REVERSE MATHEMATICS AND PROPERTIES OF FINITE CHARACTER

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Abstract. We study the reverse mathematics of the principle stating that, for every property of finite character, every set has a maximal subset satisfying the property. In the context of set theory, this variant of Tukey's lemma is equivalent to the axiom of choice. We study its behavior in the context of second-order arithmetic, where it applies to sets of natural numbers only, and give a full characterization of its strength in terms of the quantifier structure of the formula defining the property. We then study the interaction between properties of finite character and finitary closure operators, and the interaction between these properties and a class of nondeterministic closure operators.

1. Introduction

A formula φ with one free set variable is of *finite character*, and has the *finite character property*, if $\varphi(\emptyset)$ holds and, for every set A, $\varphi(A)$ holds if and only if $\varphi(F)$ holds for every finite $F \subseteq A$. In this paper, we restrict our attention to formulas of second-order arithmetic, and consider several variants and restrictions of the principle FCP (Definition 2.1) which asserts that for every formula of finite character, every subset of $\mathbb{N}$ has a maximal subset satisfying that formula. Because the empty set satisfies any formula of finite character, the soundness of this principle in second-order arithmetic can be verified in ZFC by straightforward application of Zorn's lemma. Detailed definitions of second-order arithmetic and the subsystems studed in this paper are given by Simpson [4].

The principle CE (Definition 3.3) asserts that given sets $A \subseteq B \subseteq \mathbb{N}$, a formula φ of finite character and a finitary closure operator D, such that A is a D-closed set satisfying the formula, there is a set X which is maximal with respect to the conditions that $A \subseteq X \subseteq B$, $\varphi(X)$ holds, and X is D-closed. In the third section, we give a full characterization of the strength of fragments of CE in terms of the complexity of the formulas of finite character to which they apply.

The authors are grateful to Denis Hirschfeldt, Antonio Montalbán, and Robert Soare for valuable comments and suggestions, and to an anonymous referee who suggested an improvement that strengthened Proposition 4.4. The first author was partially supported by an NSF Graduate Research Fellowship and an NSF Postdoctoral Fellowship.

We can further generalize CE by replacing the finitary closure operator with a more general kind of operator which we name a *nondeterministic closure operator*. The corresponding principle, NCE (Definition 4.2), is studied in the final section, where a full characterization of its strength is obtained.

We were led to study the reverse mathematics of FCP by our separate work [1] on the principle FIP which states that every countable family of subsets of $\mathbb{N}$ has a maximal subfamily with the finite intersection property. All the principles studied there are consequences of appropriate restrictions of FCP. Similarly, Propositions 3.7 and 4.4 below demonstrate how CE and NCE can be used to prove facts about countable algebraic objects in second-order arithmetic. In light of these applications, we find it worthwile to have a complete understanding of the reverse mathematics strengths of these principles.

Considering this paper together with our work on FIP gives a new example of two principles, FCP and FIP, which are each equivalent to the axiom of choice when formalized in set theory, but which have drastically different strengths when formalized in second-order arithmetic. The axiom scheme for FCP is equivalent to full comprehension in second-order arithmetic, while FIP is weaker than ACA_0 and incomparable with WKL_0.

2. Properties of finite character

We begin with the study of various forms of the following principle.

Definition 2.1. The following scheme is defined in RCA_0.

(FCP) For each L_2 formula φ of finite character, which may have arbitrary set parameters, every set A has a $\subseteq$-maximal subset B such that $\varphi(B)$ holds.

FCP is analogous to the set-theoretic principle $\mathsf{M}\,7$ in the catalog of Rubin and Rubin [3], which is equivalent to the axiom of choice [3, p. 34 and Theorem 4.3].

In order to better gauge the reverse mathematical strength of FCP, we consider restrictions of the formulas to which it applies. As with other such ramifications, we will primarily be interested in restrictions to classes in the arithmetical and analytical hierarchies. In particular, for each $i \in \{0,1\}$ and $n \geq 0$, we make the following definitions:

- Σ^i_n-FCP is the restriction of FCP to Σ^i_n formulas;
- Π^i_n-FCP is the restriction of FCP to Π^i_n formulas;
- Δ^i_n-FCP is the scheme which says that for every Σ^i_n formula $\varphi(X)$ and every Π^i_n formula $\psi(X)$, if $\varphi(X)$ is of finite character and
$$(\forall X)[\varphi(X) \iff \psi(X)],$$
then every set A has a $\subseteq$-maximal set B such that $\varphi(B)$ holds.

We also define QF-FCP to be the restriction of FCP to the class of quantifer-free formulas without parameters.

The following proposition demonstrates two monotonicity properties of formulas of finite character.

