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Мамсе, папе, and to Veronica.

ABSTRACT

We study the logical strength of various weak combinatorial principles, using the tools of reverse mathematics, computability theory, and effective measure theory. Our focus is on Ramsey's theorem, various equivalents of the axiom of choice, and theorems arising from problems in cognitive science. We obtain new results concerning the effective content of previously studied principles, and show how these relate to several new principles we introduce.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

Reverse mathematics is an area of mathematical logic and foundations of mathematics devoted to classifying mathematical theorems according to their logical strength. It may be considered a branch of proof theory, but has also attracted the attention of researchers in computability theory, philosophy of mathematics, and, more recently, even cognitive science. It is a common intuition among mathematicians that some theorems in disparate areas of mathematics are deeply interrelated, and that the same patterns of problem-solving crop up in solutions to otherwise seemingly unrelated problems. Reverse mathematics can be viewed as a formalization and confirmation of this intuition. Indeed, a striking fact repeatedly demonstrated by research in reverse mathematics is that a vast number of mathematical theorems can be classified into one of just five main types (as explained below). This points to a great deal of regularity underlying most mathematical arguments.

Some theorems, however, lack this regularity, and their logical strength is therefore more difficult to understand. Over the past decade, elucidating the strength of these *irregular* theorems has motivated a very active and fruitful research program in mathematical logic, with Ramsey's theorem playing a focal role. Ramsey's theorem is an important and in many ways surprising foundational result that, in broad terms, states that in any configuration or arrangement of objects, however complicated, some amount of order is necessary. Studying this order, and how it arises, has led to the discovery of a cornucopia of weak combinatorial principles of the irregular variety.

In this dissertation, we explore the logical strength of Ramsey's theorem, and of various related principles, using the tools of reverse mathematics and computability theory. This includes looking at, in Chapter 2, computability-thoeretic aspects of Ramsey's theorem, and the information content of solutions to computable instances of it; in Chapter 3, principles based on an analysis of typical and atypical properties of a form of Ramsey's theorem, using notions from effective measure theory; in Chapter 4, purely combinatorial variants of Ramsey's theorem, including analogues of it to other partial orders; and, in Chapter 5, assorted equivalents of the axiom of choice and their connections to weak versions of Ramsey's theorem. Additionally, in Chapter 6, we look at certain combinatorial principles arising from a problem in cognitive science. Although we encounter regular principles along the way, our work is ultimately aimed at understanding the strength of irregular principles, how they compare, and why their strength is so different from that of most other theorems. (Chapter 2 is based on [18], which is joint work with Carl G. Jockusch, Jr.; Chapter 3 is based on [14]; Chapter 4 is based on [16] and [17], joint work with Jeffry L. Hirst and Tamara J. Lakins; Chapter 5 is based on [19], joint work with Carl Mummert; and Chapter 6 is based on [15].)

We assume familiarity with basic computability theory and reverse mathematics, but for completeness, provide brief introductions to both below, also as a means of identifying our notation and conventions. The standard references on computability theory and reverse mathematics, to which the reader is referred for complete details, are Soare [60] and Simpson [59], respectively. We conclude with a brief survey of previous results concerning the logical strength of Ramsey's theorem.

1.1 Computability theory

The set of natural numbers, $\{0, 1, 2, \ldots\}$, we denote by ω. Without further qualification, a *set* will always refer a subset of ω, while the *complement* of a set will always refer to its complement with respect to ω. Informally, a set A is *computable* if there exists an effective procedure, or algorithm, to determine in finite time, for each $n \in \omega$, whether or not $n \in A$. For example, the set of prime numbers is computable, since we can determine whether or not $n \in \omega$ is prime by simply checking which $m < n$ divide n. (This example also illustrates that issues of speed or efficiency, however important for practical implementations, are not part of the definition of a computable set.) The concept of computability can be formalized by means of a number of formal models, most notably Turing machines, and these turn out to all be equivalent. (For a thorough discussion of models of computation, and their equivalences, see Odifreddi [51], Chapter I.) We fix such a formalization, and henceforth tacitly invoke the Church-Turing thesis to allow ourselves to use the formal and informal definitions interchangeably.

The principal theme of modern computability theory is *relative* computability: for sets A and B, we say A is *Turing reducible to* B (also, A is *computable from* B, or simply B-*computable*), written $A \leq_T B$, if there is an algorithm to decide which numbers belong to A using information about which numbers belong to B. We refer to B in such a computation as an *oracle*, and think of it intuitively as having greater information content than A. Naturally, we write $A <_T B$ if $A \leq_T B$ and $B \not\leq_T A$, and we obtain the following notion of equivalence: A and B are *Turing equivalent*, written $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$. By identifying functions with their graphs, we can extend these definitions from subsets of ω to functions $\omega \to \omega$, and even to functions on ω^n for $n > 1$. For example, we shall make frequent use of the fact that there is a computable bijection $\omega^2 \to \omega$; we denote the image of the pair (x, y) under some fixed such bijection by $\langle x, y \rangle$.

The equivalence class of a set under $\equiv_T$ is called its *Turing degree.* For Turing degrees $\mathbf{a}$ and $\mathbf{b}$, we write $\mathbf{a} \leq \mathbf{b}$ if $A \leq_T B$ for every $A \in \mathbf{a}$ and $B \in \mathbf{b}$. Under $\leq$, the Turing degrees form an upper semilattice: the least element is denoted $\mathbf{0}$, the degree of the computable sets; the join of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \cup \mathbf{b}$, is, for $A \in \mathbf{a}$, $B \in \mathbf{b}$, the degree of the *join* of A and B, i.e., $A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}$.

Since algorithms can be specified by finite sets of instructions, they can be effectively listed as $\Phi_0, \Phi_1, \ldots$. Formally, each Φ_e is a partial function $\omega \to \omega$, thereby allowing for the possibility that the algorithm does not necessarily halt on every input. For each set B, we can similarly effectively list all the algorithms that use B as an oracle, and we list these as $\Phi^B_0, \Phi^B_1, \ldots$. Thus, if $A \leq_T B$ then the characteristic function of A equals Φ^B_e for some e, which we call a $\Delta_1^{0,B}$ *index for A*. We call $\Phi_0^B, \Phi_1^B, \ldots$ the *partial B-computable functions*, and we call those that happen to be total simply the *B-computable functions.* (It is not difficult to see that this is consistent with the notion of B-computable function as defined in the preceding paragraph.) We identify Φ_e with $\Phi_e^\emptyset$ when convenient.

For each B, e, and x, $\Phi_{e,s}^B(x)$ denotes the result of running the algorithm Φ_e^B with input x for s steps, which may be undefined if the algorithm does not halt. In that case, we write $\Phi_{e,s}^B(x)\uparrow$. We say $\Phi_{e,s}^B$ *converges* to y, and write $\Phi_{e,s}^B(x)\downarrow= y$ if, in s or fewer steps, the algorithm halts, and outputs y. In this case, the *use* of this computation, denoted $\varphi_e^B(x)$, refers to the largest n whose membership in B is queried in the course of the algorithm's running. We write $\Phi_e^B(x)\downarrow= y$ if there is an s such that $\Phi_{e,s}^B(x)\downarrow= y$; $\Phi_e^B(x)\uparrow$ if for all s, $\Phi_{e,s}^B(x)\uparrow$.

