ON THE STRENGTH OF THE FINITE INTERSECTION PRINCIPLE

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Abstract. We study the logical content of several maximality principles related to the finite intersection principle ($F\mathsf{IP}$) in set theory. Classically, these are all equivalent to the axiom of choice, but in the context of reverse mathematics their strengths vary: some are equivalent to ACA_0 over RCA_0, while others are strictly weaker and incomparable with WKL_0. We show that there is a computable instance of $F\mathsf{IP}$ every solution of which has hyperimmune degree, and that every computable instance has a solution in every nonzero c.e. degree. In particular, $F\mathsf{IP}$ implies the omitting partial types principle (OPT) over RCA_0. We also show that, modulo Σ^0_2 induction, $F\mathsf{IP}$ lies strictly below the atomic model theorem (AMT).

1. Introduction

After Zermelo introduced the axiom of choice in 1904, set theorists began to obtain results proving other set-theoretic principles equivalent to it (relative to choice-free axiomatizations of set theory such as ZF). These equivalence results, and their further development, now constitute a program in set theory, which has been documented in detail by Jech [8] and by Rubin and Rubin [11, 12]. Moore [10] provides a general historical account of the axiom of choice.

In this article, we study the logical content of several such equivalences from the point of view of computability theory and reverse mathematics. Specifically, we focus on maximality principles related to the following:

Finite intersection principle. *Every family of sets has a $\subseteq$-maximal subfamily with the finite intersection property.*

This research has two closely related motivations. First, we wish to study various equivalents of the axiom of choice to determine how they compare with one another and with other mathematical principles, in the spirit of the program of reverse mathematics. This program is devoted to gauging the relative strengths of (countable analogues of) mathematical theorems by calibrating the precise set existence axioms necessary and sufficient to carry out their proofs in second-order arithmetic. Second, we wish to explore potential new connections between set-theoretic principles and computability-theoretic constructions, such as have emerged in the investigations of other theorems, looking for new insights into the underlying combinatorics of the principles. (For examples, see Hirschfeldt and Shore [5, Section 1], and also

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Section 4 below.) We refer to Soare [15] and Simpson [14], respectively, for general background in computability theory and reverse mathematics.

Various forms of the axiom of choice have been studied in the present context, including direct formalizations of choice principles in second-order arithmetic by Simpson [14, Section VII.6]; countable well-orderings by Friedman and Hirst [4] and Hirst [7]; and principles related to properties of finite character by Dzhafarov and Mummert [3]. These principles display varying strengths, but tend to be at least as strong as ACA_0. By contrast, the finite intersection principle and its variants will turn out to be strictly weaker than ACA_0 and incomparable with WKL_0. We establish a link between maximal subfamilies with the finite intersection property and sets of hyperimmune Turing degree, which allows us to closely locate the positions of these principles among the statements lying between RCA_0 and ACA_0. In particular, we show that they are closely related in strength to the atomic model theorem, studied by Hirschfeldt, Shore, and Slaman [6].

We pass to the formal definitions needed for the sequel.

Definition 1.1.

(1) A *family of sets* is a sequence $A = \langle A_n : n \in \omega \rangle$ of sets. A family A is *nontrivial* if $A_n \neq \emptyset$ for some n.
(2) Given a family of sets A, we say a set S is *in* A, and write $S \in A$, if $S = A_n$ for some n. A family of sets $B = \langle B_n : n \in \omega \rangle$ is a *subfamily* of A if every set in B is in A, that is, $(\forall n)(\exists m)[B_n = A_m]$.
(3) Two sets in A are *distinct* if they differ extensionally as sets.

Our definition of a subfamily is intentionally weak; see Proposition 2.3 below and the remarks preceding it.

Definition 1.2. Let $A = \langle A_n : n \in \omega \rangle$ be a family of sets and fix $k \geq 2$. Then A has the

- D_k *intersection property* if the intersection of any k distinct sets in A is empty;
- $\overline{D}_k$ *intersection property* if the intersection of any k distinct sets in A is nonempty;
- F *intersection property* if the intersection of any two or more distinct sets in A is nonempty.

Definition 1.3. Let A be a family of sets, let P be any of the properties in Definition 1.2, and let B be a subfamily of A with the P intersection property. We say B is a *maximal* such subfamily if for every other such subfamily C, B being a subfamily of C implies C is a subfamily of B.

It is straightforward to formalize Definitions 1.1–1.3 in RCA_0.

Given a family $A = \langle A_n : n \in \omega \rangle$ and $J \in \omega^\omega$, we use the notation $\langle A_{J(n)} : n \in \omega \rangle$ for the subfamily $\langle B_n : n \in \omega \rangle$ where $B_n = A_{J(n)}$. We call this the subfamily *defined* by J. For a finite set $\{m_0, \ldots, m_{s-1}\} \subset \omega$, we let $\langle A_{m_0}, \ldots, A_{m_{s-1}} \rangle$ denote the subfamily $\langle B_n : n \in \mathbb{N} \rangle$ where $B_n = A_{m_n}$ for $n < s$ and $B_n = A_{m_{s-1}}$ for $n \geq s$. More generally, we call a subfamily B of A *finite* if it has only finitely many distinct sets.

