AN INFINITUDE OF FINITUDES

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1. Introduction

Definitions of finiteness are set-theoretical properties **P** such that, if the axiom of choice (**AC**) is assumed, a set satisfies **P** if and only if it is finite. It is well known that the equivalence of many definitions of finiteness (to one another, and to "ordinary" finiteness, defined below) *requires* some (perhaps weaker) form of **AC**. Thus, it is of interest to study how various definitions of finiteness relate to one another in theories without **AC** (see [1] for a comprehensive overview of previous work done in this area).

For this and other reasons, a great many—but nevertheless, *finitely many*—definitions of finiteness have been invented and investigated over the years, beginning with the work of Tarski [5] in 1924. In this note, I show how a natural generalization of a particularly well-studied notion of finiteness yields *infinitely many* new, non-equivalent notions.

2. Definitions

I begin with some definitions and terminology. Along with two classical notions of finiteness, necessary to gauge the strength of any definition of finiteness in the context of set theory without **AC**, I give Lévy's notion of "amorphous sets", and from it obtain infinitely many new definitions.

The first definition of finiteness is the most commonly encountered one. It is the intuitive, or "ordinary", notion of finiteness.

Definition 2.1. A set X is *finite* if there exists an $n \in \omega$ and a bijective function $f : X \to n$. Otherwise, X is said to be *infinite*.

In 1958, Lévy [4] introduced the notion of a *Ia-finite set*, by which he meant any set which cannot be written as a union of two disjoint infinite sets. He proved that this property was equivalent to finiteness (in the sense of Definition 2.1) under **AC**, but not so without it. He called any infinite, Ia-fintie set *amorphous*, which, given the bizarre nature of such a set, should one exist (as it is consistent, say in the theory ZF, that it does), is probably a very fitting name.

In the next definition, I present a trivial generalization of Ia-finiteness, which will be seen to have the interesting consequence of yielding infinitely many definitions of finiteness.

Definition 2.2. Fix $n \in \omega$, $n \geq 1$. A set X is $\mathbf{A}^n$*-finite* if whenever $X_1, \ldots, X_n$ are pairwise disjoint sets with $X = X_1 \cup \cdots \cup X_n$, there is some i such that X_i is finite. Otherwise, X is said to be $\mathbf{A}^n$*-infinite*.

Observe, in particular, that $\mathbf{A}^1$-finiteness is precisely "ordinary" finiteness of Definition 2.1, while $\mathbf{A}^2$-finiteness is precisely Lévy's Ia-finiteness.

Definition 2.3. A set X is $\mathbf{W}$-*finite* if it is finite or else not well-orderable. Otherwise, X is said to be $\mathbf{W}$-*infinite.*

The "purpose" of $\mathbf{W}$-finiteness in investigations of definitions of finiteness is in many ways dual to that of ordinary finiteness above. Much as ordinary finiteness serves as the most restrictive notion of finiteness, implying (in set theory without **AC**) every other notion of finiteness, $\mathbf{W}$-finiteness serves as the least restrictive (cf. Section 3 for more on this).

3. Implications

It is obvious that every finite set is $\mathbf{A}^1$-finite (the first person to whom it was obvious was Lévy, of course, in [4]). It is equally obvious that every finite set is $\mathbf{A}^n$-finite for any $n \geq 1$.

Theorem 3.1. *For $n \geq 1$, ZF $\vdash$ "every $\mathbf{A}^n$-finite set is $\mathbf{A}^{n+1}$-finite".*

Proof. Let X be an $\mathbf{A}^{n+1}$-infinite set, so that $X = X_1 \cup \cdots \cup X_{n+1}$ for some pairwise disjoint infinite sets $X_1, \ldots, X_{n+1}$. Then $X_n \cup X_{n+1}$ is an infinite set disjoint from each of $X_1, \ldots, X_{n-1}$, and $X = X_1 \cup \cdots \cup X_{n-1} \cup (X_n \cup X_{n+1})$, so X is $\mathbf{A}^n$-infinite. □

