# INFINITE SATURATED ORDERS

DAMIR D. DZHAFAROV

Abstract. We generalize the notion of saturated orders to infinite partial orders and give both a set-theoretic and an algebraic characterization of such orders. We then study the proof theoretic strength of the equivalence of these characterizations in the context of reverse mathematics, showing that depending on one's choice of definitions, this equivalence is either provable in $\mathsf{RCA}_0$ or equivalent to $\mathsf{ACA}_0$.

## 1. Introduction

Saturated orders (see Definition 1.2) were introduced by Suck in [7] as a generalization of interval orders (see Definition 1.1). The latter, developed by Fishburn (see [3]), have been used extensively in the theory of measurement, utility theory, and various areas of psychophysics and mathematical psychology (see [3], Chapter 2, for examples). Suck applied the concept of saturated orders to the theory of knowledge spaces, as introduced by Doignon and Falmagne (see [1]), but he formulated it for finite orders only. Since the study of knowledge spaces in general need not be restricted to finite structures, it is natural to ask whether the concept of saturation can be formulated for partial orders of arbitrary cardinality.

In this note, we give such a formulation and show it to be equivalent to a certain algebraic characterization of partial orders. We then look at the proof theoretic strength of this equivalence within the framework of reverse mathematics. This answers questions of Suck raised at the *Reverse Mathematics: Foundations and Applications* workshop at the University of Chicago in November 2009. Beyond an interest in the underlying combinatorial principles, the motivation for this kind of analysis comes from seeking a possible new basis by which to judge and compare competing quantitative approaches to problems in cognitive science. The exploration of this interaction was one of the goals of the Chicago workshop.

**Definition 1.1.** Let $\mathbf{P} = (P, \leq_P)$ be a partial order.

(1) An *interval representation* of $\mathbf{P}$ is a map $f$ from $P$ into the set of open intervals of some linear order $\mathbf{L} = (L, \leq_L)$ such that for all $p, p' \in P$, $p <_P p'$ if and only if $\ell <_L \ell'$ for all $\ell \in f(p)$ and $\ell' \in f(p')$;
(2) $\mathbf{P}$ is an *interval order* if it admits an interval representation.

**Definition 1.2** ([7], Definitions 1 and 3)**.** Let $\mathbf{P} = (P, \leq_P)$ be a finite partial order.

(1) A *set representation* of $\mathbf{P}$ is an injective map $\varphi : P \to \mathcal{P}(Q)$ for some set $Q$ such that $p <_P p'$ if and only if $\varphi(p) \subsetneq \varphi(p')$ for all $p, p' \in P$.

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(2) A set representation $\varphi$ of $\mathbf{P}$ is *parsimonious* if $|\varphi(p)| = \left|\bigcup_{p' <_P p} \varphi(p')\right| + 1$ for all $p \in P$.
(3) $\mathbf{P}$ is *saturated* if $|P| = \left|\bigcup_{p \in P} \varphi(p)\right|$ for all parsimonious set representations $\varphi$ of $\mathbf{P}$.

Every finite partial order $\mathbf{P} = (P, \leq_P)$ admits at least one parsimonious set representation, namely $\pi : P \to \mathcal{P}(P)$ where $\pi(p) = \{p' \in P : p' \leq_P p\}$ for all $p \in P$. Suck [7, Definition 2] calls this the *principal ideal representation* of $\mathbf{P}$. The notion of saturation arose as a means of characterizing finite orders for which this is essentially the only parsimonious set representation ([7], p. 375). Indeed, suppose $\varphi : P \to \mathcal{P}(Q)$ is parsimonious, and let $\alpha_\varphi : P \to Q$ be defined by setting $\alpha_\varphi(p)$ for each $p \in P$ to be the single element of $\varphi(p) - \bigcup_{p' <_P p} \varphi(p')$. If $\mathbf{P}$ is saturated then $\alpha_\varphi$ must be a bijection between $P$ and $\bigcup_{p \in P} \varphi(p)$. Let $\leq_Q$ be an ordering of the latter set defined by setting $q \leq_Q q'$ for each $q, q' \in \bigcup_{p \in P} \varphi(p)$ if and only if $q = \alpha_\varphi(p)$ and $q' = \alpha_\varphi(p')$ for some $p, p' \in P$ with $p \leq_P p'$. Then $\alpha_\varphi$ is an isomorphism of $\mathbf{P}$ with $(\bigcup_{p \in P} \varphi(p), \leq_Q)$, and $\varphi(p) = \alpha_\varphi(\pi(p))$ for all $p \in P$. Thus, up to a renaming of elements, $\varphi$ and $\pi$ are the same representation.