Let P be any of the properties in Definition 1.2. We shall be interested in the following:

***P* intersection principle** ($P\mathsf{IP}$). *Every nontrivial family of sets has a maximal subfamily with the P intersection property.*

Following common usage, we shall refer to a given family as an *instance* of $P\mathsf{IP}$, and to a maximal subfamily with the P intersection property as a *solution* to this instance.

The classic set-theoretic analogues of $D_k\mathsf{IP}$ and $\overline{D}_k\mathsf{IP}$ in the catalogue of Rubin and Rubin [12] of equivalents of the axiom of choice are $\mathsf{M}\,8\,(D_k)$ and $\mathsf{M}\,8\,(\overline{D}_k)$, respectively; the analogue of $F\mathsf{IP}$ is $\mathsf{M}\,14$. For additional references and results concerning these forms, see [12, pp. 54–56, 60].

Remark 1.4. Although we do not make it an explicit part of the definition, all of the families $A = \langle A_n : n \in \omega\rangle$ we construct in our results will have the property that for every n, A_n contains $2n$ and otherwise contains only odd numbers. This will have the advantage that if we are given an arbitrary subfamily $B = \langle B_n : n \in \omega\rangle$ of some such family, then for every n there is a unique m such that $B_n = A_m$, and it can be found uniformly B-computably from n. If A is computable, every subfamily B will then be of the form $\langle A_{J(n)} : n \in \omega\rangle$ for some $J\colon \omega \to \omega$ with $J \equiv_T B$.

2. Basic implications and equivalences to ACA_0

The following pair of propositions establishes the basic relations that hold among the principles we have defined.

Proposition 2.1. *For any property P in Definition 1.2, $P\mathsf{IP}$ is provable in ACA_0.*

Proof. Given a nontrivial family of sets $A = \langle A_n : n \in \mathbb{N}\rangle$ we may assume that $A_0 \neq \emptyset$. Arithmetical comprehension suffices to determine whether a finite collection of sets has the P intersection property. Thus, we can build a maximal subfamily $\langle B_n : n \in \mathbb{N}\rangle$ of A with the P intersection property in ACA_0 by letting $B_0 = A_0$ and, for $n > 0$, letting B_n be A_n if $\langle B_0, \dots, B_{n-1}, A_n\rangle$ has the P intersection property, and B_{n-1} otherwise. □

Proposition 2.2. *For every $k \geq 2$, the following are provable in RCA_0:*

(1) *$F\mathsf{IP}$ implies $\overline{D}_k\mathsf{IP}$;*
(2) *$\overline{D}_{k+1}\mathsf{IP}$ implies $\overline{D}_k\mathsf{IP}$.*

Proof. To prove (1), let $A = \langle A_n : n \in \mathbb{N}\rangle$ be a nontrivial family of sets. By recursion, define a new family $\hat{A} = \langle \hat{A}_n : n \in \mathbb{N}\rangle$ with the property that for every finite set F with $|F| \geq k$,

$$\bigcap_{n\in F} \hat{A}_n \neq \emptyset \iff (\forall G \subseteq F)[\,|G| = k \implies \bigcap_{n \in G} A_n \neq \emptyset\,]. \tag{2.1}$$

As discussed in Remark 1.4, we begin by adding $2n$ to $\hat{A}_n$ for all n. Then, at stage $s \geq 0$, we consider each $F \subseteq \{0, \dots, s\}$ of size at least k, and add $2\langle F, s\rangle + 1$ to the sets $\hat{A}_n$ with $n \in F$ if for each $G \subseteq F$ of size k, $\bigcap_{n\in G} A_n$ contains an element $\leq s$. (We are identifying F with its canonical index here.)

The family $\hat{A}$ exists by Δ^0_1 comprehension, and is nontrivial by construction. It is also easily seen to satisfy (2.1). Now every subfamily $\hat{B} = \langle \hat{B}_n : n \in \mathbb{N}\rangle$ of $\hat{A}$ determines a subfamily $B = \langle B_n : n \in \mathbb{N}\rangle$ of A by defining B_n to be A_m for the unique m with $\hat{B}_n = \hat{A}_m$. Moreover, the fact that $\hat{B}_n$ contains no even numbers besides $2m$ means that the subfamily B is Δ^0_1-definable from $\hat{B}$. Finally, (2.1) ensures that if $\hat{B}$ has the F intersection property then B has the $\overline{D}_k$ intersection property, and that the latter is maximal if the former is. Thus, applying $F\mathsf{IP}$ to $\hat{A}$ allows us in RCA_0 to find a maximal subfamily of A with the $\overline{D}_k$ intersection property, as desired.

