-1}\rangle$-forest, it follows by Lemma 2.6 and the fact that d is stable that this can only happen finitely often. Hence, this definition eventually stabilizes.

To complete the proof, fix any path P through U_l. Let $\sigma_0 \prec \sigma_1 \prec \cdots \prec P$ be the strings for which we just showed that the sets $H_{\sigma_n,j}$ stabilize to $H_{P_n,j}$. For each n, there is a unique $j < k$ such that $H_{P_n,j} \neq \emptyset$. Hence, for each $i < k+1$, there is a $j < k$ such that $H_{P_n,j} \neq \emptyset$ for infinitely many $n \equiv i \bmod (k+1)$; let j_i be the least such j. Then we can fix $i < i' < k+1$ and j with $j_i = j_{i'} = j$. Since $H_{P_n,j} \neq \emptyset$ for infinitely many n, and since $H_{P_m,j} < H_{P_n,j}$ whenever $m < n$ and both sets are non-empty, it follows that $H = \bigcup_n H_{P_n,j}$ is infinite. In particular, H is an infinite homogeneous set for d with color j. Furthermore, whenever some $H_{P_n,j}$ is non-empty there is an $x \geq \max_{m<n} u_{P_m}$ such that $\Psi^{\bigcup_{m<n} H_{P_m,0} \cup F}(x) \downarrow = 1$ and $c(x) = n \bmod (k+1)$. Note that Ψ^H agrees with this computation by construction. As there are infinitely many n with $H_{P_n,j} \neq \emptyset$ and $n \equiv i \bmod (k+1)$, and infinitely many n with $H_{P_n} \neq \emptyset$ and $n \equiv i' \bmod (k+1)$, we conclude that there are infinitely many x in Ψ^H with $c(x) = i$, and infinitely many x with $c(x) = i'$. Hence, Ψ^H is not almost-homogeneous for c, as was to be shown. This completes the proof. □

4.3. **Consequences.** We can now give the the main result of this section, which is just a simplified version of the lemma just proved.

Theorem 4.3. *Fix $k \geq 1$. Let d be a partial computable coloring $[\omega]^2 \to k$ and let Ψ be a Turing functional. There exists a computable coloring $e : \omega \to \#(k)$ such that if d is total and stable, then it has an infinite homogeneous set H for which Ψ^H is not almost-homogeneous for e. Moreover, an index for e can be obtained from k, an index for d as a partial computable function, and an index for Ψ.*

Proof. Let $\langle U_l : l \in \omega\rangle$ be any uniform sequence of tree enumerations with $U_0(x) \downarrow = \{\omega \upharpoonright x\}$ for all x, and let $C = \{0, \dots, k\}$ and $C_0 = \emptyset$. Then, apply Lemma 4.2 to k, C_0 and C, d, and $\langle U_l : l \in \omega\rangle$ to get colorings $c : \omega \to k+1$ and, for each $j < k$,

$c_j : \omega \to \#(k-1)$. Define $e : \omega \to \#(k)$ by

$$e(x) = \langle c(x), c_0(x), \ldots, c_{k-1}(x)\rangle. \qquad \square$$

Corollary 4.4. *There exists a computable family of sets $\vec{X}$ with the following property. For every $k \geq 1$, if d is a computable stable coloring $[\omega]^2 \to k$ and Ψ is any Turing functional, then d has an infinite homogeneous set H such that Ψ^H is not an infinite $\vec{X}$-cohesive set.*

Proof. By the uniformity of Theorem 4.3, there is a computable sequence $\langle e_i : i \in \omega\rangle$ such that if i is the triple (of indices for) $\langle k, d, \Phi\rangle$, then e_i is the coloring $e : \omega \to \#(k)$ given by the theorem. We define a family of sets $\vec{X} = \langle X_n : n \in \omega\rangle$ as follows. Fix $i = \langle k, d, \Phi\rangle$, and for each s, let $X_{\langle i,s\rangle}(x)$ be the sth digit in the binary expansion of $e_i(x)$, regarded as a sequence of length $\lfloor \log_2 \#(k)\rfloor + 1$ by prepending 0s if necessary, or 0 if $s \geq \lfloor \log_2 \#(k)\rfloor + 1$. (For example, if $k = 2$ then $\#(k) = 12$, so if $e_i(x) = 3$ then $X_{\langle i,0\rangle}(x), X_{\langle i,1\rangle}(x), \ldots$ equal $0, 0, 1, 1, 0, 0, 0, \ldots$, respectively; if $e_i(x) = 9$ then $X_{\langle i,0\rangle}(x), X_{\langle i,1\rangle}(x), \ldots$ equal $1, 0, 0, 1, 0, 0, 0, \ldots$; etc.) Then every infinite set which is cohesive for $\langle X_{\langle i,s\rangle} : s \in \omega\rangle$ is an infinite almost-homogeneous set for e_i. So by definition of e_i, if d is total and stable then it must have an infinite homogeneous set H with Ψ^H not equal to any infinite $\vec{X}$-cohesive set. $\square$