Proposition 2.2. *Let $\varphi(X)$ be a formula of finite character. The following are provable in RCA_0:*

1. *if $A \subseteq B$ and $\varphi(B)$ holds then $\varphi(A)$ holds;*
2. *if $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a sequence of sets such that $\varphi(A_i)$ holds for each $i \in \mathbb{N}$, and $\bigcup_{i\in\mathbb{N}} A_i$ exists, then $\varphi(\bigcup_{i\in\mathbb{N}} A_i)$ holds.*

Proof. The proof of (1) is immediate from the definitions. For (2), the key point is to show that if F is a finite subset of $\bigcup_{i\in\mathbb{N}} A_i$ then there is some $j \in \mathbb{N}$ with $F \subseteq A_j$. This follows from induction on the Σ^0_1 formula $\psi(n, F) \equiv (\exists m)(\forall i < n)(i \in F \Longrightarrow i \in A_m)$, in which F is a set parameter. □

Our first theorem in this section characterizes most of the above restrictions of FCP (see Corollary 2.5). We draw particular attention to part (2) of the theorem, where Σ^0_1 does not appear in the list of classes of formulas. The reason behind this will be made apparent by Theorem 2.6.

Theorem 2.3. *For $i \in \{0, 1\}$ and $n \geq 1$, let Γ be any of Π^i_n, Σ^i_n, or Δ^i_n.*

1. *Γ-FCP is provable in Γ-CA_0;*
2. *If Γ is Π^0_n, Π^1_n, Σ^1_n, or Δ^1_n, then Γ-FCP implies Γ-CA_0 over RCA_0.*

The proof of this theorem will make use of the following technical lemma, which is needed only because there are no term-forming operations for sets in the language L_2 of second-order arithmetic. For example, there is no term in L_2 that takes a set X and a number n and returns $X \cup D_n$ where, as in the rest of this paper, D_n denotes the finite set with canonical index n, or $\emptyset$ if n is not a canonical index. The moral of the lemma is that such terms can be interpreted into L_2 in a natural way.

The coding of finite sets by their canonical indices can be formalized in RCA_0 in such a way that the predicate $i \in D_n$ is defined by a formula $\rho(i, n)$ with only bounded quantifiers, and such that the set of canonical indices is also definable by a bounded-quantifier formula [4, Theorem II.2.5]. Moreover, RCA_0 proves that every finite set has a canonical index. We use the notation $Y = D_n$ to abbreviate the formula $(\forall i)[i \in Y \Longleftrightarrow \rho(i, n)]$, along with similar notation for subsets of finite sets.

Lemma 2.4. *Let $\varphi(X)$ be a formula with one free set variable. There is a formula $\widehat{\varphi}(x)$ with one free number variable such that RCA_0 proves*

$$(\forall A)(\forall n)[A = D_n \Longrightarrow (\varphi(A) \Longleftrightarrow \widehat{\varphi}(n))]. \quad (2.4.1)$$

Moreover, we may take $\widehat{\varphi}$ to have the same complexities in the arithmetical and analytic hierarchies as φ.

Proof. Let $\rho(i, n)$ be the formula defining the relation $i \in D_n$, as discussed above. We may assume φ is written in prenex normal form. Form $\widehat{\varphi}(n)$ by replacing each occurrence $t \in X$ of φ, t a term, with the formula $\rho(t, n)$.

Let $\psi(X, \bar{Y}, \bar{m})$ be the quantifier-free matrix of φ, where $\bar{Y}$ and $\bar{m}$ are sequences of variables that are quantified in φ. Similarly, let $\widehat{\psi}(n, \bar{Y}, \bar{m})$ be the matrix of $\widehat{\varphi}$. Fix any model $\mathscr{M}$ of RCA_0 and fix $n, A \in \mathscr{M}$ such that

$\mathscr{M} \models A = D_n$. A straightforward metainduction on the structure of ψ proves that

$$\mathscr{M} \models (\forall \bar{Y})(\forall \bar{m})[\psi(A, \bar{Y}, \bar{m}) \iff \widehat{\psi}(n, \bar{Y}, \bar{m})].$$

The key point is that the atomic formulas in $\psi(A, \bar{Y}, \bar{m})$ are the same as those in $\widehat{\psi}(n, \bar{Y}, \bar{m})$, with the exception of formulas of the form $t \in A$, which have been replaced with the equivalent formulas of the form $\rho(t, n)$.

A second metainduction on the quantifier structure of φ shows that we may adjoin quantifiers to ψ and $\widehat{\psi}$ until we have obtained φ and $\widehat{\varphi}$, while maintaining logical equivalence. Thus every model of RCA_0 satisfies (2.4.1).

Because ρ has only bounded quantifiers, the substitution required to pass from φ to $\widehat{\varphi}$ does not change the complexity of the formula. □

We shall sometimes identify a finite set with its canonical index. Thus, if F is finite and n is its canonical index, we may write $\widehat{\varphi}(F)$ for $\widehat{\varphi}(n)$.

Proof of Theorem 2.3. For (1), let $\varphi(X)$ and $A = \{a_i : i \in \mathbb{N}\}$ be an instance of Γ-FCP. Define $g \colon 2^{<\mathbb{N}} \times \mathbb{N} \to \{0, 1\}$ by

$$g(\tau, i) = \begin{cases} 1 & \text{if } \widehat{\varphi}(\{a_j : \tau(j)\downarrow = 1\} \cup \{a_i\}) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

where $\widehat{\varphi}$ is as in the lemma. The function g exists by Γ comprehension. By primitive recursion, there exists a function $h \colon \mathbb{N} \to \{0, 1\}$ such that for all $i \in \mathbb{N}$, $h(i) = 1$ if and only if $g(h \upharpoonright i, i) = 1$. For each $i \in \mathbb{N}$, let $B_i = \{a_j : j < i \wedge h(j) = 1\}$. An induction on φ shows that $\varphi(B_i)$ holds for every $i \in \mathbb{N}$.