A similar argument can be used to prove (2). □

We do not know whether the implications from $F\mathsf{IP}$ to $\overline{D}_k\mathsf{IP}$ or from $\overline{D}_{k+1}\mathsf{IP}$ to $\overline{D}_k\mathsf{IP}$ are strict. Nevertheless, by the previous proposition, results in the sequel that are phrased as implications to $F\mathsf{IP}$ or implications from $\overline{D}_2\mathsf{IP}$ are optimal.

An apparent weakness of our definition of subfamily is that we cannot, in general, effectively decide which members of a family are in a given subfamily. The following proposition demonstrates that if the definition were strengthened to make this decidable, all the intersection principles would collapse to ACA_0. The subsequent proposition shows that this happens for $P = D_k$ even with the weak definition.

Proposition 2.3. *Let P be any of the properties in Definition 1.2. The following are equivalent over RCA_0:*

(1) ACA_0;
(2) *every nontrivial family of sets $\langle A_n : n \in \mathbb{N}\rangle$ has a maximal subfamily B with the P intersection property, and the set $I = \{n \in \mathbb{N} : A_n \in B\}$ exists.*

Proof. That (1) implies (2) is proved similarly to Proposition 2.1.

To show that (2) implies (1), we work in RCA_0 and let $f\colon \mathbb{N} \to \mathbb{N}$ be a given function whose range we wish to prove exists. For every n, let

$$A_n = \{2n\} \cup \{2m + 1 : (\exists a \leq m)[f(a) = n]\}.$$

Then $n \in \operatorname{range}(f)$ if and only if A_n is not a singleton, in which case A_n contains cofinitely many odd numbers. Consequently, for every finite $F \subset \mathbb{N}$ with $|F| \geq 2$, we have $\bigcap_{n\in F} A_n \neq \emptyset$ if and only if each $n \in F$ is in the range of f.

Apply (2) to the family $\langle A_n : n \in \mathbb{N}\rangle$ to find the maximal subfamily B and set I. If $P = D_k$ there can be at most $k - 1$ distinct n such that $n \in \operatorname{range}(f)$ and $A_n \in B$, so for almost all n we have $n \in \operatorname{range}(f) \iff A_n \notin B \iff n \notin I$. If instead $P = F$ or $P = \overline{D}_k$ then every set in B contains cofinitely many odd numbers and so $n \in \operatorname{range}(f) \iff A_n \in B \iff n \in I$. In any case, then, the range of f exists. □

Proposition 2.4. *For every $k \geq 2$, $D_k\mathsf{IP}$ is equivalent to ACA_0 over RCA_0.*

Proof. Fix a function $f\colon \mathbb{N} \to \mathbb{N}$, and let A be the family defined in the preceding proposition. Let $B = \langle B_n : n \in \mathbb{N}\rangle$ be the family obtained from applying $D_k\mathsf{IP}$ to A. As above, for all but finitely many n we have $n \in \operatorname{range}(f) \iff A_n \notin B \iff (\forall m)[2n \notin B_m]$, giving us a Π^0_1 definition of the range of f. Since the range is also definable by a Σ^0_1 formula, it must exist by Δ^0_1 comprehension. □

We conclude this section by showing that, by contrast, $F\mathsf{IP}$ is strictly weaker than ACA_0. We shall obtain a considerable strengthening of this fact in Theorem 4.4, but the proof here further illustrates the flexibility of our definition of subfamily.

Proposition 2.5. *Every computable nontrivial family has a low maximal subfamily with the F intersection property.*

Proof. Given $A = \langle A_n : n \in \omega \rangle$ computable and nontrivial, let $\mathbb{P}_A$ be the notion of forcing whose conditions are pairs (σ, s) where $\sigma \in \omega^{<\omega}$ and some number $\leq s$ belongs to $\bigcap_{n<|\sigma|} A_{\sigma(n)}$, and a condition (τ, t) extends (σ, s) if $\sigma \preceq \tau$ and $s \leq t$. Generic filters will then consist of conditions with pairwise comparable first coordinates, and generic objects can be defined from these as unions of first coordinates as in Cohen forcing. In particular, $\emptyset'$ can build a 1-generic object, G, which must be low by the usual argument. For every n, the set of conditions with n in their range is Σ^0_1-definable and dense above any condition (σ, s) with $A_n \cap \bigcap_{m<|\sigma|} A_{\sigma(m)} \neq \emptyset$, meaning that if G avoids this set it is because A_n does not intersect $\bigcap_{m<s} A_{G(m)}$ for some s. It follows that $\langle A_{G(n)} : n \in \omega \rangle$ is a maximal subfamily of A with the F intersection property. □

Iterating and dovetailing this argument produces an ω-model witnessing:

Corollary 2.6. *Over* RCA_0*,* $F\mathsf{IP}$ *does not imply* ACA_0*.*

3. Connections with hyperimmunity

Corollary 2.6 naturally leads to the question whether $F\mathsf{IP}$ (or any one of the principles $\overline{D}_k\mathsf{IP}$) is provable in RCA_0, or at least in WKL_0. We show in this section that the answer to both questions is no. Recall that a set has *hyperimmune* degree if it computes a function not dominated by any computable function; otherwise, the degree of this set is *hyperimmune-free*. In this section, we prove the following result:

Theorem 3.1. *There is a computable nontrivial family of sets every maximal subfamily of which with the* $\overline{D}_2$ *intersection property has hyperimmune degree.*

As there is an ω-model of WKL_0 consisting entirely of sets of hyperimmune-free degree, this yields:

Corollary 3.2. *The principle* $\overline{D}_2\mathsf{IP}$ *is not provable in* WKL_0*.*

The proof of the theorem is motivated by the failure in general of being able to find computable solutions to computable instances of $\overline{D}_2\mathsf{IP}$. Indeed, suppose we have a family $A = \langle A_n : n \in \omega \rangle$, computable and nontrivial. The most direct way to build a maximal subfamily $B = \langle B_n : n \in \mathbb{N} \rangle$ with the $\overline{D}_2$ intersection property is as in Proposition 2.1 above. Of course, this subfamily need not be computable, but we could try to temper our strategy to make it be. An obvious such attempt is the following. We first search through the members of A in some effective fashion until we find the first that is nonempty, and we let this be B_0. Then, having defined $B_0, \ldots, B_{n-1}$ for some n, we search through A again until we find the first member not among the sets we have chosen already but intersecting each of them, and we let this be B_n. The resulting subfamily will indeed be computable and have the $\overline{D}_2$ intersection property. However, it need not be maximal. For example, suppose the first nonempty set we discover is A_1, so that we set $B_0 = A_1$. It may be that A_0 intersects A_1, but that we discover this only after discovering that A_2 intersects A_1 in our effective search, thereby setting $B_1 = A_2$. It may then be that A_0 also intersects A_2, but that we discover this only after discovering that A_3 intersects A_1 and A_2, setting $B_2 = A_3$. In this fashion, $A_1, A_2, A_3, \ldots$ could successively *prevent* A_0 from entering B at every step, and B would end up *missing* A_0 even though A_0 intersects every B_n.

Turning to the proof, we build a computable $A = \langle A_n : n \in \omega \rangle$ by stages. We define a sequence $P_e^0, P_e^1, \dots$ of members of A called *prevention sets*, playing the role $A_1, A_2, A_3, \dots$ did above. We differentiate indices e by also calling these *e-prevention sets*. For certain e, we additionally define a *missing set*, denoted M_e, which will play the role of A_0. In the example above, we needed to see A_n in the subfamily before we could see $A_1, \dots, A_{n-1}$ intersect A_0. In order to produce this behavior in A, every P_e^m will be a prevention set *for* some string $\sigma \in \omega^{<\omega}$, representing that if $A_{\sigma(0)}, \dots, A_{\sigma(|\sigma|-1)}$ appear in a given maximal subfamily then so must P_e^m.

As before, we initially put $2n$ into A_n for every n, and otherwise put in only odd numbers. Whenever we speak of making two members of A intersect we shall mean adding to both the least odd number that has not previously been put into any other set. We call a set *fresh* if it contains no odd numbers. Whenever we define a new e-prevention set in the course of the construction, we shall mean fixing the least m such that P_e^m is undefined, and letting P_e^m be A_n for the least n such that A_n is fresh. When we define M_e, we let it be some fresh A_n in the same sense.

Say $\sigma \in \omega^{<\omega}$ or $J \in \omega^\omega$ *enumerates* A_n if n belongs to its range. Say σ is *bounded by* $s \in \omega$ if $|\sigma| \leq s$, $\sigma(n) \leq s$ for all $n < |\sigma|$, and for all $n < m < |\sigma|$, some number $\leq s$ has been added to $A_{\sigma(n)} \cap A_{\sigma(m)}$ by stage s. Say σ *e-extends* a string τ if σ enumerates an e-prevention set for some ρ with $\tau \preceq \rho \prec \sigma$.

Proof of Theorem 3.1. We build A along with finitely-branching trees $T_0, T_1, \dots \subseteq \omega^{<\omega}$, and for every $J \in \omega^\omega$ a partial J-computable function f_J. We aim to satisfy the following requirements:

- $\mathcal{Q}$: if $J \in \omega^\omega$ defines a maximal subfamily of A with the $\overline{D}_2$ intersection property then f_J is total;
- $\mathcal{R}_e$: if $J \in \omega^\omega$ defines a subfamily of A with the $\overline{D}_2$ intersection property, and if f_J is total and bounded by φ_e, then $J \in [T_e]$;
- $\mathcal{S}_e$: no infinite path through T_e defines a maximal subfamily of A with the $\overline{D}_2$ intersection property.

Clearly, these suffice for proving the theorem.

To satisfy $\mathcal{S}_e$, we pursue a strategy with prevention sets and missing sets based on the example above. Satisfying $\mathcal{R}_e$ will guide the construction of T_e. At stage s, we build a finite approximation $T_e[s]$ to T_e consisting of certain strings bounded by s, with every leaf of $T_e[s]$ properly extending some leaf of $T_e[s-1]$. We let T_e consist of the intersection of the upward closure of the leaves of $T_e[s]$.