A simple example of an order that is not saturated is one of type $\mathbf{2} \oplus \mathbf{2}$, i.e., one isomorphic to $(\{a, b, c, d\}, \leq)$ where $a \leq b$, $c \leq d$, $a \not\leq d$ and $c \not\leq b$ (consider the parsimonious set representation that maps $a$ and $c$ to $\{0\}$ and $\{1\}$ respectively, and $b$ and $d$ to $\{0, 2\}$ and $\{1, 2\}$ respectively). This order is also not an interval order:

**Theorem 1.3** (Fishburn [2], p. 147; Mirkin [5]). *A partial order is an interval order if and only if it does not contain a suborder of type* $\mathbf{2} \oplus \mathbf{2}$.

Suck [7, Theorem 2] extended this observation by showing that every finite interval order is a saturated order. The converse, however, fails, as it is easy to build a saturated order which admits a suborder of type $\mathbf{2} \oplus \mathbf{2}$. (See Figure 1, which also illustrates that saturated orders are not, contrary to interval orders, closed under restrictions of the domain).

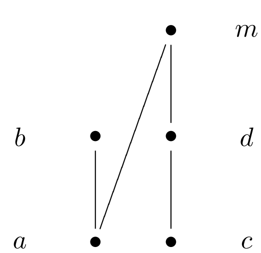


FIGURE 1.

If one recasts the condition of not containing a suborder of type $\mathbf{2} \oplus \mathbf{2}$ as

$$(\forall p_a, p_b, p_c, p_d \in P)[p_a \not<_P p_b \vee p_c \not<_P p_d \vee p_a \leq_P p_d \vee p_c \leq_P p_b],$$

then the following definition and theorem can be seen as providing an algebraic characterization of saturated orders similar to the characterization of interval orders provided by Theorem 1.3.

**Definition 1.4** ([8], Definitions 5 and 6). Let $\mathbf{P} = (P, \leq_P)$ be a finite partial order.

(1) A *fan* in $\mathbf{P}$ is a subset $F$ of $P$ with at least two elements such that $\max F$ exists under $\leq_P$ and such that no elements of $F - \{\max F\}$ are pairwise $\leq_P$-comparable.
(2) Two fans $F_0$ and $F_1$ in $\mathbf{P}$ are *parallel* if no element of $F_0$ is $\leq_P$-comparable with any element of $F_1$.
(3) Two parallel fans $F_0$ and $F_1$ in $\mathbf{P}$ are *skewly topped* if there exists some $m \in P$ and some $i \in \{0, 1\}$ such that
 (a) $m \geq_P \max F_i$,
 (b) $m \not\geq_P \max F_{1-i}$,
 (c) and $m \geq_P p$ for all $p \in F_{1-i} - \{\max F_{1-i}\}$.

**Theorem 1.5** (Suck [8], Theorem 5)**.** *A finite partial order is saturated if and only if every two parallel fans in it are skewly topped.*

We can now state the questions of Suck mentioned above.

**Question 1.6** (Suck)**.**
(1) Does (some suitable analog of) Theorem 1.5 hold for infinite partial orders?
(2) If so, what are the set theoretic axioms necessary to carry out its proof?

The second part of the question is inspired by the work of Marcone [4], who investigated the reverse mathematical content of Theorem 1.3. We refer the reader to Section 3 for a brief introduction to reverse mathematics, and Simpson [6] for a complete reference. In the next section we give an affirmative answer to part (1) of Question 1.6, and in Section 3 we consider possible answers to part (2).

## 2. Infinite saturated orders

In this section we formulate the concept of saturation for infinite partial orders and prove an analog of Theorem 1.5. To begin, notice that set representations can be defined for infinite orders just as for finite ones. The other parts of Definition 1.2, however, need to be appropriately adjusted to the infinite setting.