Sets can be classified in numerous ways in terms of their structural properties. A set A is *computably enumerable* (c.e.) if it can be effectively enumerated, i.e., if it is the image of computable function $f:\omega\to\omega$. We call $\{A_s\}_{s\in\omega}$ where $A_s=\{f(i): i\le s\}$ a *computable enumeration* of A. The canonical example of a c.e. set is $\emptyset'$, the *halting set*, which is the collection of all $\langle e,x\rangle$ such that $\Phi_e(x)\downarrow$. All c.e. sets are $\emptyset'$-computable, and while the converse is not true, all $\emptyset'$-computable sets enjoy the following weaker structural characterization, the fact of which is known as the limit lemma: a set A is $\emptyset'$-computable if and only if there exists a computable function $g:\omega^2\to\omega$ such that for every x, $A(x) = \lim_s g(x,s)$. In this case, we call $\{A_s\}_{s\in\omega}$ where $A_s=\{x : g(x,s)=1\}$ a *computable approximation* to A.

Computable, computably enumerable, and $\emptyset'$-computable sets can also be characterized in terms of the arithmetical hierarchy: the computable sets are precisely those definable by both a Σ_1^0 and Π_1^0 formula of first-order arithmetic; the c.e. sets are those definable by a Σ_1^0 formula; and the $\emptyset'$-computable sets those definable by both a Σ_2^0 and Π_2^0 formula (for this reason, $\emptyset'$-computable sets are also called Δ_2^0 *sets*). All of these notions relativize to any set A. The halting problem relative to A, i.e., the set of $\langle e,x\rangle$ such that $\Phi_e^A(x)\downarrow$, is called the *jump* of A, and is denoted A'. (The jump of a set is always of degree strictly above that of the set itself.) We also write $A^{(1)}$ for A', and for $n\ge 1$, $A^{(n+1)}$ for $(A^{(n)})'$. These definitions extend to, and are well-defined on, degrees.

Finally, we recall the definitions of trees and Π_1^0 classes. Viewing ω as an ordinal, we recall that for $n\le\omega$, $n^{<\omega}$ denotes the set of all finite sequences of numbers $<n$, while n^ω is the set of all infinite such sequences. For $\sigma\in n^{<\omega}$, $|\sigma|$ denotes the *length* of σ as a finite sequence, and for $i<|\sigma|$, $\sigma(i)$ denotes the $(i+1)$st bit of σ. If also $\tau\in n^{<\omega}$, we write $\sigma\preceq\tau$ if $|\tau|\ge|\sigma|$ and for all $i<|\sigma|$, $\sigma(i)=\tau(i)$; we write $\sigma\prec\tau$ if $\sigma\preceq\tau$ and $\sigma\ne\tau$. (We similarly define $\sigma\prec f$ if $\sigma\in n^{<\omega}$ and $f\in n^\omega$.) A *tree* is a subset T of $n^{<\omega}$ such that for every σ and τ in $n^{<\omega}$, if $\sigma\preceq\tau$ and $\tau\in T$, then $\sigma\in T$. An *infinite path* through T is an $f\in n^\omega$ such that for all $i\in\omega$, $f\upharpoonright i\in T$ (here, $f\upharpoonright i$ denotes the restriction of f to $j<i$; more generally, given a subset A of ω, $A\upharpoonright i$ denotes the set of elements of A less than i). The set of all infinite paths through T is denoted $[T]$, and $\sigma\in T$ is called *extendible* if there is $f\in[T]$ with $\sigma\prec f$. *König's lemma* asserts that if T is an infinite tree and for each $\sigma\in T$ there are only finitely many $i<n$ such that $\sigma i\in T$, then $[T]\ne\emptyset$ (see the next section for an important variant of König's lemma used in reverse mathematics).

Our particular interest below will be in the case $n = 2$. It is not difficult to see that if $T \subseteq 2^{<\omega}$ is a computable tree, then $[T]$ forms an Π^0_1 *class*, i.e., a class of sets $\mathscr{P}$ such that, for some Π^0_1 formula $P(X)$ containing X as a parameter, a set A belongs to $\mathscr{P}$ if and only if $P(A)$ holds. (See the next section for more about formulas of arithmetic with set parameters.) Conversely, every Π^0_1 class is the set of paths through some computable subtree of $2^{<\omega}$ (and we call a Δ^0_1 index for this tree an *index* for the Π^0_1 class). The question of what the Turing degrees of non-empty members of Π^0_1 classes are has been a central one in computability. It is not difficult, for example, to see that every such class has a $\emptyset'$-computable set, but in fact this bound can be considerably improved. We shall discuss numerous results of this form, including the ubiquitous low basis theorem, in Section 2.2 below.

1.2 Reverse mathematics

The setting for reverse mathematics is *second-order arithmetic*, which is a system strong enough to encompass most of classical mathematics. Fragments of this system are called *subsystems of second-order arithmetic*, and the logical strength of a theorem is then measured according to the weakest subsystem in which (the countable analog of) that theorem can be proved. This is a two-step process: the first step consists in actually finding such a subsystem, and the second in showing that the theorem "reverses", i.e., is in fact equivalent to this subsystem over a weak base system.

The *language of second-order arithmetic*, L_2, is a two-sorted one, with variables for natural numbers $(x, y, z, \ldots)$ and constants for 0 and 1, variables for sets of numbers $(X, Y, Z \ldots)$, symbols for basic arithmetical operations and relations $(+, \times, <)$, and a symbol for set membership $(\in)$. Equality for sets of numbers is then defined extensionally, with $X = Y$ standing as an abbreviation for $(\forall n)\,[n \in X \leftrightarrow n \in Y]$. The set of L_2-formulas is ramified into the arithmetical and analytical hierarchies, and these are then used to define induction and comprehension schemes.

Semantic interpretations of L_2-theories are given by L_2-*structures*. A general L_2-structure $\mathscr{M}$ includes a set $\mathbb{N}^{\mathscr{M}}$ of "numbers", a collection $\mathcal{S}^{\mathscr{M}}$ of "sets", and interpretations of the symbols of L_2 using $\mathbb{N}^{\mathscr{M}}$ and $\mathcal{S}^{\mathscr{M}}$. An L_2-structure $\mathscr{M}$ is an ω-*model* if $\mathbb{N}^{\mathscr{M}}$ is the set $\omega = \{0, 1, 2, \ldots\}$ of standard natural numbers, $\mathcal{S}^{\mathscr{M}} \subseteq \mathcal{P}(\omega)$, and all the symbols of L_2 are given their standard interpretations. We identify an ω-model with the collection of subsets of ω that it contains. As usual, the notation $\mathscr{M} \models \varphi$ indicates that the formula φ (which may have parameters from $\mathscr{M}$) is true in $\mathscr{M}$. By convention, when arguing syntactically, or equivalently, inside an arbitrary model of some particular L_2-theory, we use $\mathbb{N}$ instead of ω to refer to the natural numbers.