Observe above the dependence of H on Ψ. If this dependence could be eliminated, which is to say, if in the statement of the corollary we could interchange the universal quantifier over Ψ with the existential quantifier over H, we would have that $\mathsf{COH} \not\leq_{\mathrm{c}} \mathsf{SRT}^2_{<\infty}$. This remains an open question. We do however have the following consequence as a special case.

Corollary 4.5. $\mathsf{COH} \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{<\infty}$.

In fact, Theorem 4.3 establishes the following slightly stronger result, which is of independent interest and partially extends Theorem 2.10 (4) of Hirschfeldt and Jockusch [13]. Let $(\mathsf{RT}^n_k)^*$ be the principle asserting that every coloring $[\omega]^n \to k$ has an infinite almost-homogeneous set. It is easy to see that $(\mathsf{RT}^1_k)^* \leq_{\mathrm{W}} \mathsf{COH}$ for all k.

Corollary 4.6. *For all $k \geq 1$, we have that $(\mathsf{RT}^1_{\#(k)})^* \not\leq_{\mathrm{W}} \mathsf{SRT}^2_k$.*

Proof. Suppose to the contrary that $(\mathsf{RT}^1_{\#(k)})^* \leq_{\mathrm{W}} \mathsf{SRT}^2_k$ as witnessed by Φ and Ψ. Let f be a computable function such that if d is (an index for) a partial computable coloring $[\omega]^2 \to k$ then $f(d)$ is an index for the coloring $\omega \to \#(k)$ given by Theorem 4.3. And let g be a computable function such that if e is (an index for) a computable coloring $\omega \to \#(k)$ then $g(e)$ is an index for the stable coloring Φ^e. By the recursion theorem, let $e : \omega \to \#(k)$ have index a fixed-point for the function $f \circ g$, so that e is the function given by Theorem 4.3 for the coloring $d = \Phi^e$. Since d is stable, it has an infinite homogeneous set H for which Ψ^H is not almost-homogeneous for e, which is a contradiction. $\square$

We mention, in closing this section, that we do not know how to extend Corollary 4.5 from uniform reducibility to generalized uniform reducibility (as defined by Hirschfeldt and Jockusch [13, Definition 4.3]). Whether COH is generalized uniform reducible to $\mathsf{SRT}^2_{<\infty}$ appears also as Question 5.2 of [13].

5. COH and strong computable reducibility

For our final result, we turn to strong computable reducibility. In trying to show that, say, $\mathsf{Q} \not\leq_{\rm sc} \mathsf{P}$, one may hope to be able to keep the construction of a witnessing instance X of Q separate from the construction of a solution $\widehat{Y}$ to the computed instance Φ^X of P. This is because the "backward" reduction $\Psi^{\widehat{Y}}$ does not reference X, and so one can try to build a bit of $\widehat{Y}$, then build a bit more of X to diagonalize $\Psi^{\widehat{Y}}$, and then repeat for the next reduction procedure. This is indeed what happens for example in the proof that $\mathsf{COH} \not\leq_{\rm sc} \mathsf{D}^2_2$ in [9], which builds a limit-homogeneous set independently of a family of sets. Unfortunately, here our P will be SRT^2_2 rather than D^2_2, so we must build a homogeneous set rather than merely a limit-homogeneous one. This necessarily brings the construction of the original instance X into the construction of the solution $\widehat{Y}$. To be specific, we can try to define a forest of some sort as we did above, then extend the coloring to force the limiting color of each number in the range of this forest, and appeal to Lemma 2.6 to conclude that the range of some path in some tree in this forest is limit-homogeneous. But there is no guarantee that this range is also homogeneous, let alone homogeneous with the same color as the limiting color. If we instead try to do this in the opposite order, by first defining a finite set with certain desirable properties (like causing Ψ to converge on a new element), we can later extend the coloring to make this finite set homogeneous, but then there may be no further extension that causes all elements of this set to have the same limit.