Let $B = \{a_i : h(i) = 1\} = \bigcup_{i \in \mathbb{N}} B_i$. Because Proposition 2.2 is provable in RCA_0 and hence in Γ-CA_0, it follows that $\varphi(B)$ holds. By the same token, if $\varphi(B \cup \{a_k\})$ holds for some k then so must $\varphi(B_k \cup \{a_k\})$, and therefore $a_k \in B_{k+1}$, which means that $a_k \in B$. Therefore B is $\subseteq$-maximal, and we have shown that Γ-CA_0 proves Γ-FCP.

For (2), we assume Γ is one of Π^0_n, Π^1_n, or Σ^1_n; the proof for Δ^1_n is similar. We work in $\mathsf{RCA}_0 + \Gamma$-FCP. Let $\varphi(n)$ be a formula in Γ and let $\psi(X)$ be the formula $(\forall n)[n \in X \implies \varphi(n)]$. It is easily seen that ψ is of finite character, and it belongs to Γ because Γ is closed under universal number quantification. By Γ-FCP, $\mathbb{N}$ contains a $\subseteq$-maximal subset B such that $\psi(B)$ holds. For any y, if $y \in B$ then $\varphi(y)$ holds. On the other hand, if $\varphi(y)$ holds then so does $\psi(B \cup \{y\})$, so y must belong to B by maximality. Therefore $B = \{y \in \mathbb{N} : \varphi(y)\}$, and we have shown that Γ-FCP implies Γ-CA_0. □

The corollary below summarizes the theorem as it applies to the various classes of formulas we are interested in. Of special note is part (5), which says that FCP itself (that is, FCP for arbitrary L_2-formulas) is as strong as any theorem of second-order arithmetic can be.

Corollary 2.5. *The following are provable in* RCA_0*:*

(1) Δ^0_1-FCP, Σ^0_0-FCP, *and* QF-FCP*;*
(2) *for each* $n \geq 1$, ACA_0 *is equivalent to* Π^0_n-FCP*;*
(3) *for each* $n \geq 1$, Δ^1_n-CA_0 *is equivalent to* Δ^1_n-FCP*;*
(4) *for each* $n \geq 1$, Π^1_n-CA_0 *is equivalent to* Π^1_n-FCP *and to* Σ^1_n-FCP*;*
(5) Z_2 *is equivalent to* FCP*.*

The case of FCP for Σ^0_1 formulas is anomalous. The proof of part (2) of Theorem 2.3 does not go through for Σ^0_1 because this class is not closed under universal quantification. As the next theorem shows, this limitation is quite significant. Intuitively, the proof uses the fact that a Σ^0_1 formula φ is continuous in the sense that if $\varphi(X)$ holds then there is an N such that $\varphi(Y)$ holds for any Y with $X \cap \{0, \dots, N\} = Y \cap \{0, \dots, N\}$.

Theorem 2.6. Σ^0_1-FCP *is provable in* RCA_0*.*

Proof. Let $\varphi(X)$ be a Σ^0_1 formula of finite character. We claim that there exists some $c_\varphi \in \mathbb{N}$ such that for every set A, if $A \cap \{0, \dots, c_\varphi\} = \emptyset$ then $\varphi(A)$ holds. To show this, put $\varphi(X)$ in normal form, so that

$$\varphi(X) \equiv (\exists m)\rho(X[m])$$

where ρ is Σ^0_0. As $\varphi(\emptyset)$ holds, there is some $c = c_\varphi$ such that $\rho(\emptyset[c])$ holds. Now let A be any set such that $A \cap \{0, \dots, c\} = \emptyset$. Then $\rho(A[c])$ holds, so $\varphi(A)$ holds. This proves the claim.

Now fix any set A. By the claim, we know that $\varphi(A - \{0, \dots, c_\varphi\})$ holds. We may use bounded Σ^0_1 comprehension [4, Theorem II.3.9] to form the set I of m such that $D_m \subseteq \{0, \dots, c_\varphi\}$ and $\varphi(D_m \cup (A - \{0, \dots, c_\varphi\}))$ holds. We may then choose $m \in I$ such that D_m has maximal cardinality among the sets with indices in I. It follows immediately that $D_m \cup (A - \{0, \dots, c_\varphi\})$ is a maximal subset of A satisfying φ. □

The above proof contains an implicit non-uniformity in choosing a finite set of maximal cardinality. The next proposition shows that this non-uniformity is essential, by showing that a sequential form of Σ^0_1-FCP is a strictly stronger principle.