The function f_J is defined along with a sequence $r_{-1} < r_0 < \cdots$ of numbers. Let $r_{-1} = 0$. Having defined r_{i-1} for some i, let r_i be least such that for each $j \leq i$, $J \upharpoonright r_i$ j-extends $J \upharpoonright r_{i-1}$. Let $f_J(i)$ be the least number that bounds $J \upharpoonright r_i$. As we will define P_e^m for every m and e, and prevention sets will always be defined fresh, no set will be fresh forever. Thus, checking whether a given A_n is a prevention set will be computable, making f_J partial J-computable.

We effectively label every stage $s > 0$ as an *e-stage* for some e in such a way that there are infinitely many e-stages for every e. For every i, let $s_{e,i} = (\mu s)[\varphi_e(i)[s]\downarrow]$, with the convention that if $j < i$ and $s_{e,i}$ is defined then so is $s_{e,j}$ and $s_{e,j} < s_{e,i}$. We may further assume that every $s_{e,i}$ is an e-stage.

Construction. Initially, let $T_e[0] = \{\emptyset\}$ for all e, and let all prevention sets and missing sets be undefined. Next, assume we are at an e-stage $s > 0$. The construction is split into three steps.

Step 1: defining T_e. If $s \neq s_{e,i}$ for any i, let $T_e[s] = T_e[s-1]$. If $s = s_{e,i}$, let $T_e[s]$ be the downward closure of all σ bounded by s of minimal length for which there exists a leaf $\tau \in T_e[s-1]$ such that for each $j \leq i$, σ j-extends τ.

Step 2: dealing with e-prevention sets. Define a new e-prevention set for each σ bounded by s. Now consider each P_e^m defined at a stage before s, say as a prevention set for σ. For each $\tau \succeq \sigma$ bounded by s that only enumerates M_e if σ does, intersect P_e^m with every set enumerated by τ.

Step 3: dealing with M_e. If $s = s_{e,e}$, define M_e. If M_e is defined, check whether there is an r such that each leaf $\sigma \in T[s]$ e-extends $\sigma \upharpoonright r$ with witness P_e^m disjoint from M_e and not enumerated by any string in $T[s]$ of length r. If so, find the largest such r, and intersect all sets enumerated by the strings in $T[s]$ of length r with M_e.

End construction.

Verification. The family A is clearly computable and nontrivial. We now verify that the requirements have been satisfied.

Lemma 3.3. *Requirement $\mathcal{Q}$ is satisfied.*

Proof. Suppose $J \in \omega^\omega$ defines a maximal subfamily of A with the $\overline{D}_2$ intersection property. Let r_{i-1} be as in the definition of f_J, which is always defined at least for $i = 0$. We claim that r_i is defined, which implies that $f(i)$ is. It suffices to exhibit an initial segment of J that, for each $j \leq i$, j-extends $J \upharpoonright r_{i-1}$. The length of the shortest such string is by definition r_i. Let $\sigma_{-1} = J \upharpoonright r_{i-1}$, and suppose we have defined $\sigma_{j-1} \prec J$ for some $j \leq i$. Let τ be σ_{j-1} if M_j is not enumerated by J or if it is enumerated by σ_{j-1}, and let τ be an initial segment of J enumerating M_j otherwise. Then at step 2 of the first j-stage that bounds τ, some prevention set P_j^m for τ is defined, and is eventually intersected with every set enumerated by J. By maximality, let σ_j be an initial segment of J enumerating P_j^m, which j-extends τ and so also σ_{j-1}. Continuing, σ_i will be the desired initial segment of J. $\square$

Lemma 3.4. *For every e, requirement $\mathcal{R}_e$ is satisfied.*

Proof. Suppose $J \in \omega^{<\omega}$ defines a subfamily of A with the $\overline{D}_2$ intersection property and that f_J is total and bounded by φ_e. Let $r_0 < r_1 < \cdots$ be as in the definition of f_J. Then for every i, $J \upharpoonright r_i$ is bounded by $\varphi_e(i)$, and so, by usual conventions, also by $s_{e,i}$. By induction on i and by the definition of f_J, it follows that $J \upharpoonright r_i$ is a leaf of $T_e[s_{e,i}]$. Hence, $J \upharpoonright r_i \in T_e$ for all i, and $J \in [T_e]$. $\square$

Lemma 3.5. *For every e, requirement $\mathcal{S}_e$ is satisfied.*

Proof. Suppose $J \in [T_e]$. By construction, the subfamily of $\mathcal{A}$ defined by J has the $\overline{D}_2$ intersection property. We show that this subfamily does not contain M_e even though every set in it intersects M_e, and hence that it is not maximal.