The above is a serious obstacle. To get around it, we abandon the generalized Seetapun framework from Section 2, and instead present the following alternative argument, which uses an entirely new method, which we call *tree labeling*, for building homogeneous sets. The drawback is that, unlike in our results above, where we constructed instances that were computable or close to computable, here our instance is far more complicated. We begin with a definition.

Definition 5.1. Let $\mathbb{C}$ be the following notion of forcing. A condition is a sequence $p = \langle \sigma^p_0, \ldots, \sigma^p_{n^p-1}, \ell^p \rangle$, in which $\sigma^p_0, \ldots, \sigma^p_{n^p-1}$ are finite binary sequences of the same length, and ℓ^p is a function $n^p \to (2 \times \omega) \cup \{u\}$ such that if $\ell^p(n) = \langle i, k \rangle$ then $\sigma^p_n(x) = i$ for all x with $k \leq x < |\sigma^p_n|$. A condition q extends p if $n^q \geq n^p$, $\sigma^q_n \succeq \sigma^p_n$ for all $n < n^p$, and $\ell^q \supseteq \ell^p$.

We call each σ^p_n a *column* of p. The role of ℓ^p is to either *lock* a column to some $i < 2$ from some point k onward, or to *unlock* it, meaning that it can never later be locked. Note that any sufficiently generic filter $\mathscr{G}$ for $\mathbb{C}$ gives rise to a family $\vec{X} = \langle X_n : n \in \omega \rangle$ of sets, with the nth column of any condition in $\mathscr{G}$ being an initial segment of X_n. More generally, we say $\vec{X}$ *extends* p provided each of $X_0, \ldots, X_{n^p-1}$ extend $\sigma^p_0, \ldots, \sigma^p_{n^p-1}$, respecting all locks. This means that if $\vec{X}$ extends p, which has locked its nth column to i from k on, then $X_n(x) = i$ for all $x \geq k$.

In what follows, we use $\vec{X}$ (respectively, X_n for some $n \in \omega$) both for a family of sets we are building (respectively, for its nth column) and as a name in the $\mathbb{C}$ forcing language for a generic family (respectively, a name for the nth column of a generic family). We define what it means for a condition to force an arithmetical statement relative to this name in the obvious way: thus, we say p forces $x \in X_n$ if $x < |\sigma^p_n|$ and $\sigma^p_n(x) = 1$, and from there we proceed inductively in the usual manner. (See e.g., Shore [23, Chapter 3] for details.) Given any set P, we can also extend our forcing language and relation to include P as a parameter.

Theorem 5.2. *There exist a family of sets $\vec{X} = \langle X_n : n \in \omega \rangle$ and a collection Y of infinite sets such that no $(\vec{X} \oplus P)$-computable infinite set is $\vec{X}$-cohesive for any $P \in Y$, and every stable coloring $[\omega]^2 \to 2$ computable from $\vec{X}$ either has an $(\vec{X} \oplus P)$-computable infinite homogeneous set for some $P \in Y$, or else, for each $j < 2$, an infinite homogeneous set with color j that computes no infinite $\vec{X}$-cohesive set.*

Proof. The idea is to make $\vec{X}$ generic over the collection Y, where Y will be obtained by repeatedly taking paths through certain non-well-founded trees of $\omega^{<\omega}$. We define

- a sequence of $\mathbb{C}$-conditions $p_0 \geq p_1 \geq \cdots$ with $\lim_s n^{p_s} = \infty$;
- a sequence of finite sets $H^\Phi_{j,0} \subseteq H^\Phi_{j,1} \subseteq \cdots$ for each Turing functional Φ and each $j < 2$;
- a sequence of infinite sets $I_0 \supseteq I_1 \supseteq \cdots$ with $H_{j,s} < I_s$ for each j and all s;
- a sequence of finite families $Y_0 \subseteq Y_1 \subseteq \cdots$ of infinite subsets of ω.