Proposition 2.7. *The following are equivalent over* RCA_0*:*

1. ACA_0*;*
2. *for every family* $A = \langle A_i : i \in \mathbb{N} \rangle$ *of sets, and every* Σ^0_1 *formula* $\varphi(X, x)$ *with one free set variable and one free number variable such that for all* $i \in \mathbb{N}$*, the formula* $\varphi(X, i)$ *is of finite character, there exists a family* $B = \langle B_i : i \in \mathbb{N} \rangle$ *of sets such that for all* i*,* B_i *is a* $\subseteq$*-maximal subset of* A_i *satisfying* $\varphi(X, i)$*.*

Proof. The forward implication follows by a straightforward modification of the proof of Theorem 2.3. For the reversal, let a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$ be given. For each $i \in \mathbb{N}$, let $A_i = \{i\}$, and let $\varphi(X, x)$ be the formula

$$(\exists y)[x \in X \implies f(y) = x].$$

Then, for each i, $\varphi(X, i)$ has the finite character property, and for every set S that contains i, $\varphi(S, i)$ holds if and only if $i \in \text{range}(f)$. Thus, if $B = \langle B_i : i \in \mathbb{N} \rangle$ is the subfamily obtained by applying part (2) to the family $A = \langle A_i : i \in \mathbb{N} \rangle$ and the formula $\varphi(X, x)$, then

$$i \in \text{range}(f) \iff B_i = \{i\} \iff i \in B_i.$$

It follows that the range of f exists. □

Remark 2.8. Proposition 2.7 would not hold with the class of bounded-quantifier formulas of finite character in place of the class of Σ^0_1 such formulas, because in that case part (2) is provable in RCA_0. Thus, in spite of the similarity between the two classes suggested by the proof of Theorem 2.6, they do not coincide.

3. Finitary closure operators

We can strengthen FCP by imposing additional requirements on the maximal set being constructed. In particular, we now consider requiring the maximal set to satisfy a finitary closure property as well as a property of finite character.

Definition 3.1. A *finitary closure operator* is a set of pairs $\langle F, n \rangle$ in which F is (the canonical index for) a finite (possibly empty) subset of $\mathbb{N}$ and $n \in \mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *closed* under a finitary closure operator D, or D*-closed*, if for every $\langle F, n \rangle \in D$, if $F \subseteq A$ then $n \in A$.

This definition of a closure operator is not the standard set-theoretic definition presented by Rubin and Rubin [3, Definition 6.3]. However, it is easy to see that for each operator of the one kind there is an operator of the other such that the same sets are closed under both. Our definition has the advantage of being readily formalizable in RCA_0.

The following principle expresses the monotonicity of finitary closure operators. The proof follows directly from definitions.

Proposition 3.2. *It can be proved in* RCA_0 *that if* D *is a finitary closure operator and* $A_0 \subseteq A_1 \subseteq A_2 \cdots$ *is a sequence of sets such that* $\bigcup_{i\in\mathbb{N}} A_i$ *exists and each* A_i *is* D*-closed, then* $\bigcup_{i\in\mathbb{N}} A_i$ *is* D*-closed.*

The principle in the next definition is analogous to principle $\mathsf{AL}'\,3$ of Rubin and Rubin [3], which is equivalent to the axiom of choice in the context of set theory [3, p. 96, and Theorems 6.4 and 6.5].

Definition 3.3. The following scheme is defined in RCA_0.

(CE) If D is a finitary closure operator, φ is an L_2 formula of finite character, and A is any set, then every D-closed subset of A satisfying φ is contained in a maximal such subset.

In the terminology of Rubin and Rubin [3], this is a "primed" statement, meaning that it asserts the existence not merely of a maximal subset of a given set, but the existence of a maximal extension of any given subset. Primed versions of FCP and its restrictions can be formed, and are equivalent to the unprimed versions over RCA_0. By contrast, CE has only a primed form. This is because if A is a set, φ is a formula of finite character, and D is a finitary closure operator, A need not have any D-closed subset of which φ holds. For example, suppose φ holds only of $\emptyset$, and D contains a pair of the form $\langle \emptyset, a\rangle$ for some $a \in A$.

This leads to the observation that the requirements in the CE scheme that the maximal set must both be D-closed and satisfy a property of finite character are, intuitively, in opposition to each other. Satisfying a finitary closure property is a positive requirement, in the sense that forming the closure of a set usually requires adding elements to the set. Satisfying a property of finite character can be seen as a negative requirement in light of part (1) of Proposition 2.2.

We consider restrictions of CE as we did restrictions of FCP above. By analogy, if Γ is a class of formulas, we use the notation Γ-CE to denote the restriction of CE to the formulas in Γ. We begin with the following analogue of part (1) of Theorem 2.3 from the previous section.

Theorem 3.4. *For* $i \in \{0,1\}$ *and* $n \geq 1$*, let* Γ *be* Π^i_n*,* Σ^i_n*, or* Δ^1_n*. Then* Γ-CE *is provable in* Γ-CA_0*.*

Proof. Let φ be a formula of finite character in Γ, which may have parameters, and let D be a finitary closure operator. Let A be any set and let C be a D-closed subset of A such that $\varphi(C)$ holds.