**Definition 2.1.** Let $\mathbf{P} = (P, \leq_P)$ be a partial order.
(1) A set representation $\varphi : P \to \mathcal{P}(Q)$ of $\mathbf{P}$ is *parsimonious* if for all $p \in P$
 (a) $\left|\varphi(p) - \bigcup_{p' <_P p} \varphi(p')\right| = 1$,
 (b) and for all $q \in \varphi(p)$, $\{q\} = \varphi(p') - \bigcup_{p'' <_P p'} \varphi(p'')$ for some $p' \leq_P p$.
(2) Given a parsimonious set representation $\varphi : P \to \mathcal{P}(Q)$ of $\mathbf{P}$, define $\alpha_\varphi : P \to Q$ by $\alpha_\varphi(p) = q$ for $p \in P$ if and only if $\{q\} = \varphi(p) - \bigcup_{p' <_P p} \varphi(p')$.
(3) $\mathbf{P}$ is *saturated* if and only if $\alpha_\varphi$ is injective for all parsimonious set representations $\varphi$ of $\mathbf{P}$.

Part 1 (a) above is a straightforward modification of Definition 1.2 (2). Part 1 (b) is intended to express the idea that for a set representation $\varphi : P \to \mathcal{P}(Q)$ to be parsimonious, $\bigcup_{p \in P} \varphi(p)$ should comprise a minimal number of elements from $Q$.

It is not difficult to check that for finite partial orders the new definitions agree with the old:

**Proposition 2.2.** *Let $\mathbf{P} = (P, \leq_P)$ be a finite partial order.*
(1) *A set representation of $\mathbf{P}$ is parsimonious according to Definition 1.2 if and only if it is parsimonious according to Definition 2.1.*

(2) **P** *is saturated according to Definition 1.2 if and only if it is saturated according to Definition 2.1.*

In particular, the discussion following Definition 1.2 holds verbatim for infinite partial orders if parsimony and saturation are understood according to Definition 2.1. Thus infinite saturated orders admit essentially only one parsimonious set representation, and so the preceding definition does indeed capture the "spirit" of the concept.

We next generalize the notion of fans from Definition 1.4; we shall see at the end of the section why fans alone would not suffice.

**Definition 2.3.** A *bouquet* in **P** is a subset $B$ of $P$ with at least two elements such that $\max B$ exists under $\leq_P$. We define what it means for two bouquets to be *parallel* and *skewly topped* just as for fans.

If **P** is finite, or even just a partial order in which every element has only finitely many $\leq_P$-successors, then every two parallel bouquets $B_0$ and $B_1$ can be replaced by parallel fans $F_0$ and $F_1$ with the same respective maxima. Namely, let

$$F_i = \{b \in B_i : (\forall b' >_P b)[b' \in B_i \implies b' = \max B_i]\}$$

for each $i$. Then an element of $P$ skewly tops $B_0$ and $B_1$ if and only if it skewly tops $F_0$ and $F_1$, and conversely. Thus we have:

**Proposition 2.4.** *If* $\mathbf{P} = (P, \leq_P)$ *is a finite partial order, then every two parallel fans in* **P** *are skewly topped if and only if every two parallel bouquets in* **P** *are skewly topped.*

The following is the analog of Theorem 1.5 for infinite partial orders. Along with the preceding two propositions, it also gives an alternative proof of Theorem 1.5, Suck's original one having been by induction on the size of the partial order.