In the end, we take $X_n = \bigcup_s \sigma^{p_s}_n$ for each n, and let $\vec{X} = \{X_n : n \in \omega\}$, let $H^\Phi_j = \bigcup_s H^\Phi_{j,s}$ for each $j < 2$, and let $Y = \bigcup_s Y_s$. Our goal is to ensure the following requirements, for all $s \in \omega$, all Turing functionals Φ and Ψ, and each $j < 2$:

- $\mathcal{P}_s$: the sequence $p_0 \geq p_1 \geq \cdots$ is 3-generic relative to each $P \in Y_s$;
- $\mathcal{Q}_{\Phi,s}$: if $\Phi^{\vec{X}}$ is a stable coloring of pairs, it either has an $(\vec{X} \oplus P)$-computable infinite homogeneous set for some $P \in Y$, or both H^Φ_0 and H^Φ_1 are infinite;
- $\mathcal{R}_{\Phi,\Psi,j}$: if $\Phi^{\vec{X}}$ is a stable coloring of pairs, it either has an $(\vec{X} \oplus P)$-computable infinite homogeneous set for some $P \in Y$, or if $\Psi^{H^\Phi_j}$ defines an infinite set then this set has infinite intersection with both X_0 and $\overline{X_0}$.

The $\mathcal{P}$ requirements will ensure that for all $P \in Y$ there is no $(\vec{X} \oplus P)$-computable infinite $\vec{X}$-cohesive set. To see this, fix any $P \in Y$ and any Turing functional Γ. Let W be the set of conditions p that force one of the following two statements:

- $\Gamma^{\vec{X} \oplus P}$ does not define an infinite set;
- there is an $n \in \omega$ such that for every $k \in \omega$ and every $i < 2$, there is an $x \geq k$ with $X_n(x) = i$ and $\Gamma^{\vec{X} \oplus P}(x) \downarrow = 1$.

Then W is Σ^0_3-definable in P, and we claim that it is dense in $\mathbb{C}$. Hence, some condition p_s in our sequence must meet it, from which it follows that $\Gamma^{\vec{X} \oplus P}$ is either not an infinite set or not $\vec{X}$-cohesive, as desired. So let p be any condition, and assume it has no extension forcing the first statement above. Let q be any extension of p with $n^q = n^p + 1$ and $\ell^q(n^p) = u$; we claim that q forces the second statement, witnessed by $n = n^p$. Assume not, so that for some $k \in \omega$, some $i < 2$, and some condition r extending q, no extension of r forces that there is an x with $X_n(x) = i$ and $\Gamma^{\vec{X} \oplus P}(x) \downarrow = 1$. Then in particular, there is no sequence of strings $\tau_0, \ldots, \tau_{b-1}$ as follows:

(1) $b \geq n^r + 1$;
(2) each τ_m has the same length;

(3) each τ_m with $m < n^r$ extends σ^r_m and respects any lock on this column;
(4) $\tau_{n^p}(x) = i$ for all x with $\max\{|\sigma^r_0|, k\} \leq x < |\tau_{n^p}|$;
(5) $\Gamma^{\langle \tau_0,\ldots,\tau_{b-1}\rangle \oplus P \upharpoonright |\tau_0|}(x) \downarrow= 1$ for some x with $\max\{|\sigma^r_0|, k\} \leq x < |\tau_{n^p}|$.

Let r' be the condition that differs from r only in that $\ell^{r'}(n^p)$ is $\langle i, \max\{|\sigma^r_0|, k\}\rangle$ rather than u. Then sequences of strings satisfying properties (1)–(4) above are precisely the sequences of columns of extensions of r'. Thus, the fact that no such sequence can also satisfy (5) means r' forces that $\Gamma^{\vec{X} \oplus P}(x) \simeq 0$ for all $x \geq 0$, which is to say that this computation does not define an infinite set. Since r' is an extension of p, this is impossible.

It follows that the $\mathcal{Q}$ and $\mathcal{R}$ requirements ensure that every $\vec{X}$-computable stable coloring of pairs has an infinite homogeneous set that computes no infinite $\vec{X}$-cohesive set, as desired.

Construction. Distribute all the requirements between the stages of the construction in such a way that there are infinitely many stages dedicated to satisfying each requirement. We begin by letting p_0 be any condition with $n^{p_0} = 1$ and $\ell^{p_0}(0) = u$. Thus, we will be free to extend the 0th column of any of our conditions p_s arbitrarily. Let $H^{\Phi}_{0,0} = H^{\Phi}_{1,0} = \emptyset$ for all Φ, let $I_0 = \omega$, and let $Y_0 = \emptyset$. At the beginning of stage $s+1$, assume we are given p_s, $H^{\Phi}_{0,s}$ and $H^{\Phi}_{1,s}$ for all Φ, I_s, and Y_s. Assume inductively that if $H^{\Phi}_{0,s}$ or $H^{\Phi}_{1,s}$ is non-empty for some Φ, then p_s forces that $\Phi^{\vec{X}}$ is a stable coloring of pairs and $\Phi^{\vec{X}}(x, y) = j$ for all $x \in H^{\Phi}_{j,s}$ and all $y \in I_s$. At the conclusion of the stage, if we did not explicitly define p_{s+1}, $H^{\Phi}_{j,s+1}$ for some Φ or j, I_{s+1}, or Y_{s+1}, we mean to let these be p_s, $H^{\Phi}_{j,s}$, I_s, and Y_s, respectively.