The theory of *second-order arithmetic*, Z_2, is the axiom scheme consisting of:

1. the usual *Peano axioms* for a discrete ordered semiring, e.g., $(\forall x)[x + 1 \neq 0]$ and $(\forall x)(\forall y)(\forall z)[x < y \rightarrow x + z < y + z]$;

2. the *full second-order induction* scheme, containing, for every L_2-formula $\phi(x)$, the axiom
$$(\phi(0) \wedge (\forall x)[\phi(x) \to \phi(x+1)]) \to (\forall x)[\phi(x)];$$

3. the *full second-order induction* scheme, containing, for every L_2-formula $\phi(x)$, the axiom
$$(\exists X)(\forall x)[x \in X \leftrightarrow \phi(x)].$$

If Γ is Σ^i_n or Π^i_n, where $i \in \{0,1\}$ and $n \geq 0$, the scheme of Γ *comprehension* (Γ-CA) consists of the restriction of the induction scheme to formulas in Γ. Additionally, the scheme of Δ^i_n *comprehension* (Δ^i_n-CA) contains, for every Σ^i_n formula $\phi(x)$ and Π^i_n formula $\psi(x)$ not mentioning X, the axiom

$$(\forall x)[\phi(n) \leftrightarrow \psi(n)] \to (\exists X)(\forall x)[x \in X \leftrightarrow \phi(x)].$$

We define the scheme of Γ *induction* ($\mathsf{I}\Gamma$) as the analogous restriction of the induction scheme.

Fragments of Z_2 are called *subsystems of second-order arithmetic*, the weakest of which that will be of interest to us here (and, indeed, the standard base system for arguments in reverse mathematics) is the *recursive comprehension axiom* (RCA_0). RCA_0 consists of the discrete ordered semiring axioms, together with Δ^0_1-CA_0 and $\mathsf{I}\Sigma^0_1$. Intuitively, this subsystem corresponds to computable mathematics: it is satisfied by the ω-model REC containing only computable sets, and in fact, its ω-models are precisely the *Turing ideals*, i.e., classes of sets closed under Turing reducibility and join. In this sense, RCA_0 is rather weak. Nevertheless, it is able to establish many elementary properties of the natural numbers. Some of the theorems provable in RCA_0 include the Baire category theorem, the intermediate value theorem, Urysohn's lemma and the Tietze extension theorem, the soundness theorem from mathematical logic, and the Banach/Steinhaus uniform boundedness principle.

Stronger systems are obtained by adding stronger set-existence axioms to RCA_0. The main ones of interest to us will be the following:

- *arithmetical comprehension axiom* (ACA_0) is the subsystem obtained by adding the restriction of the comprehension scheme to arithmetical formulas;

and for each $n \geq 1$,

- Π^1_n *comprehension axiom* (Π^1_n-CA_0) is the subsystem obtained by adding Π^1_n-CA;

- Δ^1_n *comprehension axiom* (Δ^1_n-CA_0) is the subsystem obtained by adding Δ^1_n-CA.

Over RCA_0, ACA_0 is provably equivalent to a number of important results, including: the result that every bounded sequence of real numbers has a least upper bound; the fact that every sequence of points in a compact metric space has a convergent subsequence; the Ascoli lemma, the Bolzano/Weierstrass theorem; and the prime ideal theorem. Intuitively, ACA_0 corresponds to the assertion of the existence of the jump of every set, and its ω-models are precisely the *jump ideals*, i.e., the Turing ideals closed under jump. In practice, we rarely deal with ACA_0 directly, working instead with the following principle which is equivalent to it: for every $f : \omega \to \omega$, the range of f exists.

There are two important subsystems that do not directly correspond to restrictions of the second-order comprehension scheme. The first of these, WKL_0, consists of RCA_0 along with a single axiom known as *weak König's lemma*, asserting that any infinite subtree of $2^{<\mathbb{N}}$ contains an infinite path. The second is *arithmetical transfinite recursion* (ATR_0), and consists of RCA_0 along with an axiom scheme that states that any arithmetically-defined functional $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ may be iterated along any countable well-ordering, starting with any set. We shall not make use of ATR_0 below. WKL_0, by contrast, plays a key role in investigations of weak combinatorial principles. This system is already powerful enough to accommodate a sizable fragment of analysis, algebra, and topology, and among the theorems equivalent to it over RCA_0, are: the result that every continuous real-valued function on a compact metric space is bounded and attains a maximum value; the fact that every countable commutative ring has a prime ideal; Gödel's compactness theorem; the separable Hahn/Banach theorem; and the Heine/Borel theorem. In the language of computable model theory, the ω-models of WKL_0 are called *Scott sets*.

The following theorem summarizes the well-known relations between the subsystems mentioned above:

Theorem 1.2.1. *We have:*

$$\Pi^1_1\text{-}\mathsf{CA}_0 \implies \mathsf{ATR}_0 \implies \Delta^1_1\text{-}\mathsf{CA}_0 \implies \mathsf{ACA}_0 \implies \mathsf{WKL}_0 \implies \mathsf{RCA}_0;$$

and for each $n \geq 1$,

$$\Pi^1_{n+1}\text{-}\mathsf{CA}_0 \implies \Delta^1_{n+1}\text{-}\mathsf{CA}_0 \implies \Pi^1_n\text{-}\mathsf{CA}_0;$$

and all the implications are strict.

In the sequel, when it is straightforward to formalize the statement of a principle in L_2 or the proof of a result in RCA_0, we present it classically. One common obstacle to our doing so will be the implicit use in some proofs of the Σ^0_2 *bounding scheme*, $\mathsf{B}\Sigma^0_2$, a principle that, despite being extremely weak, is nevertheless not provable in RCA_0. More generally, if Γ is either Σ^0_n or Π^0_n for some n, we define the Γ *bounding scheme* to contain, for every formula $\phi(x, y)$ in Γ, the axiom

$$(\forall n)[(\forall x < n)(\exists y)[\phi(x, y)] \to (\exists m)(\forall x < n)(\exists y < m)[\phi(x, y)]].$$

It is known that $\mathsf{B}\Sigma^0_2$ is equivalent to $\mathsf{B}\Pi^0_1$ (see [25, Lemma 2.10]), and by Theorem 6.4 of Hirst [30] also to the infinitary pigeonhole principle, asserting that every partition of an infinite set into finitely many pieces must have an infinite part (in the notation of the next section, this is RT^1). By Lemma 10.6 of Cholak, Jockusch, and Slaman [5], $\mathsf{B}\Sigma^0_2$ also ollows from the principle SRT^2_2 defined below.