We claim that for every $i \geq e$, each leaf σ of $T_e[s_{e,i}]$ enumerates a set that is not intersected with M_e at stage $s_{e,i}$, meaning σ cannot enumerate M_e. In fact, we claim that this set is an e-prevention set. We proceed by induction on i, assuming the claim for all $j < i$. If $i = e$, choose any witness P_e^m to σ e-extending a leaf of $T_e[s_{e,i} - 1]$, and note that σ is put into $T_e[s_{e,e}]$ before M_e is defined.

If $i > e$, let P_e^m witness that the claim holds for the leaf of $T_e[s_{e,i-1}]$ that σ extends. In either case, since σ is bounded, P_e^m is a prevention set for some string that does not enumerate M_e. Hence, P_e^m cannot be intersected with M_e before step 3 of stage $s_{e,i}$. But if P_e^m is so intersected at this step, then by construction σ enumerates another e-prevention set that is kept disjoint from M_e through the end of the stage. This proves the claim. Since, for every i, some initial segment of J is a leaf of $T_e[s_{e,i}]$, J cannot enumerate M_e.

Now fix r. We claim that every set enumerated by $J \upharpoonright r$ intersects M_e. Let $i \geq e$ be large enough that each leaf of $T_e[s_{e,i-1}]$ has length $\geq r$. For every $j \geq i$, each leaf of $T_e[s_{e,j}]$ e-extends a leaf of $T_e[s_{e,j-1}]$, which is in turn an extension of a leaf of $T_e[s_{e,i-1}]$. By an argument similar to that of the previous claim, we can consequently choose j such that each leaf of $T_e[s_{e,j}]$ enumerates a P_e^m that is not enumerated by any leaf of $T_e[s_{e,i-1}]$, and not intersected with M_e at stage $s_{e,j}$. Then at step 3 of this stage, the sets enumerated by the strings of length r in $T_e[s_{e,j}]$, and in particular those enumerated by $J \upharpoonright r$, are intersected with M_e. □

□

4. Relationships with other principles

By the preceding results, $F\mathsf{IP}$ and the principles $\overline{D}_k\mathsf{IP}$ are of the irregular variety that do not admit reversals to any of the main subsystems of Z_2. Many principles of this kind have been studied in the literature, and collectively they form a rich and complicated structure. (A partial summary is given by Hirschfeldt and Shore [5, p. 199], with additional discussions by Montalbán [9, Section 1] and Shore [13].) In this section, we show that the intersection principles lie near the bottom of this structure.

Theorem 3.1 gives us a lower bound on the strength of $\overline{D}_2\mathsf{IP}$. Examining the proof, we note that the construction there is computable, and it and the verification can be carried out using only Σ_1^0 induction. (See [14, Definition VII.1.4] for the formalizations of Turing reducibility and equivalence in RCA_0.) We thus obtain the following:

Corollary 4.1. *Over* RCA_0, $\overline{D}_2\mathsf{IP}$ *implies the principle* HYP*, which asserts that for every* S*, there is a set of degree hyperimmune relative to* S*.*

The reverse mathematical strength of HYP was examined by Hirschfeldt, Shore, and Slaman [6] in their investigation of certain model-theoretic principles related to the atomic model theorem (AMT). Specifically, they showed [6, Theorem 5.7] that HYP is equivalent to the omitting partial types principle (OPT), a weaker form of AMT asserting that every complete, consistent theory has a model omitting the nonprincipal members of a given set of partial types. (See [6, pp. 5808, 5831] for complete definitions, and [14, Section II.8] for a general development of model theory in RCA_0.)

Thus, Corollary 4.1 provides a connection between model-theoretic principles on the one hand, and set-theoretic principles, namely the intersection principles, on the other. We can extend this to an even firmer relationship. The following principle was introduced by Hirschfeldt, Shore, and Slaman [6, p. 5823]. They showed that it strictly implies AMT over RCA_0, but that AMT implies it over $\mathsf{RCA}_0 + \mathsf{I}\Sigma_2^0$ [6, Theorem 4.3, Corollary 4.5, and p. 5826].

Π^0_1 genericity principle ($\Pi^0_1\mathsf{G}$)**.** *For any uniformly Π^0_1-definable collection of sets U_n, each of which is dense in $2^{<\mathbb{N}}$, there is a set G that meets every U_n (i.e., $G \upharpoonright r \in U_n$ for some r).*

Proposition 4.2. $\Pi^0_1\mathsf{G}$ *implies* $F\mathsf{IP}$ *over* RCA_0.

Proof. Let a nontrivial family $A = \langle A_n : n \in \mathbb{N}\rangle$ be given. Consider the notion of forcing $\mathbb{P}_A$ defined in Proposition 2.5. The set of conditions for this forcing exists by Δ^0_1 comprehension, and RCA_0 suffices to show there is an order-preserving map f from $2^{<\mathbb{N}}$ into this set with an upward dense image. (For example, define f by recursion. Let $f(\emptyset) = (\emptyset, 0)$, and assume we have defined $f(\rho) = (\sigma, s)$. Fix $i \in \{0, 1\}$. If $i = 1$ and, under some fixed identification of $\mathbb{P}_A$ with $\mathbb{N}$, $|\rho| = (\tau, t) \leq (\sigma, t)$, then let $f(\rho i) = (\tau, t)$. Otherwise, let $f(\rho i) = (\sigma, s)$.)