$\mathcal{P}$ *requirements.* These are satisfied in a straightforward manner. Suppose s is dedicated to requirement $\mathcal{P}_t$ for some $t < s$, and that it is the $\langle n, m\rangle$th such stage. If $n > |Y_t|$, do nothing. Otherwise, let P be the nth member of the finite set Y_t in some fixed listing, and let W be the mth $\Sigma^0_3(P)$ set. If p_s has an extension in W, choose one and let it be p_{s+1}, and otherwise do nothing.

$\mathcal{Q}$ *requirements.* Suppose s is dedicated to $\mathcal{Q}_{\Phi,t}$. By extending p_s if necessary, we may assume that it decides whether or not $\Phi^{\vec{X}}$ is a stable coloring of pairs. If p_s forces that it is not such a coloring, we can do nothing. So assume otherwise. If, for some $j < 2$ and some $k \in \omega$, there is no extension of p_s forcing that $\lim_y \Phi^{\vec{X}}(x, y) = j$ for some $x \in I_s$ with $x \geq k$, then $P = \{x \in I_s : x \geq k\}$ will be limit-homogeneous for $\Phi^{\vec{X}}$ with color $1 - j$. Hence, $\vec{X} \oplus P$ will compute an infinite homogeneous set for $\Phi^{\vec{X}}$ via the uniform thinning algorithm for obtaining a homogeneous set from a limit-homogeneous one. We thus define $Y_{s+1} = Y_s \cup \{P\}$, and otherwise do nothing. If there are no j and k as above, we can find numbers $x_0, x_1 \in I_s$ and an extension of p_s forcing that $H^{\Phi}_{j,s} \cup \{x_j\}$ is homogeneous and $\lim_y \Phi^{\vec{X}}(x_j, y) = j$ for each $j < 2$; we then let p_{s+1} be this extension, and let $H^{\Phi}_{j,s+1} = H^{\Phi}_{j,s} \cup \{x_j\}$. Thus, we have added an element to each of H^{Φ}_0 and H^{Φ}_1, so if $\Phi^{\vec{X}}$ does not end up having an $(\vec{X} \oplus P)$-computable infinite homogeneous set for any $P \in Y$, both H^{Φ}_0 and H^{Φ}_1 will be infinite.

$\mathcal{R}$ *requirements.* Suppose s is dedicated to $\mathcal{R}_{\Phi,\Psi,j}$. As above, we may assume p_s forces that $\Phi^{\vec{X}}$ is a stable coloring of pairs. Let k be the length of the columns of p_s, and define T to be the tree of all $\alpha \in \mathit{Inc}(I_s)$ such that there is no finite

$F \subseteq \operatorname{ran}(\alpha \upharpoonright |\alpha|-1)$ and no $w \geq k$ with $\Psi^{H^{\Phi}_{j,s} \cup F}(w) \downarrow = 1$. If T is not well-founded, we let I_{s+1} be the range of any infinite path through it. Then provided we ensure that all $x \in H^{\Phi}_j - H^{\Phi}_{j,s}$ come from I_s, any set defined by $\Psi^{H^{\Phi}_j}$ will contain no numbers $w \geq k$, and so will not be infinite.

We thus turn to dealing with the case that T is well-founded. Let $i < 2$ be such that the number of prior stages dedicated to requirement $\mathcal{R}_{\Phi,\Psi,j}$ is congruent to i modulo 2. Our strategy is to either define $H^{\Phi}_{j,s+1}$ so that $\Psi^{H^{\Phi}_{j,s+1}}(w) \downarrow = 1$ for some $w \geq k$ and ensure that $X_0(x) = i$, or else to find a set P such that $\Phi^{\vec{X}}$ has an $(\vec{X} \oplus P)$-computable infinite homogeneous set, and add this P to Y.

Notice each $\alpha \in T$ is either terminal, in which case there is an $F \subseteq \operatorname{ran}(\alpha)$ and a $w \geq k$ with $\Psi^{H^{\Phi}_{j,s} \cup F}(w) \downarrow = 1$, or else every 1-extension of α is also in T.