1.3 Ramsey's theorem

In this section, we collate some of the major results concerning the computability-theoretic and reverse-mathematical investigation of Ramsey's theorem. We begin by reviewing some of the terminology specific to its study.

Definition 1.3.1. Fix an infinite set $S \subseteq \omega$ and $n, k \in \omega$.

1. We denote by $[S]^n$ the collection of all sets $T \subseteq S$ of cardinality n.
2. A *k-coloring of S of exponent n* is a map $f : [S]^n \to k$, where k is identified with the set of its predecessors in ω. If $S = \omega$ and $n = 2$, we call f a k-coloring of *pairs*.
3. A set $H \subseteq S$ is said to be *homogeneous for* a k-coloring f of $[S]^n$ if f is constant on $[H]^n$.

For notational convenience, we tacitly assume that the members of every n-element subset $\{x_0, \dots, x_{n-1}\}$ of a set S are indexed such that $x_0 < \cdots < x_{n-1}$, and given a coloring $f : [S]^n \to k$, write $f(x_0, \dots, x_{n-1})$ in place of $f(\{x_0, \dots, x_{n-1})$.

The formal statement of Ramsey's theorem, along with its classical combinatorial proof, are as follows:

Ramsey's Theorem (RT). *For all $n, k \geq 1$, every coloring $f : [\omega]^n \to k$ has an infinite homogeneous set.*

Proof. We proceed by induction on n. For $n = 1$, this is simply the infinitary pigeonhole principle, and is clear. Assume the result, then, for some $n \geq 1$, and let $f : [\omega]^{n+1} \to k$ be given. We inductively define a sequence of sets $A_0 \supset A_1 \supset \cdots$ and a sequence of numbers $c_0, c_1, \dots < k$, such that for all i:

- $\min A_i < a$ for all $a \in A_{i+1}$;
- $f(\min A_i, a_0, \dots, a_{n-1}) = c_{i+1}$ for all $a_0, \dots, a_{n-1} \in A_{i+1}$.

Let $A_0 = \omega$ and $c_0 = 0$, and assume that for some $i \geq 0$, we have defined A_i and $c_i < k$. Define a coloring $g : [A_i - \{\min A_i\}]^n \to k$ by $g(a_0, \dots, a_{n-1}) = f(\min A_i, a_0, \dots, a_{n-1})$. By RT^n_k, relativized to $A_i - \{\min A_i\}$, there exists an infinite homogeneous set for this coloring contained in $A_i - \{\min A_i\}$, and we take this for A_{i+1}. Clearly, the resulting sequence has the desired properties. Now apply RT^1_k to the k-coloring of exponent 1 that maps $i \mapsto c_i$ for all i, to get an infinite set S and a $c < k$ such that $c_i = c$ for all $i \in S$. Then $H = \{\min A_i : i \in S\}$ is homogeneous for f with color c. □

We let RT^n denote the restriction of RT to colorings of exponent of n, and RT^n_k the restriction to k-colorings of exponent n.

As is common with Π^1_2 statements like RT, we refer to a coloring $f : [\omega]^n \to k$ as an *instance* of this theorem, and to an infinite homogeneous set for this coloring as a *solution* to this instance. The computability-theoretic study of Ramsey's theorem proceeds by restricting attention to computable instances, and analyzing the complexity of its solutions. Historically, the first result in this direction was the following:

Theorem 1.3.2 (Specker [61])**.** *There exists a computable stable coloring $f : [\omega]^2 \to 2$ with no computable infinite homogeneous set.*

It follows that RT fails in the ω-model REC, and hence that it is not provable in RCA_0. Specker's theorem was considerably sharpened by the following results of Jockusch:

Theorem 1.3.3 (Jockusch [34], Theorems 5.5, Lemma 5.9, and Theorems 5.6 and 5.1; [35], Corollary 1)**.** *Fix $n, k \geq 2$.*

1. *Every computable $f : [\omega]^n \to k$ has a Π^0_n infinite homogeneous set.*
2. *There exists a computable $f : [\omega]^n \to 2$ with no Δ^0_n infinite homogeneous set, and hence also no Σ^0_n infinite homogeneous set.*
3. *Every computable $f : [\omega]^n \to k$ has an infinite homogeneous H set with Δ^0_{n+1} jump, i.e., $H' \leq_T \emptyset^{(n)}$.*
4. *There exists a computable $f : [\omega]^n \to 2$ every infinite homogeneous set of which computes $\emptyset^{(n-2)}$.*
5. *For every computable $f : [\omega]^n \to k$, the degrees of infinite homogeneous sets of f are closed upwards.*

Since every Π^0_n set is Δ^0_{n+1} and hence $\emptyset^{(n)}$-computable, part (1) can be formalized to show that for all n and k, RT^n and RT^n_k is provable in ACA_0. Part (4), in turn, can be formalized to show that for all $n \geq 3$, RT^n_k implies, and hence is equivalent to, ACA_0.

For $n = 2$, the situation is more complicated. In particular, the preceding theorem leaves open the question of whether RT^2 implies ACA_0. The answer, in the form of the following result, was not found until twenty years after the question was posed:

Theorem 1.3.4 (Seetapun, see [56], Theorem 2.1)**.** *Given a sequence $C_0, C_1, \ldots$ of non-computable sets, every computable 2-coloring of pairs has an infinite homogeneous set H that does not compute any of the C_i.*

The theorem shows that specific information cannot be coded into the solutions of computable instances of Ramsey's theorem for pairs. It can be iterated and dovetailed to produce an ω-model of $\mathsf{RCA}_0 + \mathsf{RT}^2$ that contains no set computing $\emptyset'$. Since every ω-model of ACA_0 is a jump ideal, ACA_0 cannot hold in this model.

A natural question to ask in light of Seetapun's theorem is whether solutions to computable colorings of pairs can at least be made to code general, non-specific information, i.e., to be "far away" from being computable in some sense. A minor step in this direction can be obtained from Specker's theorem, because since every subset of a homogeneous set is homogeneous, and since every infinite c.e. set has an infinite computbale subset, it follows that computable instances of Ramsey's theorem need not even have c.e. solutions. Also, by part (2) of Theorem 1.3.3, it follows that computable instances of RT^2 need not have low solutions, i.e., solutions H satisfying $H' \leq_T \emptyset'$. Nevertheless, as the following theorem shows, computable instances of Ramsey's theorem for pairs necessarily have solutions with comparatively low information:

Theorem 1.3.5 (Cholak, Jockusch, and Slaman [5], Theorem 3.1)**.** *Every computable* $f : [\omega]^2 \to 2$ *has an infinite homogeneous set* H *which is* low$_2$*, i.e.,* $H'' \leq_T \emptyset''$*.*

(For further motivation for this theorem, see the introduction to Chapter 2.) Iterating this theorem produces an ω-model of RT^2 all of whose sets are low$_2$, and since no low$_2$ set can compute $\emptyset'$, this yields another proof that RT^2 does not imply ACA_0.