**Theorem 2.5.** *A partial order is saturated if and only if every two parallel bouquets in it are skewly topped.*

*Proof.* ($\Longrightarrow$) Suppose $B_0$ and $B_1$ are two parallel bouquets in **P** that are not skewly topped. Let $q^*$ be a symbol not in $P$, and let $Q = P \cup \{q^*\} - \{\max B_0, \max B_1\}$. Let $\pi$ be the principal ideal representation of **P**, and define $\varphi : P \to \mathcal{P}(Q)$ as follows. If $p \geq_P \max B_0$ or $p \geq_P \max B_1$ let $\varphi(p) = \pi(p) \cup \{q^*\} - \{\max B_0, \max B_1\}$, and otherwise let $\varphi(p) = \pi(p)$. We claim, first of all, that $\varphi$ is a set representation. So fix distinct $p_0, p_1 \in P$ and note that if $p_0 \not\geq_P \max B_0, \max B_1$ and $p_1 \not\geq_P \max B_0, \max B_1$ then $\varphi(p_i) = \pi(p_i)$ for each $i$, meaning $\varphi(p_0) \neq \varphi(p_1)$ and $p_i <_P p_{1-i}$ if and only if $\varphi(p_i) \subsetneq \varphi(p_{1-i})$. This leaves the following cases to consider.

*Case 1: for some* $i, j \in \{0, 1\}$,

- $p_i \geq_P \max B_j$,
- $p_{1-i} \not\geq_P \max B_0, \max B_1$.

Clearly $\varphi(p_0) \neq \varphi(p_1)$ since $q^* \in \varphi(p_i)$ and $q^* \notin \varphi(p_{1-i})$. If $p_0$ and $p_1$ are $\leq_P$-comparable, it must be that $p_{1-i} <_P p_i$, so $\varphi(p_{1-i}) = \pi(p_{1-i}) \subsetneq \pi(p_i)$. And since $p_{1-i} \not\geq \max B_0, \max B_1$ we have $\max B_0, \max B_1 \notin \pi(p_{1-i})$, implying that $\varphi(p_{1-i}) \subseteq \pi(p_i) - \{\max B_0, \max B_1\} \subsetneq \varphi(p_i)$. Conversely, if $\varphi(p_0)$ and $\varphi(p_1)$ are comparable under inclusion, it must be that $\varphi(p_{1-i}) \subsetneq \varphi(p_i)$. Thus $\pi(p_{1-i}) \subseteq \varphi(p_i) - \{q^*\} \subseteq \pi(p_i)$. However, it cannot be that $\pi(p_{1-i}) = \pi(p_i)$ since this would mean that $\max B_j \leq_P p_{1-i}$, so we must have $\pi(p_{1-i}) \subsetneq \pi(p_i)$ and hence $p_{1-i} <_P p_i$.

*Case 2: for some $i, j \in \{0,1\}$,*

- $p_i \geq_P \max B_j$,
- $p_i \not\geq_P \max B_{1-j}$,
- $p_{1-i} \geq_P \max B_{1-j}$,
- $p_{1-i} \not\geq_P \max B_j$.

In this case we clearly cannot have $p_{1-i} <_P p_i$ or $p_i <_P p_{1-i}$. We show that neither $\varphi(p_{1-i}) \subseteq \varphi(p_i)$ nor $\varphi(p_i) \subseteq \varphi(p_{1-i})$ can happen. Indeed, suppose it were the case that $\varphi(p_{1-i}) \subseteq \varphi(p_i)$ (the other case being symmetric). Then every $p \in B_{1-j} - \{\max B_{1-j}\}$, being an element of $\pi(p_{1-i})$, would belong to $\varphi(p_i)$ and, not being $q^*$, also to $\pi(p_i)$. Thus, we would have $p \leq_P p_i$, so $p_i$ would skewly top $B_0$ and $B_1$, a contradiction.