We wish to define a certain subtree T_0 of T. To this end, we first label each node of T by either a finite number or the symbol ∞. We do this by induction on rank. If $\alpha \in T$ is terminal, label it by the least $w \geq k$ such that $\Psi^{H^{\Phi}_{j,s} \cup F}(w) \downarrow = 1$ for some $F \subseteq \operatorname{ran}(\alpha)$. Now suppose $\alpha \in T$ is not terminal, and that we have labeled all nodes of T of smaller rank. In particular, this means that we have labeled all the 1-extensions of α in T. If there is a $w \in \omega$ such that infinitely many of the 1-extensions of α were labeled by w, we label α by the least such w. Otherwise, we label α by ∞.

Now define T_0 as follows. Add $\emptyset$ to T_0, and suppose we have added some $\alpha \in T$ to T_0, and that this α is not terminal in T. If α was labeled by a finite number w, add to T_0 all 1-extensions of α that were also labeled by w. If, on the other hand, α is labeled by ∞, and there are infinitely many 1-extensions of α labeled by finite numbers, add these extensions to T_0. In this case, for each w, cofinitely many of the 1-extensions β of α must satisfy $\Psi^{H^{\Phi}_{j,s} \cup F}(x) \simeq 0$ for all x with $k \leq x < w$ and all $F \subseteq \operatorname{ran}(\beta)$. If cofinitely many of the 1-extensions of α were labeled by ∞, add these to T_0 instead. Thus, in any case, each non-terminal node in T_0 has infinitely many 1-extensions in T_0, and all of these extensions have the same kind of label: they are either all labeled by one and the same finite number, provided α itself is labeled by this number; they are labeled by finite numbers, and only finitely many are labeled by any given number; or they are all labeled by ∞. Note also that every terminal node in T_0 is terminal in T.

We now attempt to define a sequence $p_s \geq q_0 \geq q_1 \geq \cdots$ of conditions, and a sequence $\emptyset = \alpha_0 \preceq \alpha_1 \cdots$ of 1-extensions in T_0, such that for each n, the condition q_n forces that $\lim_y \Phi^{\vec{X}}(x,y) = j$ for all $x \in \operatorname{ran}(\alpha_n)$, and in fact, that $\Phi^{\vec{X}}(x,y) = j$ for all $x \in \operatorname{ran}(\alpha_n)$ and all $y \geq \alpha_{n+1}(n)$. Hence, q_n forces that $\operatorname{ran}(\alpha_n)$ is homogeneous for $\Phi^{\vec{X}}$ with color j. If we get stuck in this process, we will be able to add some P to Y; otherwise, we will succeed in defining $H^{\Phi}_{j,s+1}$.

If $\alpha_0 = \emptyset$ is labeled by some $w \in \omega$, let q_0 be any extension of p_s with $\sigma_0^{q_0}(w) = i$, which exists since $w \geq k = |\sigma_0^{p_s}|$. If α_0 is labeled by ∞, let $q_0 = p_s$. Next, assume we have defined q_n and α_n for some n, and that α_n is not terminal in T_0. By assumption, q_n forces that there is an $m \in \omega$ such that $\Phi^{\vec{X}}(x,y) = j$ for all $x \in \operatorname{ran}(\alpha_n)$ and all $y \geq m$. Let S be the set of all 1-extensions β of α_n with $\beta(n) \geq m$, so that S is infinite. We consider the following two cases.

Case 1: α_n is labeled by ∞, but its 1-extensions are all labeled by finite numbers. Let P be the set of all pairs $\langle x, w \rangle$ such that x is equal to $\beta(n)$ for some $\beta \in S$ with label $w \geq |\sigma_0^{q_n}|$. As noted above, there are infinitely many numbers w with

$\langle x, w\rangle \in P$ for some x. Now if there is an extension q of q_n and an $\langle x, w\rangle \in P$ such that $\sigma_0^q(w) = i$ and q forces that $\lim_y \Phi^{\vec{X}}(x, y) = j$, let $q_{n+1} = q$, and let α_{n+1} be any $\beta \in S$ with $\beta(n) = x$ and label w. Otherwise, we have that if q is any condition extending q_n with $\sigma_0^q(w) = i$ for some $\langle x, w\rangle \in P$, then no extension of q can force that $\lim_y \Phi^{\vec{X}}(x, y) = j$. This means that q itself forces that $\lim_y \Phi^{\vec{X}}(x, y) \simeq 1 - j$. But notice that for every e, the set