An important variant of Ramsey's theorem of pairs is provided by the following definition:

Definition 1.3.6. A k-coloring f of pairs is *stable* if for each $x \in \omega$, $\lim_s f(x, s)$ exists.

Stable Ramsey's Theorem (SRT^2)**.** *For all* $k \geq 1$*, every stable coloring* $f : [\omega]^2 \to k$ *has an infinite homogeneous set.*

Again, SRT^2_k denotes the restriction of SRT^2 to k-colorings. Stable colorings are in many ways simpler to work with and better understood than general colorings, in large part due to the following characterization, the proof of which is a straightforward application of the limit lemma:

Lemma 1.3.7.

1. *For any computable stable coloring* $f : [\omega]^2 \to k$ *there exist disjoint* Δ^0_2 *sets* $A_0, \ldots, A_{k-1}$ *such that* $\omega = A_0 \cup \cdots \cup A_{k-1}$ *and any infinite subset of any* A_i *computes an infinite homogeneous set for* f*.*

2. *For any disjoint* Δ^0_2 *sets* $A_0, \ldots, A_{k-1}$ *with* $\omega = A_0 \cup \cdots \cup A_{k-1}$ *there exists a computable stable coloring* $f : [\omega]^2 \to k$ *such that every infinite homogneoeus set for* f *is a subset of some* A_i*.*

In particular, if $k = 2$, each Δ^0_2 set A gives rise to a computable stable coloring f such that every infinite subset and co-subset (i.e., subset of the complement) of A computes an infinite homogeneous set for f, and conversely, each such coloring f gives rise to a Δ^0_2 set A such that every infinite homogeneous set of f is an infinite subset or co-subset of A.

It follows that every computable stable coloring $f : [\omega]^2 \to k$ has a Δ^0_2 infinite homogeneous set, whis provides a degree-theoretic distinction between SRT^2 and RT^2. The two principles are nevertheless deeply connected. Lemma 1.3.8 below allows us in many cases to transfer results about computable stable colorings to computable colorings in general. Recall that, given a sequence $\vec{R} = \{R_0, R_1, \ldots\}$ of sets, an infinite set S is called $\vec{R}$*-cohesive* if for each $i \in \omega$ either $S \subseteq^* R_i$ or $S \subseteq^* \overline{R_i}$. (Here, $A \subseteq^* B$ means that there exists an x so that for all $y > x$, $y \in A$ implies $y \in B$.) If $\vec{R}$ contains exactly the computable sets then an $\vec{R}$-cohesive set S is called *r-cohesive*, while if $\vec{R}$ contains exactly the c.e. sets then

S is called *cohesive*. It was shown by Jockusch and Stephan in [40], Corollary 2.4, that the r-cohesive and cohesive degrees, i.e., the degrees of r-cohesive and cohesive sets, coincide.

Lemma 1.3.8. *The restriction of a computable coloring of pairs to an r-cohesive set is stable.*

Proof. Fix $k \geq 1$ a computable $f : [\omega]^2 \to k$. For each $x \in \omega$, let R_x be the computable set $\{y > x : f(x, y) = 0\}$. Then if S is r-cohesive and $x \in S$, we have either that $S \subseteq^* R_x$, in which case $f(x, y) = 0$ for all sufficiently large $y \in S$, or $S \subseteq^* \overline{R_x} = \{y : y > x \to f(x, y) = 1\}$, in which case $f(x, y) = 1$ for all such y. Thus, $f \upharpoonright [S]^2$ is stable. $\square$

The formal analogue of the existence of an r-cohesive set in the language of second-order arithmetic is the following principle:[1]

Cohesive principle (COH)**.** *For every sequence $\vec{R}$ of sets, there exists an $\vec{R}$-cohesive set.*

Reverse-mathematically, Lemma 1.3.8 gives us a useful characterization of RT^2_k in terms of SRT^2_k and COH. We shall make frequent use of this proposition in the sequel.

Proposition 1.3.9 (Cholak, Jockusch, and Slaman [5], Lemma 7.11; Mileti [46], Section A.1)**.** *Over* RCA_0*,* RT^2_2 *is equivalent to* $\mathsf{SRT}^2_2 + \mathsf{COH}$.

While RT^2_k clearly implies SRT^2_k, it is unknown whether the reverse implication holds. It is also unknown whether SRT^2_k or even RT^2_k implies WKL_0 (see the discussion preceding Proposition 3.4.5 below). The precise strength of these principles is gauged by comparing them with other weak principles lying strictly between RCA_0 and ACA_0, as in, most notably, the work of Hirschfeldt and Shore [27]. We shall mention such results as are relevant to our work as we encounter them. A more complete summary, including the contributions from this dissertation, appears in Appendix A.

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[1]More precisely, COH corresponds to the assertion of the existence of a *p-cohesive* set. See the comment preceding Corollary 3.5.6 below.

CHAPTER 2
RAMSEY'S THEOREM AND CONE AVOIDANCE

2.1 Introduction

Seetapun and Slaman [56], p. 580, asked whether Seetapun's theorem (Theorem 1.3.4), for the special case where $C_i \equiv_T \emptyset'$ for all $i \in \omega$, could be effectivized to produce an infinite non-high homogeneous set not computing $\emptyset'$. A positive answer (for arbitrary non-computable C_i) was given by Cholak, Jockusch, and Slaman [5], Theorem 12.2. In turn, they asked whether this could be extended from non-high to low$_2$, and thus whether their low$_2$ bound (Theorem 1.3.5) could be combined with Seetapun's theorem:

Question 2.1.1 (Cholak, Jockusch, and Slaman [5], Question 13.11)**.** Given a non-computable set C, does every computable 2-coloring of pairs admit a low$_2$ infinite homogeneous set H with $C \not\leq_T H$?

In this chapter, which is joint work with Carl G. Jockusch, Jr., we give an affirmative answer to this question, thereby showing that for computable colorings of pairs the infinite homogeneous set asserted to exist by Ramsey's theorem can be chosen to code neither general nor specific information.

Our solution relies on a new proof of Seetapun's theorem (Theorem 1.3.4) which is considerably more straightforward than both the original (see [56], Section 2) as well as the simplification of Hummel and Jockusch (see [32], Theorem 2.1). This proof is based in part on the method used by Cholak, Jockusch, and Slaman in [5], Section 4, to prove Theorem 1.3.5, which featured control of the first jump of the homogeneous set being constructed. It established the stronger result ([5], Corollary 12.6) that every computable 2-coloring of pairs has an infinite homogeneous set whose jump has degree at most $\mathbf{d}$, where $\mathbf{d}$ is a given degree $\gg \mathbf{0}'$. (See Definition 2.2.7 and the subsequent discussion. Also see the remarks after the proof of Theorem 2.3.3 for an explanation of a missing point in the proof of this stronger result.) The proof proceeds by first reducing the result to the stable case, and this reduction is straightforward. In our proof, the crucial new element is the proof is Lemma 2.4.7, which handles cone avoidance in the stable case. By keeping track of the effective content in this new proof, we are able to answer Question 2.1.1.