For any $X \subseteq A$, let $\mathrm{cl}_D(X)$ denote the D*-closure* of X. That is, $\mathrm{cl}_D(X) = \bigcup_{i\in\mathbb{N}} X_i$, where $X_0 = X$ and for each $i \in \mathbb{N}$, X_{i+1} is the set of all $n \in \mathbb{N}$ such that either $n \in X_i$ or there is a finite set $F \subseteq X_i$ such that $\langle F, n\rangle \in D$. Because we take D to be a set, $\mathrm{cl}_D(X)$ can be defined using a Σ^0_1 formula with parameter D. Define a formula $\psi(k, X)$ by

$$\psi(k,X) \Longleftrightarrow (\forall n)[(D_n \subseteq \mathrm{cl}_D(X \cup D_k) \Longrightarrow \widehat{\varphi}(n)] \\ \wedge\, \mathrm{cl}_D(X \cup D_k) \subseteq A,$$

where $\widehat{\varphi}$ is as in Lemma 2.4. Note that ψ is arithmetical if Γ is Π^0_n or Σ^0_n, and is in Γ otherwise.

Define a function $f\colon \mathbb{N} \to \{0,1\}$ inductively such that $f(i) = 1$ if and only if $\psi(\{j < i : f(j) = 1\} \cup \{i\}, C)$ holds. The characterization of the complexity of ψ ensures that this f can be constructed using Γ comprehension, by first forming the oracle $\{k : \psi(k, C)\}$.

Now, for each $i \in \mathbb{N}$, let

$$B_i = \mathrm{cl}_D(C \cup \{j < i : f(j) = 1\}),$$

and let $B = \bigcup_{i \in \mathbb{N}} B_i$. The construction of f ensures that $\varphi(B_i)$ implies $\varphi(B_{i+1})$ for all $i \in \mathbb{N}$, and we have assumed that φ holds of $B_0 = \mathrm{cl}_D(C) = C$. Therefore, an instance of induction shows that φ holds of B_i for all $i \in \mathbb{N}$, and thus also of B by Proposition 2.2. This also shows that $B \subseteq A$. Similarly, because each B_i is D-closed, the formalized version of Proposition 3.2 implies B is D-closed.

Finally, we check that B is maximal. Suppose that H is a D-closed set such that $B \subseteq H \subseteq A$ and $\varphi(H)$ holds. Fixing $i \in H$, because $B_i \subseteq B \subseteq H$ and H is D-closed, we have $\mathrm{cl}_D(B_i \cup \{i\}) \subseteq H$. Thus, $\varphi(F)$ holds for every finite subset F of $\mathrm{cl}_D(B_i \cup \{i\})$, so by construction $f(i) = 1$ and $B_{i+1} = \mathrm{cl}_D(B_i \cup \{i\})$. Because $B_{i+1} \subseteq B$, we conclude that $i \in B$. Thus $B = H$, as desired. □

It follows that for most standard syntactical classes Γ, Γ-CE is equivalent to Γ-FCP. Indeed, for any class Γ we have that Γ-CE implies Γ-FCP, because any instance of the latter can be regarded as an instance of the former by adding an empty finitary closure operator. Conversely, if Γ is Π^0_n, Π^1_n, Σ^1_n, or Δ^1_n, then Γ-FCP is equivalent to Γ-CA_0 by Theorem 2.3 (2), and hence equivalent to Γ-CE. Thus, in particular, parts (2)–(5) of Corollary 2.5 hold for CE in place of FCP, and the full scheme CE itself is equivalent to Z_2.

The proof of the preceding theorem does not work for $\Gamma = \Delta^0_1$, because then Γ-CA_0 is just RCA_0, and we need at least ACA_0 to prove the existence of the function f defined there (the formula $\psi(\sigma, X)$ being arithmetical at best). The next theorem shows that this cannot be avoided, even for a class of considerably weaker formulas.

Theorem 3.5. QF-CE *implies* ACA_0 *over* RCA_0.

Proof. Assume a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$ is given. Let $\varphi(X)$ be the quantifier-free formula $0 \notin X$, which trivially has finite character, and let $\langle p_i : i \in \mathbb{N}\rangle$ be an enumeration of all primes. Let D be the finitary closure operator consisting, for all $i, n \in \mathbb{N}$, of all pairs of the form

- $\langle \{p_i^{n+1}\}, p_i^{n+2}\rangle$;
- $\langle \{p_i^{n+2}\}, p_i^{n+1}\rangle$;
- $\langle \{p_i^{n+1}\}, 0\rangle$, if $f(n) = i$.

The set D exists by Δ^0_1 comprehension relative to f and our enumeration of primes.

Note that $\emptyset$ is a D-closed subset of $\mathbb{N}$ and $\varphi(\emptyset)$ holds. Thus, we may apply CE for quantifier-free formulas to obtain a maximal D-closed subset B of $\mathbb{N}$ such that $\varphi(B)$ holds. By definition of D, for every $i \in \mathbb{N}$, B either contains every positive power of p_i or no positive power. Now if $f(n) = i$ for some n, then no positive power of p_i can be in B, because otherwise p_i^{n+1} would necessarily be in B and hence so would 0. On the other hand, if $f(n) \neq i$ for all n then $B \cup \{p_i^{n+1} : n \in \mathbb{N}\}$ is D-closed and satisfies φ, so by maximality p_i^{n+1} must belong to B for every n. It follows that $i \in \text{range}(f)$ if and only if $p_i \notin B$, so the range of f exists. □

The next corollary can be contrasted with 2.5 part (1) and Theorem 2.6 to illustrate a difference between CE from FCP in terms of some of their weakest restrictions.