$$W_e = \{q \in \mathbb{C} : \exists \langle x, w\rangle \in P\ (w \geq e \wedge \sigma_0^q(w) = i)\}$$

is dense in $\mathbb{C}$, and that it is Σ^0_1-definable from P. In this case, we claim that $\vec{X} \oplus P$ will compute an infinite homogeneous set for $\Phi^{\vec{X}}$, and so we let $Y_{s+1} = Y_s \cup \{P\}$. To prove the claim, note that since the sequence of conditions $p_0 \geq p_1 \geq \cdots$ will be generic relative to P, for each e there will be some condition in this sequence meeting W_e. Thus, for each e there will be an $\langle x, w\rangle \in P$ with $w \geq e$ such that $X_0(w) = i$. This means that to compute an infinite limit-homogeneous set for $\Phi^{\vec{X}}$ (and from there an infinite homogeneous one), $\vec{X} \oplus P$ will only need to search for pairs $\langle x, w\rangle \in P$ with $X_0(w) = i$, assured that these can be found for arbitrarily large w, and that $\lim_y \Phi^{\vec{X}}(x, y)$ must be $1 - j$.

Case 2: otherwise. Let P be the set of all x equal to $\beta(n)$ for some $\beta \in S$. If there is an extension of q_n that, for some $x \in P$, forces that $\lim_y \Phi^{\vec{X}}(x, y) = j$, we let q_{n+1} be this extension, and let α_{n+1} be any $\beta \in S$ with $\beta(n) = x$. If, on the other hand, there is no extension of q_n as above, then P will be an infinite limit-homogeneous set for $\Phi^{\vec{X}}$ with color $1 - j$. Hence, $\vec{X} \oplus P$ will compute an infinite homogeneous set for $\Phi^{\vec{X}}$. In this case, we let $Y_{s+1} = Y_s \cup \{P\}$.

We complete this stage of the construction as follows. If we ended up adding some P to Y, we do nothing more. Otherwise, we must have succeeded in defining α_{n+1} from every non-terminal α_n. But since T_0, being a subtree of T, is well-founded, some α_n must be terminal in T_0. Choose any $F \subseteq \alpha_n$ such that $\Psi^{H^\Phi_{j,s} \cup F}(w) \downarrow = 1$ for some $w \geq k$, say with use u. Let $p_{s+1} = q_n$, $H^\Phi_{j,s+1} = H^\Phi_{j,s} \cup F$, and $I_{s+1} = \{x \in I_s : x > u\}$.

Verification. It is clear from the construction that the $\mathcal{P}$ and $\mathcal{Q}$ requirements are all satisfied. For the $\mathcal{R}$ requirements, suppose that $\Phi^{\vec{X}}$ has no $(\vec{X} \oplus P)$-computable infinite homogeneous set for any $P \in Y$, and fix Ψ and j. Then at every stage dedicated to $\mathcal{R}_{\Phi,\Psi,j}$ we succeed in adding one more element w to the set defined by H^Φ_j, while ensuring that $X_0(w)$ is 0 or 1, depending as the number of previous such stages was even or odd. Hence, H^Φ_j intersects both X_0 and $\overline{X_0}$ infinitely often, as needed. This completes the proof. □

Corollary 5.3. $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_2$.

We do not know how to extend our technique to show more generally that $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_{<\infty}$, or even that $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_k$ for any $k \geq 2$. The reason is that when more colors are involved, it no longer follows that if no element of some set limits to a given color, then all the elements of that set limit to a given other color. Thus, adding such a set to the collection Y above does not appear to produce a homogeneous set. Now it is not difficult to generalize our argument to get around this problem when adding a set to Y at a stage dedicated to a $\mathcal{Q}$ requirement, or in Case 2 of a stage dedicated to a $\mathcal{R}$ requirement. But Case 1 seems more involved.