Cholak, Jockusch, and Slaman gave a second proof of Theorem 1.3.5 which featured direct control of the second jump of H. It was adapted in [5], Theorem 10.2, to models of second-order arithmetic to show that every Π^1_1 sentence provable from $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{RT}^2_2$ is provable from just $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. In Section 2.6, we use an extension this technique, together with a new basis theorem for Π^0_1 classes, to extend our main result to show that every computable 2-coloring of pairs has a pair of low$_2$ infinite homogeneous sets whose degrees form a minimal pair. Along the way, we extend Theorem 2.5 of Jockusch and Stephan [40] to show that there is a minimal pair of low$_2$ cohesive degrees.

2.2 Basis theorems

The low basis theorem of Jockusch and Soare [39], Theorem 2.1, played a crucial role in the proof of Theorem 1.3.5. Hence, it is not surprising that a cone-avoiding version of the low basis theorem will play a central role in answering Question 2.1.1. The needed result is due to Linda Lawton (see [26], Theorem 4.1).

Definition 2.2.1. Given a sequence $C_0, C_1, \ldots$ of sets, we call a set S *cone-avoiding* (for this sequence) provided $C_i \not\leq_T S$ for all $i \in \omega$.

Theorem 2.2.2 (Lawton)**.** *Let $T \subseteq 2^{<\omega}$ be a computable infinite tree, let $C_0, C_1, \ldots$ be a sequence of non-computable sets, and let $C = \oplus_{i\in\omega} C_i$. Then T has a cone-avoiding infinite path f such that $f' \leq_T \emptyset' \oplus C$. Furthermore, an index for f' as a $(\emptyset' \oplus C)$-computable set can be found $(\emptyset' \oplus C)$-effectively from an index for T.*

The theorem is proved by forcing with Π^0_1 classes, alternating between forcing the jump and applying the following lemma:

Lemma 2.2.3 (Lawton)**.** *Let $T \subseteq 2^{<\omega}$ be a computable infinite tree, let C be any non-computable set, and let $e \in \omega$. Then T has a computable infinite subtree $\widetilde{T}$ such that $\Phi^f_e \neq C$ for all $f \in [\widetilde{T}]$, and an index for $\widetilde{T}$ can be found $(\emptyset' \oplus C)$-effectively from e and an index for T.*

Proof. For each $n \in \omega$, define the computable tree

$$U_n = \{\sigma \in T : \neg(\Phi^\sigma_{e,|\sigma|}(n) \downarrow = C(n))\},$$

and notice that U_n must be infinite for some n. Indeed, if each U_n was finite, then to compute $C(n)$ for a given n we would need only to search for $m, k \in \omega$ such that $\Phi^\sigma_{e,|\sigma|}(n) = k$ for all $\sigma \in T$ of length m, from which we conclude that $C(n) = k$. This contradicts the assumption that C is non-computable. Clearly, an index for U_n as a computable tree can be found C-effectively from e and an index for T, and $\emptyset'$ can determine whether or not U_n is infinite. (For the latter, use that $U_n \subseteq 2^{<\omega}$.) Thus, $\emptyset' \oplus C$ can find the least n such that U_n is infinite, and we set $\widetilde{T} = U_n$. $\square$

Note, in particular, that if the sequence $C_0, C_1, \ldots$ is taken uniformly Δ^0_2, Theorem 2.2.2 asserts the existence, in any non-empty Π^0_1 class of sets, of a low member which avoids the upper cone of each C_i. The variation on this result below for the case where the Π^0_1 class in question contains no computable members is included here for general interest. As a special case, we get that any non-empty Π^0_1 class of sets with no computable members contains a low set Turing incomparable with a given non-computable low set.

Theorem 2.2.4. *Let $T \subseteq 2^{<\omega}$ be a computable infinite tree with no computable infinite paths, let $C_0, C_1, \ldots$ be a sequence of non-computable sets, and let $D = \oplus_{i\in\omega} C_i'$. Then T contains an infinite path f such that $f \mid_T C_i$ for all $i \in \omega$, and $f' \leq_T D$. Furthermore, an index for f' as a D-computable function can be found D-effectively from an index for T.*

We can obtain the theorem from the next lemma, by interspersing the construction used in its proof into that of Theorem 2.2.2.

Lemma 2.2.5. *Let $T \subseteq 2^{<\omega}$ be a computable infinite tree with no computable infinite paths, let C be any set, and let $e \in \omega$. Then T has a computable infinite subtree $\widetilde{T}$ such that $\Phi_e^C \neq f$ for all $f \in [\widetilde{T}]$, and an index for $\widetilde{T}$ can be found C'-effectively from e and an index for T.*

Proof. Since $T \subseteq 2^{<\omega}$ is infinite and computable and has no computable infinite paths, T has at least two distinct infinite paths, and thus there exist incompatible strings σ_1 and σ_2 such that each σ_i is extendible to an infinite path through T. Such σ_1 and σ_2 can be found $\emptyset'$-effectively since the set of extendible nodes of T is $\emptyset'$-computable. Fix n such that $\sigma_1(n)\downarrow \neq \sigma_2(n)\downarrow$. Use C' to determine whether $\Phi_e^C(n)\downarrow$. If not, let $\widetilde{T} = T$. If so, choose $j \in \{1, 2\}$ such that $\Phi_e^C(n)\downarrow \neq \sigma_j(n)\downarrow$, and let $\widetilde{T}$ be the set of all strings $\sigma \in T$ which are compatible with σ_j. □

Finally, we present the following "minimal pair basis theorem" which will be central to our work in Section 2.6 (recall that two degrees **a** and **b** are said to form a *minimal pair* if $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$; we do not require that **a** and **b** be non-zero):

Theorem 2.2.6. *Let $T \subseteq 2^{<\omega}$ be a computable infinite tree, and let B be any set. Then T has an infinite path f with $f'' \leq_T \emptyset'' \oplus B'$ whose degree makes a minimal pair with that of B. Furthermore, an index for f'' as a $(\emptyset'' \oplus B')$-computable set can be found $(\emptyset'' \oplus B')$-effectively from an index for T.*