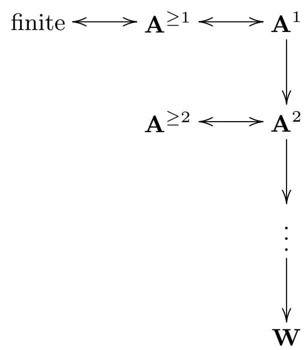
Theorem 3.1 permits us to give a nice alternative characterization of the notions $\mathbf{A}^n$ of Definition 2.2. For $n \geq 1$, let us say that a set is $\mathbf{A}^{\geq n}$-*finite* if it is $\mathbf{A}^k$-finite for all $k \geq n$. Thus any $\mathbf{A}^{\geq n}$-finite set is $\mathbf{A}^n$-finite, and the preceding theorem gives us precisely the converse, so that a set is $\mathbf{A}^n$-finite if and only if it is $\mathbf{A}^{\geq n}$-finite. As a special case, a set is finite if and only if it is $\mathbf{A}^{\geq 1}$-finite, if and only if it $\mathbf{A}^n$-finite for every $n \geq 1$. In some sense, this captures the intuition that $\mathbf{A}^n$-finite sets get "more and more infinite" as n grows.

Theorem 3.2. *For $n \geq 1$, ZF $\vdash$ "every $\mathbf{A}^n$-finite set is $\mathbf{W}$-finite".*

Proof. Let X be a $\mathbf{W}$-infinite set, so that X is infinite and well-orderable. Then X has order type $\geq \omega$, so there is an injective map $f : \omega \to X$. Let $p_1 < \cdots < p_n$ be a listing of the least n prime numbers, and for each i, $1 \leq i \leq n$, let $Y_i = \left\{p_i^k : k \geq 1\right\}$. Now for $1 \leq i < n$, define $X_i = f(Y_i)$, and let $X_n = f(Y_n) \cup (X - f(\omega))$. Then clearly the X_i are each infinite, pairwise disjoint, and $X = X_1 \cup \cdots \cup X_n$. Hence, X is $\mathbf{A}^n$-infinite. □ □

In ZFC, every set is well-orderable, hence every $\mathbf{W}$-finite set is finite. Theorem 3.2 thus implies that, in ZFC, a set is finite if and only if it is $\mathbf{A}^n$-finite, for any $n \in \omega$.

This section can be summarized with the following diagram, in which an arrow from the name of one notion of finiteness to the name of another means that every set finite in the sense of the former is (provably in ZF) finite in the sense of the latter.



I shall prove in the next section that the unidirectional arrows in the above diagram cannot be reversed in ZF. As was remarked above, all the arrows become bidirectional in ZFC.

4. Non-implications

For the proofs of Theorems 4.2 and 4.3, I assume familiarity with the theory ZFA, and with permutation (or so-called Fraenkel-Mostowski) models thereof. A full treatment may be found in [3] and a brief introduction in [2]. By the Jech-Sochor Embedding Theorem (mentioned in both sources), both the independence results of this section transfer from ZFA to the stronger theory ZF.

Fix $n \in \omega$ and a model $\mathcal{M}$ of ZFA + AC+"A is infinite". Within $\mathcal{M}$, let $A_1, \ldots, A_n$ be pairwise disjoint infinite sets with $A = A_1 \cup \cdots \cup A_n$.

Lemma 4.1. *Let $B_1, \ldots, B_{n+1}$ be a pairwise disjoint infinite sets such that $A = B_1 \cup \cdots \cup B_{n+1}$. Then if E is a finite subset of A, there exist $a, b \in A - E$ which belong to same member of $\{A_i : 1 \leq i \leq n\}$ but to different members of $\{B_j : 1 \leq j \leq n+1\}$.*

Proof. Assume not. Then there exists a finite subset E of A such that for all $a, b \in A - E$, if a and b belong to the same member of $\{A_i\}$ they also belong to the same member of $\{B_j\}$. Thus if i is fixed and $a \in A_i - E$, with j such that $a \in B_j$, then for all $b \in A - E_i$, $b \in B_j$; in other words, $A_i - E \subseteq B_j$. It follows by the pigeonhole principle that there exists a j such that for all i, $(A_i - E) \cap B_j = \varnothing$, and hence, since $A = A_1 \cup \cdots \cup A_n = B_1 \cup \cdots \cup B_n$, that $B_j \subseteq E$. But then B_j is finite, which is a contradiction. □

Now let G be the subgroup of $\mathrm{Aut}\,(V)$ consisting of all those permutations which preserve each of the A_i, and let F be the base of finite supports. Let $\mathcal{N}$ be the permutation submodel of $\mathcal{M}$ obtained from G and F; it is a model of ZFA.