Corollary 3.6. *The following are equivalent over* RCA_0*:*

(1) ACA_0*;*
(2) Σ^0_1-CE*;*
(3) Σ^0_0-CE*;*
(4) QF-CE*.*

We conclude this section with one additional illustration of how formulas of finite character can be used in conjunction with finitary closure operators. Recall the following concepts from order theory:

- a *countable join-semilattice* is a countable poset $\langle L, \leq_L \rangle$ with a maximal element 1_L and a join operation $\vee_L \colon L \times L \to L$ such that for all $a, b \in L$, $a \vee_L b$ is the least upper bound of a and b;
- an *ideal* on a countable join-semilattice L is a subset I of L that is downward closed under $\leq_L$ and closed under $\vee_L$.

The principle in the following proposition is the countable analogue of a variant of $\mathsf{AL}'1$ in Rubin and Rubin [3]; compare with Proposition 4.4 below. For more on the computability theory of ideals on lattices, see Turlington [5].

Proposition 3.7. *Over* RCA_0*,* QF-CE *implies that every proper ideal on a countable join-semilattice extends to a maximal proper ideal.*

Proof. Let L be a countable join-semilattice. Let φ be the formula $1 \notin X$, and let D be the finitary closure operator consisting of all pairs of the form

- $\langle \{a, b\}, c \rangle$ where $a, b \in L$ and $c = a \vee b$;
- $\langle \{a\}, b \rangle$, where $b \leq_L a$.

Because we define a join-semilattice to come with both the order relation and the join operation, the set D is Δ^0_0 with parameters, so RCA_0 proves D exists. It is immediate that a set X is closed under D if and only if X is an ideal in L. □

We have not been able to prove a reversal corresponding to the previous proposition.

Question 3.8. What is the strength of the principle asserting that every proper ideal on a countable join-semilattice extends to a maximal proper ideal?

This question is further motivated by work of Turlington [5, Theorem 2.4.11] on the similar problem of constructing prime ideals on computable lattices. However, because a maximal ideal on a countable lattice need not be a prime ideal, Turlington's results do not directly resolve our question.

4. Nondeterministic finitary closure operators

It appears that the underlying reason that the restriction of CE to arithmetical formulas is provable in ACA_0 (and more generally, why Γ-CE is provable in Γ-CA_0 if Γ is as in Theorem 3.4) is that our definition of finitary closure operator is very constraining. Intuitively, if D is such an operator and φ is an arithmetical formula, and we seek to extend some D-closed subset B satisfying φ to a maximal such subset, we can focus largely on ensuring that φ holds. Achieving closure under D is relatively straightforward, because at each stage we only need to search through all finite subsets F of our current extension, and then adjoin all n such that $\langle F, n\rangle \in D$. This closure process becomes far less trivial if we are given a choice of which elements to adjoin. We now consider the case when each finite subset F can be associated with a possibly infinite set of numbers from which we must choose at least one to adjoin. Intuitively, this change adds an aspect of dependent choice when we wish to form the closure of a set. We will show that this weaker notion of closure operator leads to a strictly stronger analogue of CE.

Definition 4.1. A *nondeterministic finitary closure operator* is a sequence of sets of the form $\langle F, S\rangle$ where F is (the canonical index for) a finite (possibly empty) subset of $\mathbb{N}$ and S is a nonempty subset of $\mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *closed* under a nondeterministic finitary closure operator N, or N-closed, if for each $\langle F, S\rangle$ in N, if $F \subseteq A$ then $A \cap S \neq \emptyset$.

Note that if D is a *deterministic* finitary closure operator, that is, a finitary closure operator in the stronger sense of the previous section, then for any set A there is a unique $\subseteq$-minimal D-closed set extending A. This is not true for nondeterministic finitary closure operators. For example, let N be the operator such that $\langle \emptyset, \mathbb{N}\rangle \in N$ and, for each $i \in \mathbb{N}$ and each $j > i$, $\langle \{i\}, \{j\}\rangle \in N$. Then any N-closed set extending $\emptyset$ will be of the form $\{i \in \mathbb{N} : i \geq k\}$ for some k, and any set of this form is N-closed. Thus there is no $\subseteq$-minimal N-closed set.

In this section we study the following nondeterministic version of CE.

Definition 4.2. The following scheme is defined in RCA_0.

(NCE) If N is a nondeterministic closure operator, φ is an L_2 formula of finite character, and A is any set, then every N-closed subset of A satisfying φ is contained in a maximal such subset.

Because the union of a chain of N-closed sets is again N-closed, NCE can be proved in set theory using Zorn's lemma. Restrictions of NCE to various syntactical classes of formulas are defined as for CE and FCP.

Remark 4.3. We might expect to be able to prove NCE from CE by suitably transforming a given nondeterministic finitary closure operator N into a deterministic one. For instance, we could go through the members of N one by one, and for each such member $\langle F, S\rangle$ add $\langle F, n\rangle$ to D for some $n \in S$ (e.g., the least n). All D-closed sets would then indeed be N-closed. The converse, however, would not necessarily be true, because a set could have F as a subset for some $\langle F, S\rangle \in N$, yet it could contain a different $n \in S$ than the one chosen in defining D. In particular, a maximal D-closed subset of a given set might not be maximal among N-closed subsets. The results of this section demonstrate that it is impossible, in general, to reduce nondeterministic closure operators to deterministic ones in weak systems.