Proof. We force with Π_1^0 classes. Set $T_0 = T$ and suppose that for some $e \geq 0$, $T_e \subseteq 2^{<\omega}$ is a computable infinite tree of known index. For each $n \in \omega$, define

$$U_n = \{\sigma \in T_e : \Phi_{e,|\sigma|}^\sigma(n)\uparrow\},$$

the index of which as a computable tree can be found effectively from an index for T_e. First, ask $\emptyset''$ whether U_n is infinite for some n, and if so, search $\emptyset'$-effectively for the least such n and set $T_{e+1} = U_n$. In this case, Φ_e^f is not total for any $f \in [T_{e+1}]$. Otherwise, we know that Φ_e^f is total for all $f \in [T_e]$. In this case, we ask $\emptyset''$ if there exist extendible nodes $\sigma, \tau \in T_e$ and an $n \in \omega$ such that $\Phi_{e,|\sigma|}^\sigma(n)\downarrow \neq \Phi_{e,|\tau|}^\tau(n)\downarrow$. If there do, find the least such σ, τ, and n in some fixed effective listing and inquire of B' whether $\Phi_e^B(n)$ converges. If it does, let T_{e+1} consist of all nodes in T_e comparable with ρ, where ρ is the lexicographically smaller of σ and τ such that $\Phi_e^\rho(n) \neq \Phi_e^B(n)$. Clearly $\Phi_e^f \neq \Phi_e^B$ for all $f \in [T_{e+1}]$ in this case. If $\Phi_e^B(n)\uparrow$, or if there do not exist any such σ, τ, and n, set $T_{e+1} = T_e$. Suppose now that Φ_e^B and Φ_e^f are equal and total, where $f \in [T_{e+1}]$. Then each U_n is finite and σ, τ and n as above did not exist. Given n, $\Phi_e^B(n)$ can now be computed as follows: first, search for the least k such that $\Phi_{e,|\sigma|}^\sigma(n)\downarrow$ for all $\sigma \in T$ of length k, and then search for the least $\ell \geq k$ and $m \in \omega$ such that $\Phi_{e,|\sigma|}^\sigma(n) = m$ for all $\sigma \in T$ of length k having an extension on T of length ℓ. The second search succeeds because for any extendible σ on T of length k it must be that $\Phi_{e,|\sigma|}^\sigma(n) = \Phi_e^B(n)$, so m must be equal to $\Phi_e^B(n)$.

As usual, we iterate the construction to build a sequence $2^{<\omega} = T_0 \supseteq T_1 \supseteq \cdots$ of computable infinite trees, and we take $f \in \bigcap_{e\in\omega} T_e$. Since the construction was $(\emptyset'' \oplus B')$-effective, and since it decided for each e the totality of Φ_e^f, it follows that $f'' \leq_T \emptyset'' \oplus B'$. $\square$

We shall apply this result when B satisfies $B' \leq_T \emptyset''$, in which case of course the infinite path f of the previous theorem is low$_2$.

We conclude this section with the following definition due to Simpson [58], p. 648:

Definition 2.2.7. Let $\mathbf{a}$ and $\mathbf{b}$ be Turing degrees. We say $\mathbf{a}$ is a *PA degree relative to* $\mathbf{b}$, and write $\mathbf{a} \gg \mathbf{b}$, if every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$. (If $\mathbf{b} = \mathbf{0}$, we call $\mathbf{a}$ simply a *PA degree*.)

It is known that a degree $\mathbf{a}$ is PA if and only if every computable, computably bounded infinite tree $T \subseteq \omega^{<\omega}$ has an infinite path of degree $\leq \mathbf{a}$, if and only if every $\{0,1\}$-valued partial computable function has a $\{0,1\}$-valued total extension of degree $\leq \mathbf{a}$ (see [10, Section 6] for proofs of these and other equivalences). The latter characterization easily produces a non-empty Π^0_1 class $\mathscr{P}$ every element of which has PA degree: consider the class of all $\{0,1\}$-valued (total) functions g such that $g(\langle e,i\rangle) = \Phi_e(i)$ whenever $\Phi_e(i)\downarrow\leq 1$. The low basis theorem applied to $\mathscr{P}$ consequently implies that there exists a low degree $\mathbf{a} \gg \mathbf{0}$. Theorem 2.2.2, on the other hand, gives for each sequence $C_0, C_1, \ldots$ of non-computable sets the existence of a cone-avoiding degree $\mathbf{a} \gg \mathbf{0}$ whose jump is below $\mathbf{0}' \cup \deg(\oplus_{i\in\omega} C_i)$. And similarly Theorem 2.2.6 implies that for each degree $\mathbf{b}$ there is a degree $\mathbf{a} \gg \mathbf{0}$ with $\mathbf{a}'' \leq \mathbf{0}'' \cup \mathbf{b}'$ and $(\mathbf{a}, \mathbf{b})$ a minimal pair.

We shall make frequent use of the following technical lemma:

Lemma 2.2.8. *Let* $\mathbf{a}$ *be a PA degree, and let* $\psi_0, \psi_1, \ldots$ *be an effective listing of all* Π^0_1 *formulas of first-order arithmetic. If* f *is a computable function such that for every* $e \in \omega$ *there is an* i *in the finite set* $D_{f(e)}$ *such that* ψ_i *is true, then there is a function* p *of degree* $\leq \mathbf{a}$ *such that for every* e, $p(e) \in D_{f(e)}$ *and* $\psi_{p(e)}$ *is true.*

Proof. For every $i \in \omega$, let θ_i be a Σ^0_0 formula such that $\psi_i \equiv (\forall x)[\theta_i(x)]$. Consider the computable tree $T \subseteq \omega^{<\omega}$ where $\sigma \in T$ if and only if

$$(\forall e < |\sigma|)[\sigma(e) \in D_{f(e)} \wedge (\forall x \leq |\sigma|)[\theta_{\sigma(e)}(x)]].$$

It is clear that T is computably bounded, and it is easy to see that $p : \omega \to \omega$ is an infinite path through T if and only if for all e, $p(e) \in D_{f(e)}$ and $\psi_{p(e)}$ is true. By assumption, T is non-empty, and so it must have an infinite path of degree $\leq \mathbf{d}$. $\square$

2.3 Cone avoidance and first jump control

Theorem 1.3.5 was proved by using a low$_2$ r-cohesive set to reduce it to the stable case, and then proving the stable case by transforming it to a statement about Δ^0_2 sets. We proceed analogously here, but adding cone avoidance. We state two key lemmas, delaying their proofs until Section 2.5 below, and derive from them our main results. For the remainder of this section, let a degree $\mathbf{d} \gg \mathbf{0}'$ be fixed. Additionally, except where otherwise stated, let $C_0, C_1, \dots$ be a fixed sequence of non-computable sets, and let $C = \oplus_{i\in\omega} C_i$. "Cone avoiding" below refers to this sequence.

Lemma 2.3.1. *There exists a cone-avoiding r-cohesive set X, and if $C \leq_T \emptyset'$ then X can be chosen with jump of degree at most $\mathbf{d}$.*

Lemma 2.3.2.