*Case 3: for some $j \in \{0,1\}$, $p_0, p_1 \geq_P \max B_j$.* Since $p_0$ and $p_1$ are distinct, we must have $p_i >_P \max B_j$ for some $i \in \{0,1\}$. Since $\max B_0$ and $\max B_1$ are $\leq_P$-incomparable, this means that $p_i \in \varphi(p_i)$. So if $p_i \notin \varphi(p_{1-i})$ then $\varphi(p_i) \neq \varphi(p_{1-i})$. And if $p_i \in \varphi(p_{1-i})$ then $p_i <_P p_{1-i}$ and hence $p_{1-i} \in \varphi(p_{1-i}) - \varphi(p_i)$, so again $\varphi(p_i) \neq \varphi(p_{1-i})$. Now suppose $p_{1-i} <_P p_i$ for some $i$, so that $\pi(p_{1-i}) \subsetneq \pi(p_i)$. Then as $\varphi(p_0) = \pi(p_0) \cup \{q^*\} - \{\max B_0, \max B_1\}$ and $\varphi(p_1) = \pi(p_1) \cup \{q^*\} - \{\max B_0, \max B_1\}$, we have $\varphi(p_{1-i}) \subseteq \varphi(p_i)$ and hence $\varphi(p_{1-i}) \subsetneq \varphi(p_i)$ since $\varphi(p_{1-i}) \neq \varphi(p_i)$. Conversely, suppose $\varphi(p_{1-i}) \subsetneq \varphi(p_i)$. The only way it could fail to be the case that $\pi(p_{1-i}) \subsetneq \pi(p_i)$ is if $\max B_{1-j} \in \pi(p_{1-i}) - \pi(p_i)$. But every $p <_P \max B_{1-j}$ belongs to $\varphi(p_{1-i}) - \{q^*\}$ and hence to $\varphi(p_i) - \{q^*\} \subseteq \pi(p_i)$, meaning $p \leq_P p_i$. So, in this case, $p_i$ would skewly top $B_0$ and $B_1$. It must thus be that $\pi(p_{1-i}) \subsetneq \pi(p_i)$ and hence that $p_{1-i} <_P p_i$, as desired.

Our next claim is that $\varphi$ is parsimonious. Fixing $p \in P$, we first verify condition (1a) of Definition 2.1. If $p \not\geq_P \max B_0, \max B_1$, then there is nothing to show since $\varphi(p) = \pi(p)$. If $p = \max B_j$ for some $j \in \{0,1\}$, then $\varphi(p) = \bigcup_{p' <_P p} \varphi(p') \cup \{q^*\}$ and $q^* \notin \varphi(p')$ for any $p' <_P p$ since necessarily $p' \not\geq_P \max B_0, \max B_1$. If $p >_P \max B_j$ for some $j$, then $\varphi(p) = \bigcup_{p' <_P p} \varphi(p') \cup \{p\}$. In this case, $p \notin \varphi(p')$ for any $p' <_P p$ since as $p \neq q^*$ this would mean that $p \in \pi(p')$ and hence that $p \leq_P p' <_P p$. In any case, then, $\left|\varphi(p) - \bigcup_{p' <_P p} \varphi(p')\right| = 1$.

We now verify condition (1b) of Definition 2.1. Given $q \in \varphi(p)$, we either have that $q = q^*$ and $\max B_j \leq_P p$ for some $j \in \{0,1\}$, or that $q \in P$ and $q \leq_P p$. If we apply the argument just given to $q$ instead of to $p$ then it follows that in the former case $\{q\} = \varphi(\max B_j) - \bigcup_{p' <_P \max B_j} \varphi(p')$, and that in the latter case $\{q\} = \varphi(q) - \bigcup_{p' <_P q} \varphi(p')$.

Finally, it follows that $\mathbf{P}$ is not saturated. Indeed, as the preceding argument shows, $\alpha_\varphi(\max B_0) = q^* = \alpha_\varphi(\max B_1)$. Hence, $\alpha_\varphi$ is not injective.

($\Longleftarrow$) Fix a partial order $\mathbf{P} = (P, \leq_P)$. Fix a parsimonious set representation $\varphi : P \to \mathcal{P}(Q)$ and suppose that $\alpha_\varphi$ is not injective, so that $\alpha_\varphi(p_0) = \alpha_\varphi(p_1)$ for some distinct $p_0, p_1 \in P$. Then by definition of $\alpha_\varphi$, it follows that $p_0$ and $p_1$ are $\leq_P$-incomparable and not minimal in $P$. For $i \in \{0,1\}$, let $I_i$ be the set of all $p <_P p_i$ in $P$ which are $\leq_P$-incomparable with $p_{1-i}$, and let $C_i$ consist of all $p <_P p_i$ in $P$ which are $\leq_P$-comparable with $p_{1-i}$. Note that necessarily $p <_P p_{1-i}$ for all $p \in C_i$. This implies that each $I_i$ must be nonempty as otherwise we would have $\varphi(p) \subsetneq \varphi(p_{1-i})$ for all $p <_P p_i$ by virtue of $\varphi$ being a set representation, which would mean that $\varphi(p_i) \subseteq \varphi(p_{i-1})$, and hence that $p_i \leq_P p_{1-i}$.