Recall that an *ideal* on a countable poset $\langle P, \leq_P\rangle$ is a subset I of P downward closed under $\leq_P$ and such that for all $p, q \in I$ there is an $r \in I$ with $p \leq_P r$ and $q \leq_P r$. The next proposition is similar to Proposition 3.7 above, which dealt with ideals on countable join-semilattices. In the proof of that proposition, we defined a deterministic finitary closure operator D in such a way that D-closed sets were closed under the join operation. For this we relied on the fact that for every two elements in the semilattice there is a unique element that is their join. The reason we need nondeterministic finitary closure operators below is that, for ideals on countable posets, there are no longer unique elements witnessing the relevant closure property.

Proposition 4.4. *Over* RCA_0*,* $\mathsf{QF\text{-}NCE}$ *implies that every ideal on a countable poset can be extended to a maximal ideal.*

Proof. Let $\langle P, \leq_P\rangle$ be a countable poset; without loss of generality we may assume P is infinite. Form an extended poset $\widehat{P}$ by adjoining a new element t to P and declaring $q <_{\widehat{P}} t$ for all $q \in P$. It follows immediately that the ideals on P correspond exactly to the ideals of $\widehat{P}$ that do not contain t, and each ideal on $\widehat{P}$ which is maximal among ideals not containing t corresponds to a maximal ideal on P.

Fix an enumeration $\{p_i : i \in \mathbb{N}\}$ of $\widehat{P}$. We form a nondeterministic closure operator $N = \langle N_i : i \in \mathbb{N}\rangle$ such that, for each $i \in \mathbb{N}$,

- if $i = 2\langle j, k\rangle$ and $p_j \leq_{\widehat{P}} p_k$ then $N_i = \langle\{p_k\}, \{p_j\}\rangle$;
- if $i = 2\langle j, k, l\rangle + 1$ and $p_j \leq_{\widehat{P}} p_l$ and $p_k \leq_{\widehat{P}} p_l$ then
$$N_i = \langle\{p_j, p_k\}, \{p_n : (p_j \leq_{\widehat{P}} p_n) \wedge (p_k \leq_{\widehat{P}} p_n)\}\rangle;$$
- otherwise, $N_i = \langle\{p_i\}, \{p_i\}\rangle$.

This construction gives a quantifier-free definition of each N_i uniformly in i, so RCA_0 is able to construct N. Moreover, a subset of $\widehat{P}$ is N-closed if and only if it is an ideal.

Let $\varphi(X)$ be the formula $t \notin X$, which is of finite character. Fix an ideal $I \subseteq P$. Viewing I as a subset of $\widehat{P}$, we see that I is N-closed and $\varphi(I)$ holds. Thus, by QF-NCE, there is a maximal N-closed extension $J \subseteq \widehat{P}$ satisfying φ. This immediately yields a maximal ideal on P extending I. □

Mummert [2, Theorem 2.4] showed that the proposition that every ideal on a countable poset extends to a maximal ideal is equivalent to Π^1_1-CA_0 over RCA_0, which leads to the following corollary. This contrasts sharply with Theorem 3.4, which showed that CE for arithmetical formulas is provable in ACA_0.

Corollary 4.5. QF-NCE *implies* Π^1_1-CA_0 *over* RCA_0.

We will state the precise strength of QF-NCE in Corollary 4.7 below. We must first prove the following upper bound. The proof uses a technique involving countable coded β-models, parallel to Lemma 2.4 of Mummert [2]. In ACA_0, a *countable coded β-model* is defined as a sequence $\mathscr{M} = \langle M_i : i \in \mathbb{N}\rangle$ of subsets of $\mathbb{N}$ such that for every Σ^1_1 formula φ with parameters from $\mathscr{M}$, φ holds if and only if $\mathscr{M} \models \varphi$. Π^1_1-CA_0 proves that every set is included in some countable coded β-model. Complete information on countable coded β-models is given by Simpson [4, Section VII.2].

Theorem 4.6. Σ^1_1-NCE *is provable in* Π^1_1-CA_0.

Proof. Let φ be a Σ^1_1 formula of finite character (possibly with parameters) and let N be a nondeterministic closure operator. Let A be any set and let C be an N-closed subset of A such that $\varphi(C)$ holds.

Let $\mathscr{M} = \langle M_i : i \in \mathbb{N}\rangle$ be a countable coded β-model containing A, C, N, and any parameters of φ. Using Π^1_1 comprehension, we may form the set $\{i : \mathscr{M} \models \varphi(M_i)\}$.

Working outside $\mathscr{M}$, we build an increasing sequence $\langle B_i : i \in \mathbb{N}\rangle$ of N-closed extensions of C. Let $B_0 = C$. Given i, ask whether there is a j such that

- M_j is an N-closed subset of A;
- $B_i \subseteq M_j$;
- $i \in M_j$;
- and $\varphi(M_j)$ holds.

If there is, choose the least such j and let $B_{i+1} = M_j$. Otherwise, let $B_{i+1} = B_i$. Finally, let $B = \bigcup_{i \in \mathbb{N}} B_i$.