Theorem 4.2. *For $n \geq 1$, ZF $\nvdash$ "every $\mathbf{A}^{n+1}$-finite set is $\mathbf{A}^n$-finite".*

Proof. First note that each A_i is in $\mathcal{N}$; it is supported by $\varnothing$ since every permutation in G sends A_i into itself. Hence, A is $\mathbf{A}^n$-infinite in $\mathcal{N}$. However, I claim that A is $\mathbf{A}^{n+1}$-finite in this model. Indeed, suppose not, and let $B_1, \ldots, B_{n+1}$ be pairwise disjoint infinite sets in $\mathcal{N}$ with $A = B_1 \cup \cdots \cup B_{n+1}$. Say $E_1, \ldots, E_n \in F$ are such that E_i supports B_i for $1 \leq i \leq n+1$. Then $E = E_1 \cup \cdots \cup E_n$ is a finite subset of A, so by the lemma there are $a, b \in A - E$ such that $a \in A_i \cap B_j$ and $b \in A_i \cap B_k$ for some i, j, k with $j \neq k$. Now consider the transposition $\pi = (ab)$ of $\mathrm{Aut}\,(V)$; it lies in G since $\pi \restriction A_{i'} = \mathsf{id}$ for $i' \neq i$ and $\pi A_i = A_i$. Moreover, G lies in $\mathsf{fix}_G\,(E_j)$

since $a, b \notin E_j$, but $\pi B_j \neq B_j$ since $\pi(a) = b \in \pi B_j - B_j$. Hence $\mathsf{fix}_G(E_j) \nsubseteq G_{B_j}$, contradicting the assumption that E_j supports B_j. Consequently, there are no $(n+1)$-many infinite disjoint subsets of A in $\mathcal{N}$ whose union is all of A, and hence that A is $\mathbf{A}^{n+1}$-finite. □

One way of viewing Theorem 4.2 is that Lévy's definition of "amorphous set" generalizes along with Ia-finiteness. We are thus free to define a set to be $(n+1)$-amorphous if it is $\mathbf{A}^{n+1}$-finite but $\mathbf{A}^n$-infinite, and the preceding theorem tells us that it is conceivable (read as "consistent with the theory ZF") that $(n+1)$-amorphous (highly, highly, highly bizarre objects though they are) exist.

The same model $\mathcal{N}$ allows us to see that the notions $\mathbf{A}^n$ do not collapse "downward" into the notion $\mathbf{W}$.

Theorem 4.3. *For $n \geq 1$, ZF $\nvdash$ "every $\mathbf{W}$-finite set is $\mathbf{A}^n$-finite".*

Proof. As Theorem 4.2 showed, the set A of atoms is $\mathbf{A}^n$-infinite in $\mathcal{N}$. However, A is not well-orderable in $\mathcal{N}$, and hence is $\mathbf{W}$-finite; indeed, A is not even *linearly* orderable in $\mathcal{N}$. For suppose $<$ in $\mathcal{N}$ is a linear ordering of A, and say $E \in F$ supports $<$. Since E is finite and each A_i infinite, there exist $a, b \in A - E$ both lying in the same A_i. Then the transposition $\pi = (ab)$ is in $\mathsf{fix}_G(E)$, but $\pi < \neq <$ since $(a, b) \in <$ and $(b, a) = (\pi a, \pi b) \in \pi <$, contradicting that E supports $<$. □

Thus, my claims about the arrows in the diagram at the end of Section 3 are justified.

References

[1] O. De la Cruz, *Finiteness and choice*, Fund. Math. 173 (2002), 57–76.

[2] O. De la Cruz, D. D. Dzhafarov, and E. J. Hall, *Definitions of finiteness based on order properties*, Fund. Math. 189 (2006), 155–172.

[3] T. J. Jech, *The Axiom of Choice*, Stud. Logic Found. Math. 75, North-Holland, Amsterdam, 1973.

[4] A. Lévy, *The independence of various definitions of finiteness*, Fund. Math. 46 (1958), 1–13.

[5] A. Tarski, *Sur les ensembles finis*, Fund. Math. 6 (1924), 45–95.

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