6. Summary and questions

We summarize the principal consequences of our results, and how they fit in with known ones in Figure 2 below. Here, an arrow from P to Q means that Q is reducible to P under strong Weihrauch reducibility, $\leq_{\mathrm{sW}}$ (and hence also under $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sc}}$), and all the arrows present are straightforward to establish. No additional

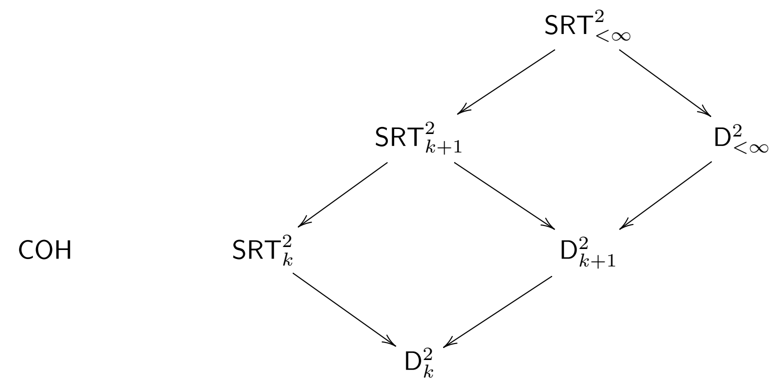

Figure 2. Relationships between SRT^2_k, D^2_k, and COH, with $k \geq 2$ arbitrary. All arrows represent reductions under $\leq_{\mathrm{sW}}$. No additional arrows hold under either $\leq_{\mathrm{W}}$ or $\leq_{\mathrm{sc}}$.

arrows hold under either $\leq_{\mathrm{sc}}$ or $\leq_{\mathrm{W}}$, as explained below.

(1) That there is no arrow from $\mathsf{D}^2_{<\infty}$ to SRT^2_k under $\leq_{\mathrm{W}}$ is by Corollary 3.3. That there is no arrow from $\mathsf{D}^2_{<\infty}$ to SRT^2_k under $\leq_{\mathrm{sc}}$ is by by Corollary 3.6.
(2) That there is no arrow from SRT^2_k to D^2_{k+1} under either $\leq_{\mathrm{W}}$ or $\leq_{\mathrm{sc}}$ is by a result of Patey [20, Corollary 3.6].
(3) That there is no arrow from COH to D^2_k under $\leq_{\mathrm{W}}$ or $\leq_{\mathrm{sc}}$ is by results of Hirschfeldt et al. [14, Theorems 2.3 and 3.7].
(4) That there is no arrows from $\mathsf{SRT}^2_{<\infty}$ to COH under $\leq_{\mathrm{W}}$ is by Corollary 4.5. That there is no arrow from SRT^2_2 to COH under $\leq_{\mathrm{sc}}$ is by Corollary 5.3, which has been extended since the submission of this article by Dzhafarov, Patey, Solomon, and Westrick [11] to show there is no arrow from $\mathsf{SRT}^2_{<\infty}$ to COH under $\leq_{\mathrm{sc}}$ (see Note 6.4 below).

We conclude with some of the questions left over from, and raised by, our work.

Question 6.1. Is it the case that COH is generalized uniformly reducible to $\mathsf{SRT}^2_{<\infty}$? (By [13, Propositions 4.7 and 4.8], we can replace $\mathsf{SRT}^2_{<\infty}$ here by D^2_2.)

In connection with Corollary 4.6, it is natural to ask the following.

Question 6.2. For $j < k$, is it the case that $(\mathsf{RT}^1_k)^* \leq_{\mathrm{W}} \mathsf{SRT}^2_j$?

As remarked at the end of Section 5, we do not know how to extend the proof of Corollary 5.3 from SRT^2_2 to $\mathsf{SRT}^2_{<\infty}$.

Question 6.3. Is it the case that $\mathsf{COH} \leq_{\mathrm{sc}} \mathsf{SRT}^2_{<\infty}$, or even that $\mathsf{COH} \leq_{\mathrm{sc}} \mathsf{SRT}^2_k$ for some $k > 2$?

Note 6.4. Since the submission of this article, Question 6.3 has been answered by Dzhafarov, Patey, Solomon, and Westrick [11], who extended the tree labeling method and showed that $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_{<\infty}$.

Finally, we would like to know more about the complexity of the construction the proof of Theorem 5.2. Hirschfeldt and Jockusch [13, Theorem 3.9] gave another argument about $\leq_{\mathrm{sc}}$ involving higher levels of the hyperarithmetical hierarchy, that Patey [20, Theorem 3.2] obtained independently using a Δ^0_2 construction. Our argument is very different from the one in [13], but we can ask the same question about whether the repeated use of hyperjumps there is really necessary.

Question 6.5. Can the family $\vec{X}$ constructed in the proof of Theorem 5.2 be chosen to be arithmetical, or at least hyperarithmetical?

Note 6.6. Since the submission of this article, the author has shown that $\vec{X}$ in Question 6.5 can be chosen to be computable in $\emptyset^{(\omega)}$.

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