For every n, let U_n be the preimage under f of the set of all conditions (σ, s) such that either σ has n in its range, or A_n does not intersect $\bigcap_{m<|\sigma|} A_{\sigma(m)}$. The U_n are then uniformly Π^0_1-definable and dense, so by $\Pi^0_1\mathsf{G}$, we may fix a set G that meets each of them. For every n, let $(\sigma_n, s_n) = f(G \upharpoonright n)$, noting that the sequence $\sigma_0, \sigma_1, \ldots$ is increasing since f is order-preserving. Letting $J = \bigcup_n \sigma_n$, we see that $B = \langle A_{J(n)} : n \in \mathbb{N}\rangle$ is a maximal subfamily of A with the F intersection property. □

By Corollary 3.9 of [6], there is an ω-model of AMT, and hence of $\Pi^0_1\mathsf{G}$, that is not a model of WKL_0. Hence, $F\mathsf{IP}$ does not imply WKL_0, and so in view of Corollary 3.2 the two are incomparable. $F\mathsf{IP}$ also inherits from $\Pi^0_1\mathsf{G}$ conservativity for restricted Π^1_2 sentences, i.e., those of the form $(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$, where Φ is arithmetical and Ψ is Σ^0_3. (This fact can also be established directly, by replacing the forcing notion in the proofs of Proposition 3.14 and Corollary 3.15 of [6] by $\mathbb{P}_A$.) It follows, for example, that $F\mathsf{IP}$ does not imply any of the combinatorial principles related to Ramsey's theorem for pairs studied by Cholak, Jockusch, and Slaman [1] or Hirschfeldt and Shore [5].

We do not know whether the preceding proposition can be strengthened to show that $F\mathsf{IP}$ follows from AMT over RCA_0. We also do not know whether HYP implies $F\mathsf{IP}$, although the following proposition and theorem provide partial steps in this direction.

Proposition 4.3. *Let $A = \langle A_n : n \in \mathbb{N}\rangle$ be a computable nontrivial family of sets. Every set S of degree hyperimmune relative to $\mathbf{0}'$ computes a maximal subfamily of A with the F intersection property.*

Proof. We may assume that A has no finite maximal subfamily with the F intersection property and that $A_0 \neq \emptyset$. Fix an S-computable function f not dominated by any $\emptyset'$-computable one, and define a function $J \leq_T f$ inductively as follows. Let $J(0) = 0$, and suppose we have defined $J \upharpoonright s$ for some $s > 0$. If there exists an $n \leq s$ not yet in the range of J such that $A_n \cap \bigcap_{m<s} A_{J(m)}$ contains an element $\leq f(s)$, let $J(s) = n$. Otherwise, let $J(s) = 0$. We then have that $J(s) \leq s$ for all s, and the subfamily B of A defined by J has the F intersection property.

We claim that if n intersects $\bigcap_{m<s} A_{J(m)}$ for all s then n is in the range of J, and hence that B is maximal. To see this, fix n and assume the claim for all numbers $< n$. Consider the $\emptyset'$-computable function g where $g(s)$ is least such that for all finite sets $F \subseteq \{0, \ldots, s\}$, if $\bigcap_{m \in F} A_m$ is nonempty then it contains an element $\leq g(s)$. Then f cannot be dominated by g, so we can find an $s \geq n$ such that $f(s) > g(s)$ and, by choice of n, large enough that every number $< n$ in the range of J is in the range of $J \restriction s$. Now $A_n \cap \bigcap_{m<s} A_{J(m)}$ is nonempty, so $g(s)$, and hence $f(s)$, bounds an element of this intersection. Thus, if n is not in the range of $J \restriction s$, then $J(s) = n$. □

Theorem 4.4. *Let $A = \langle A_n : n \in \mathbb{N}\rangle$ be a computable nontrivial family of sets. Every noncomputable c.e. set W computes a maximal subfamily of A with the F intersection property.*

Proof. As usual, assume A has no finite maximal subfamily with the F intersection property. We build a Turing reduction Φ such that Φ^W defines a maximal subfamily of A with the F intersection property. For convenience, we regard Φ as a monotone partial computable map $\omega^{<\omega} \to \omega^{<\omega}$ with domain closed under initial segment. We write Φ^σ in place of $\Phi(\sigma)$, and initially define $\Phi^\emptyset = \emptyset$. Thus, if we define $\Phi^\sigma = \tau$ we mean that $\Phi^\sigma(n) = \tau(n)$ for all $n < |\tau|$ with use bounded by $|\sigma|$.