Thus, $I_0 \cup \{p_0\}$ and $I_1 \cup \{p_1\}$ are parallel bouquets in $\mathbf{P}$ with $p_0$ and $p_1$ as their respective maxima. Towards a contradiction, suppose $m \in P$ skewly tops these bouquets, i.e., there is an $i \in \{0,1\}$ such that $p_i <_P m$, $p_{1-i} \not\leq_P m$, and $p <_P m$ for all $p <_P p_{1-i}$. Then $\alpha_\varphi(p_{1-i}) \in \varphi(p_i) \subsetneq \varphi(m)$ and $\varphi(p) \subsetneq \varphi(m)$ for all $p <_P p_{1-i}$ and thus

$$\varphi(p_{1-i}) = \{\alpha_\varphi(p_{1-i})\} \cup \bigcup_{p <_P p_{1-i}} \varphi(p) \subseteq \varphi(m),$$

which gives $p_{1-i} \leq_P m$, a contradiction. Thus, $I_0 \cup \{p_0\}$ and $I_1 \cup \{p_1\}$ are not skewly topped. □

The theorem shows why the move from fans in the finite case to bouquets in the infinite case was necessary. For consider the partial order $\mathbf{P}$ with domain

$$P = \{l_i : i \in \mathbb{N}\} \cup \{l\} \cup \{r_i : i \in \mathbb{N}\} \cup \{r\} \cup \{t_i : i \in \mathbb{N}\},$$

and ordering $\leq_P$ defined by (the transitive closure of) the following: for all $i <_\mathbb{N} j$,

- $l_i <_P l_j <_P l$,
- $r_i <_P r_j <_P r <_P t_i <_P t_j$,
- $l_i <_P t_i$.

(See Figure 2.) Then if $F_0$ and $F_1$ are parallel fans in $\mathbf{P}$, it must be that $|F_0| =$

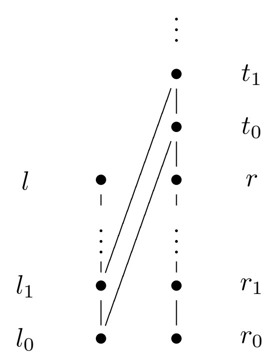


FIGURE 2.

$|F_1| = 2$, and that one of the two fans, say $F_0$, only contains elements $\leq_P$-incomparable with $r$, while the other only contains elements $\leq_P$-incomparable with $l$. Thus either $F_0 = \{l_i, l_j\}$ for some $i <_\mathbb{N} j$, or $F_0 = \{l_i, l\}$ for some $i$. In either case, $F_1$ must consist of some elements $<_P t_i$, and $t_i$ must consequently skewly top $F_0$ and $F_1$. On the other hand, $B_0 = \{l_0, l_1, \ldots\} \cup \{l\}$ and $B_1 = \{r_0, r_1, \ldots\} \cup \{r\}$ are parallel bouquets in $\mathbf{P}$ which are clearly not skewly topped by any element of $P$. By the theorem, $\mathbf{P}$ is not saturated.

## 3. Reverse mathematics

Reverse mathematics is an area of mathematical logic devoted to classifying mathematical theorems according to their proof theoretic strength. The goal is to calibrate this strength according to how much comprehension is needed to establish the existence of the sets needed to prove the theorem (i.e., according to how complicated the formulas specifying such sets must be allowed to be). This is a two-step process. The first involves searching for some weak comprehension scheme sufficient to prove the theorem, while the second gives sharpness by showing that the theorem is in fact equivalent to this comprehension scheme.