Because the inductive construction only asks arithmetical questions about $\mathscr{M}$, it can be carried out in Π^1_1-CA_0, and so Π^1_1-CA_0 proves that B exists. Clearly $C \subseteq B \subseteq A$. An arithmetical induction shows that for all $i \in \mathbb{N}$, $\varphi(B_i)$ holds and B_i is N-closed. Therefore, the formalized version of Proposition 2.2 shows that $\varphi(B)$ holds, and the analogue of Proposition 3.2 for nondeterministic finitary closure operators shows that B is N-closed.

Now suppose that H is an N-closed set such that $B \subseteq H \subseteq A$ and $\varphi(H)$ holds. Fix $i \in H$. Because φ is Σ^1_1, the property

$$(\exists X)[X \text{ is } N\text{-closed} \wedge B_i \subseteq X \subseteq A \wedge i \in X \wedge \varphi(X)] \tag{4.6.1}$$

is expressible by a Σ^1_1 sentence with parameters from $\mathscr{M}$, and H witnesses that it is true. Thus, because $\mathscr{M}$ is a β-model, this sentence must be satisfied by $\mathscr{M}$, which means that some M_j must also witness it. The inductive construction must therefore have selected such an M_j to be B_{i+1}, which means $i \in B_{i+1}$ and hence $i \in B$. It follows that B is maximal. ☐

We can now characterize the strength of Σ^1_1-NCE and its restrictions.

Corollary 4.7. *For each $n \geq 1$, the following are equivalent over RCA_0:*

(1) Π^1_1-CA_0;
(2) Σ^1_1-NCE;
(3) Σ^0_n-NCE;
(4) QF-NCE.

Proof. Theorem 4.6 shows that (1) implies (2), and it is obvious that (2) implies (3) and (3) implies (4). Corollary 4.5 shows that (4) implies (1). ☐

Our final results characterize the strength of NCE for formulas higher in the analytical hierarchy.

Theorem 4.8. *For each $n \geq 1$,*

(1) Σ^1_n-NCE *and* Π^1_n-NCE *are provable in* Π^1_n-CA_0;
(2) Δ^1_n-NCE *is provable in* Δ^1_n-CA_0.

Proof. We prove part (1), the proof of part (2) being similar. Let $\varphi(X)$ be a Σ^1_n formula of finite character, respectively a Π^1_n such formula. Let N be a nondeterministic closure operator, let A be any set, and let C be an N-closed subset of A such that $\varphi(C)$ holds.

By Lemma 4.5, let $\widehat{\varphi}$ be a Σ^1_n formula, respectively a Π^1_n formula, such that

$$(\forall X)(\forall n)[X = D_n \Longrightarrow (\varphi(X) \Longleftrightarrow \widehat{\varphi}(n))].$$

We may use Π^1_n comprehension to form the set $W = \{n : \widehat{\varphi}(n)\}$. Define $\psi(X)$ to be the arithmetical formula $(\forall n)[D_n \subseteq X \Longrightarrow n \in W]$.

We claim that for every set X, $\psi(X)$ holds if and only if $\varphi(X)$ holds. The definitions of W and ψ ensure that $\psi(X)$ holds if and only if $\varphi(D_n)$ holds for every finite $D_n \subseteq X$, which is true if and only if $\varphi(X)$ holds because φ has finite character. This establishes the claim.

By the claim, ψ is a property of finite character and $\psi(C)$ holds. Using Σ^1_1-NCE, which is provable in Π^1_1-CA_0 by Theorem 4.6 and thus is provable in Π^1_n-CA_0, there is a maximal N-closed subset B of A extending C with property ψ. Again by the claim, B is a maximal N-closed subset of A extending B with property φ. ☐

Corollary 4.9. *The following are provable in* RCA_0*:*

(1) *for each* $n \geq 1$, Δ^1_n-CA_0 *is equivalent to* Δ^1_n-NCE*;*
(2) *for each* $n \geq 1$, Π^1_n-CA_0 *is equivalent to* Π^1_n-NCE *and to* Σ^1_n-NCE*;*
(3) Z_2 *is equivelent to* NCE*.*

Proof. The implications from Δ^1_n-CA_0, Π^1_n-CA_0, and Z_2 follow by Theorem 4.8. On the other hand, each restriction of NCE trivially implies the corresponding restriction of FCP, so the reversals follow by Corollary 2.5. □

Remark 4.10. The characterizations in this section shed light on the role of the closure operator in the principles CE and NCE. For $n \geq 1$, we have shown that Σ^1_n-FCP, Σ^1_n-CE, and Σ^1_n-NCE are all equivalent over RCA_0. However, QF-FCP is provable in RCA_0, QF-CE is equvalent to ACA_0 over RCA_0, and QF-NCE is equivalent to Π^1_1-CA_0 over RCA_0. Thus the closure operators in the stronger principles serve as a sort of replacement for arithmetical quantification in the case of CE, and for Σ^1_1 quantification in the case of NCE. This allows these principles to have greater strength than might be suggested by the property of finite character alone. At higher levels of the analytical hierarchy, the principles become equivalent because the complexity of the property of finite character overtakes the complexity of the closure notions.

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