Let $\langle W[s] : s \in \omega\rangle$ be a computable enumeration of W, viewed as a sequence of increasing strings in $\omega^{<\omega}$. Without loss of generality, $W[s] \neq W[s-1]$ for all $s > 0$, with r_s denoting the least r such that $W[s] \restriction r \neq W[s-1] \restriction r$. At stage $s > 0$, let $\sigma_s \in \omega^{<\omega}$ be the longest initial segment of $W[s] \restriction r_s$ in the domain of Φ. Choose the least $n \leq s$ not in the range of Φ^{σ_s}, if it exists, such that some number $\leq s$ belongs to $A_n \cap \bigcap_{m<|\Phi^{\sigma_s}|} A_{\Phi^{\sigma_s}(m)}$. Then let τ be the least initial segment of $W[s]$ properly extending σ_s, and define $\Phi^\tau = \Phi^{\sigma_s} n$.

Clearly, Φ is a reduction, and $\bigcap_{n<|\Phi^\sigma|} A_{\Phi^\sigma(n)} \neq \emptyset$ for all σ in its domain. It is also not difficult to verify that every initial segment of W is in the domain of Φ, and hence that Φ^W is total. Thus, the subfamily of A defined by Φ^W has the F intersection property, and we claim that it is also maximal. To see this, fix n such that A_n intersects $\bigcap_{m<s} A_{\Phi^W(m)}$ for all s. We show that if n is not in the range of Φ^W then W is computable. Let r be such that the ranges of Φ^W and $\Phi^{W \restriction r}$ agree below n, and let s_0 be a stage by which $W \restriction r$ is in the domain of Φ. Given any k, we can effectively find $s > s_0$ such that $W \restriction r \preceq \sigma_s \restriction k$ and some number $\leq s$ belongs to $A_n \cap \bigcap_{m<k} \Phi^{\sigma_s}(m)$. But now $\sigma_s \restriction k = W[s] \restriction k$ must equal $W \restriction k$, else n will end up in the range of Φ^W. □

A first attempt at showing that HYP implies FIP might be the following. Given a family $A = \langle A_n : n \in \omega\rangle$ and function f that is not computably dominated, define the subfamily by putting A_n in at stage s if n is least such that $f(s)$ bounds a witness for the intersection of A_n with all sets put in so far. Then, for every n, define a function g_n by letting $g_n(s)$ be the least such witness for A_n. Now if A_n intersects all members of our subfamily, g_n must be total, and so A_n must eventually be put in provided there are infinitely many s such that $g_n(s) \leq f(s)$. Such would be the case if g_n was computable, but in general it needs only to be computable in our approximation to the subfamily.

Our final result shows that FIP does not imply Π^0_1G, or even AMT.

Corollary 4.5. *Over* RCA_0*, FIP does not imply* AMT.

Proof. Csima, Hirschfeldt, Knight, and Soare [2, Theorem 1.5] showed that for every set $D \leq_T \emptyset'$, if every complete atomic decidable theory has an atomic model computable from D, then D is nonlow$_2$ (i.e., $D'' \not\leq_T \emptyset''$). Thus AMT cannot hold in any ω-model all of whose sets have degree below a fixed low$_2$ Δ^0_2 degree. By contrast, using Theorem 4.4, we can build such a model of FIP. For example, take

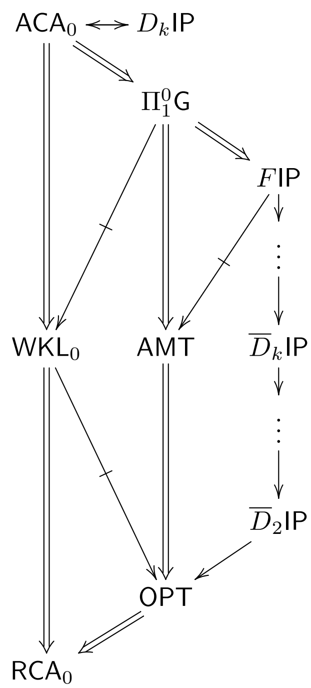

FIGURE 1. Locations of the intersection principles below ACA_0, with $k \geq 2$ arbitrary. Arrows denote implications in RCA_0, double arrows strict implications, and negated arrows nonimplications.

any sequence $\emptyset <_T V_0 <_T V_1 <_T \cdots <_T W$ of low$_2$ c.e. sets, and take the Turing ideal generated by the V_n. □

Our results are summarized in Figure 1. We conclude by stating the questions left open by our investigation. We conjecture the answer to part (3) to be no.

Question 4.6.

(1) For any $k \in \omega$, does $\overline{D}_k\mathsf{IP}$ imply $F\mathsf{IP}$, or at least $\overline{D}_{k+1}\mathsf{IP}$, over RCA_0?
(2) Does AMT imply $\overline{D}_2\mathsf{IP}$?
(3) Does every computable nontrivial family of sets have a maximal subfamily with the F intersection property computable in a given set of hyperimmune degree? A relativizable affirmative answer would show that OPT implies $\overline{D}_2\mathsf{IP}$.

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