In practice, we use for these comprehension schemes certain subsystems of second-order arithmetic. As our base theory, we use a weak subsystem called $\mathsf{RCA}_0$, which roughly corresponds to computable or constructive mathematics. A strictly stronger system is $\mathsf{WKL}_0$, obtained by adding to the axioms of $\mathsf{RCA}_0$ the comprehension scheme asserting that every infinite binary tree has an infinite branch; stronger still is $\mathsf{ACA}_0$, which adds to $\mathsf{RCA}_0$ comprehension for sets described by arithmetical formulas (i.e., formulas whose quantifiers range over only number variables). Many theorems are known to be either provable in $\mathsf{RCA}_0$ or else equivalent over $\mathsf{RCA}_0$ to one of $\mathsf{WKL}_0$ or $\mathsf{ACA}_0$; see [6], Chapter 1 for a partial list of examples, and for an overview of other subsystems of second order arithmetic.

We turn to analyzing the proof theoretic strength of Theorem 2.5, assuming familiarity with the subsystems mentioned above. For interval orders, the equivalences between various set-theoretic and algebraic characterizations were studied in this context by Marcone [4]. For example, Theorem 1.3 is provable in $\mathsf{RCA}_0$ ([4], Theorems 2.13 and 4.2), but other characterizations of interval orders are equivalent to stronger subsystems (recall the notion of interval representation from Definition 1.1):

**Theorem 3.1** (Marcone [4], Theorem 5.6)**.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{WKL}_0$*;*
(2) *a partial order is an interval order if and only if it admits an interval representation that is injective.*

For our purposes, we begin by formalizing the concept of set representation in the language of second order arithmetic.

**Definition 3.2.** The following definitions are made in $\mathsf{RCA}_0$. Let $\mathbf{P} = (P, \leq_P)$ be a partial order. A *set representation* of $\mathbf{P}$ is a subset $\varphi$ of $P \times Q$ for some set $Q$ such that if we abbreviate $\{q \in Q : (p, q) \in \varphi\}$ by $\varphi(p)$, then for all $p, p' \in P$

(1) $p \neq p' \implies \varphi(p) \neq \varphi(p')$,
(2) and $p <_P p' \iff \varphi(p) \subsetneq \varphi(p')$.

Parsimony is then formalized in a straightforward way, along with all the combinatorial notions from Definitions 1.4 and 2.3. Formalizing saturation, on the other hand, presents us with two options (we deliberately use the same term for both):

**Definition 3.3.** The following definitions are made in $\mathsf{RCA}_0$. Let $\mathbf{P} = (P, \leq_P)$ be a partial order.

(1) $\mathbf{P}$ is *saturated* if for every parsimonious set representation $\varphi \subseteq P \times Q$ of $\mathbf{P}$, it holds that for all $p_0, p_1 \in P$ and all $q_0, q_1 \in Q$, if $p_0 \neq p_1$ and $\{q_i\} = \varphi(p_i) - \bigcup_{p' <_P p_i} \varphi(p')$ for each $i \in \{0, 1\}$, then $q_0 \neq q_1$.
(2) $\mathbf{P}$ is *saturated* if for every parsimonious set representation $\varphi \subseteq P \times Q$ of $\mathbf{P}$, the map $\alpha_\varphi : P \to Q$ exists and is injective.

Classically, the two definitions are, of course, equivalent. But in the present context they need not be because the existence of the map $\alpha_\varphi$ may not always be provable in $\mathsf{RCA}_0$. The following pair of propositions show that this can indeed happen. Thus, while formulating saturation according to Definition 3.3 (2) may be more natural, the set theoretic assumptions necessary to carry out the proof of Theorem 2.5 become much higher.

**Proposition 3.4.** *It is provable in* $\mathsf{RCA}_0$ *that a partial order is saturated according to Definition 3.3 (1) if and only if every two parallel bouquets in it are skewly topped.*

*Proof.* $\mathsf{RCA}_0$ suffices to carry out the left-to-right direction of the proof of Theorem 2.5. For the right-to-left direction, fix a partial order $\mathbf{P} = (P, \leq_P)$ and a parsimonious set representation $\varphi \subseteq P \times Q$. Suppose there exists $p_0 \neq p_1$ in $P$ such that $\varphi(p_0) - \bigcup_{p' <_P p_0} \varphi(p') = \varphi(p_1) - \bigcup_{p' <_P p_1} \varphi(p')$. Then we can argue as in the right-to-left direction of the proof of Theorem 2.5 that there exist parallel bouquets in $\mathbf{P}$ which are not skewly topped. □

**Proposition 3.5.** *Over* $\mathsf{RCA}_0$*, the following are equivalent:*

(1) $\mathsf{ACA}_0$*;*
(2) *for every parsimonious set representation $\varphi$ of a partial order, the map $\alpha_\varphi$ exists;*
(3) *a partial order is saturated according to Definition 3.3 (2) if and only if every two parallel bouquets in it are skewly topped;*
(4) *a partial order is saturated according to Definition 3.3 (1) if and only if it is saturated according to Definition 3.3 (2).*

*Proof.* For every parsimonious set representation $\varphi$ of a partial order $(P, \leq_P)$, we have that $\alpha_\varphi$ is arithmetically definable, so (1) implies (2). By Proposition 3.4 it follows that (2) implies (3), and obviously the equivalence of (1) and (3) implies the equivalence of (1) and (4).

It thus remains only to show that (3) implies (1). To this end, we prove from (3) that the range of every injective function $f : \mathbb{N} \to \mathbb{N}$ exists (this is equivalent to $\mathsf{ACA}_0$; see [6], Theorem III.1.3). So fix an injective function $f$ and define a partial order $\mathbf{P} = (P, \leq_P)$ as follows. Let $P = \{p_{i,s} : i, s \in \mathbb{N}\}$.

- For all $i <_{\mathbb{N}} j$, let $p_{i,s} >_P p_{j,t}$ for all $s, t \in \mathbb{N}$.
- For each $i$ and all $s <_{\mathbb{N}} t$, let $p_{i,s} <_P p_{i,t}$ if $s > 0$ and $f(t-2) = i$, and let $p_{i,s} >_P p_{i,t}$ otherwise.

In other words, fixing $i$, if $f(t) \neq i$ for all $t$, then we have $p_{i,s} >_P p_{i,t}$ for all $s <_{\mathbb{N}} t$; while if $f(t) = i$ for some $t$, then we have $p_{i,0} >_P p_{i,t+2} >_P p_{i,s} >_P p_{i,s'}$ for all $s <_{\mathbb{N}} s'$ in $\mathbb{N} - \{0, t+2\}$. $\mathsf{RCA}_0$ suffices to show that $\mathbf{P}$ exists, that it is a linear order, and that every element has an immediate $\leq_P$-predecessor. In particular, linearity implies that there are no parallel bouquets in $\mathbf{P}$, so $\mathbf{P}$ must be saturated according to Definition 3.3 (2) by part (3) above.

Define

$$\varphi = \{(p, p') \in P \times P : p >_P p' \wedge (\forall i \in \mathbb{N})[\, p' \neq p_{i,0}\,]\},$$

which exists by $\Sigma^0_0$ comprehension and is clearly a set representation of $\mathbf{P}$. If we let $p^-$ denote the immediate $\leq_P$-predecessor of each $p \in P$, then we see that $\{p^-\} = \varphi(p) - \bigcup_{p' <_P p} \varphi(p')$. Furthermore, if $q \in \varphi(p)$ for some $p = p_{i,s}$, then $p >_P q$ and $q = p_{j,t}$ for some $j \leq_{\mathbb{N}} i$ and $t >_{\mathbb{N}} 0$, so $q = p^-_{j,t'}$ for some $p_{j,t'} \leq_P p$. Thus, $\varphi$ is parsimonious.

Applying Definition 3.3 (2) to $\varphi$, it follows that $\alpha_\varphi : P \to P$ exists and is injective, and by the preceding discussion we have $\alpha_\varphi(p) = p^-$ for all $p$. Let $R = \{i \in \mathbb{N} : \alpha_\varphi(p_{i,0}) \neq p_{i,1}\}$, which exists by $\Sigma^0_0$ comprehension. Then by construction of $\leq_P$, we have that $i \in R$ if and only if $p^-_{i,0} = p_{i,t+2}$ for some $t$ such that $f(t) = i$, which in turn holds if and only if $i \in \operatorname{ran} f$. Hence, the range of $f$ is equal to $R$ and so consequently exists. This completes the proof. □

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University of Chicago
*E-mail address*: damir@math.uchicago.edu