1. *For $k \geq 2$, if $A_1, \dots, A_k$ is a sequence of disjoint sets whose union is ω, then for some $i \leq k$, A_i contains an infinite cone-avoiding subset.*
2. *If $A_1, \dots, A_k, C \leq_T \emptyset'$, then for some $i \leq k$, A_i contains an infinite cone-avoiding subset with jump of degree at most $\mathbf{d}$.*

For clarity, we now restate Seetapun's theorem, and show how it follows from Lemmas 2.3.1 and 2.3.2. The effectiveness of Lemmas 2.3.1 and 2.3.2 allows us to add an effectiveness result here for the case where the sequence $C_0, C_1, \dots$ is uniformly Δ^0_2. The latter is already a partial answer to Question 2.1.1.

Theorem 2.3.3.

1. (Seetapun's theorem) *Every computable 2-coloring of pairs has a cone-avoiding infinite homogeneous set H.*
2. *If $C \leq_T \emptyset'$, then H can be chosen so that* $\deg(H)' \leq \mathbf{d}$.

Proof. To prove (1), let f be a computable 2-coloring of pairs. By Lemma 2.3.1, fix a cone-avoiding r-cohesive set X, and let $p : \omega \to X$ be its principal function, i.e., the function that lists the elements of X in increasing order. Define an X-computable 2-coloring g of pairs by $g(\{s,t\}) = f(\{p(s), p(t)\})$, for all $\{s,t\} \in [\omega]^2$, which must be stable by Lemma 1.3.8. By part (2) of the same lemma, relativized to X, there exists a $\Delta^{0,X}_2$ set A, such that for any infinite subset S of A or $\overline{A}$, $X \oplus S$ computes an infinite homogeneous set for f. On the other hand, by Lemma 2.3.2 (1), relativized to X, there is an infinite subset S of either A of $\overline{A}$ whose join with X is cone-avoiding. Hence, g has an infinite homogeneous set $\widetilde{H} \leq_T X \oplus S$, and so by definition of g, $H = p(\widetilde{H})$ is homogeneous for f. Now we have

$$H \leq_T X \oplus \widetilde{H} \leq_T X \oplus S,$$

implying that H is cone-avoiding.

For (2), assume $C \leq_T \emptyset'$. By Theorem 6.5 of Simpson [58], there exists a degree $\mathbf{e}$ satisfying $\mathbf{d} \gg \mathbf{e} \gg \mathbf{0}'$, so by Lemma 2.3.1 we may choose X to satisfy $\deg(X)' \leq \mathbf{e}$. In particular, $\mathbf{d} \gg \deg(X)'$, and so we may use part (2) of Lemma 2.3.2 above, relativized to X, in place of part (1) to find S with $X \oplus S$ not only cone-avoiding but also with jump of degree at most $\mathbf{d}$. Defining H as before, we thus have $\deg(H)' \leq \deg(X \oplus S)' \leq \mathbf{d}$, as desired. □

We remark that using density of the degrees under $\gg$ in the proof of part (2) above is also necessary to obtain Theorem 12.6 of [5] mentioned in the introduction, and is missing from the exposition in [5]. This was pointed out by Joseph Mileti.

An affirmative answer to Question 2.1.1 now easily follows.

Theorem 2.3.4. *Let C be non-computable. Every computable 2-coloring f of pairs has a low$_2$ infinite homogeneous set H such that $C \not\leq_T H$.*

Proof. We consider several cases.

Case 1: $C \not\leq_T \emptyset''$. By Theorem 1.3.5, f has a low$_2$ infinite homogeneous set H, meaning C can not be computable from H'', and hence also not in H.

Case 2: $C \leq_T \emptyset'$. Fix $\mathbf{a}$ satisfying $\mathbf{a} \gg \mathbf{0}'$ and $\mathbf{a}' \leq \mathbf{0}''$, as in the discussion at the end of Section 2.2. Now the desired result follows by Theorem 2.3.3 (2) using $\mathbf{a}$ in place of $\mathbf{d}$.

Case 3: $C \not\leq_T \emptyset'$ and $C \leq_T \emptyset''$. By Lemma 2.2.2, relativized to $\emptyset'$, there exists a cone-avoiding degree $\mathbf{a} \gg \mathbf{0}'$ with $\mathbf{a}' \leq \mathbf{0}'' \cup \deg(C) = \mathbf{0}''$. By Theorem 1.3.5 (or rather, by the extension of this theorem mentioned following its statement above), f has an infinite homogeneous set H such that $\deg(H)' \leq \mathbf{a}$, so H is low$_2$. And since $\deg(C) \not\leq \mathbf{a}$, it follows that C is not computable from H. □

We do not know whether Theorem 2.3.4 can be strengthened to assert that for every $C >_T 0$ every computable 2-coloring of pairs has an infinite homogeneous set $H \not\geq_T C$ with $\deg(H') \leq \mathbf{d}$. That is, we do not know whether the condition in Theorem 2.3.3 (2) that C be Δ^0_2 can be omitted in the situation where "cone-avoiding" means simply $H \not\geq_T C$.

2.4 Strategies for meeting requirements

Lemmas 2.3.1 and 2.3.2 will be proved using Mathias forcing with suitable restrictions on the infinite part of the conditions. In this section we show that each requirement for these lemmas is dense for the relevant version of Mathias forcing, and we calculate the oracle needed to find the extensions needed to witness this density. We then complete the proofs of these lemmas in the next section. Throughout this section, we assume fixed a degree $\mathbf{d} \gg \mathbf{0}'$. Lemma 2.4.4 below is a variation on the proof in Section 4.1of [5] of Theorem 2.5 of Jockusch and Stephan [40]. Lemmas 2.4.6, 2.4.7, and 2.4.8 are all variations of Lemma 4.6 of Cholak, Jockusch, and Slaman [5].

We begin by recalling and fixing some terminology.

Definition 2.4.1.

1. A *(Mathias) precondition* is a pair of sets (D, L) with D finite and $\max D < \min L$. A *(Mathias) condition* is a precondition (D, L) such that L is infinite.

2. A set G *satisfies* a precondition (D, L) if $D \subseteq G \subseteq D \cup L$.

3. A precondition $(\widetilde{D}, \widetilde{L})$ *extends* a precondition (D, L), written $(\widetilde{D}, \widetilde{L}) \prec (D, L)$ if $\widetilde{L} \subseteq L$ and $\widetilde{D}$ satisfies (D, L). We call such an extension a *finite extension* if $L - \widetilde{L}$ is finite.

4. We say a precondition (D, L) is *cone-avoiding*, *computable*, or *low* if L is, respectively, cone-avoiding, computable, or low. If (D, L) is computable or low, an *index* for this precondition is a pair of numbers (d, ℓ) such that d is the canonical index of D and ℓ is, respectively, a Δ^0_1 index for L or a $\Delta^{0,\emptyset'}_1$ index for L'.

Definition 2.4.2. Fix $e \in \omega$ and a precondition (D, L).

1. We say (D, L) *forces agreement on* e if there is no $x \in \omega$, and there are no finite sets F_0, F_1 satisfying (D, L) such that $\Phi_e^{F_0}(x) \downarrow \neq \Phi_e^{F_1}(x) \downarrow$.

2. We say (D, L) *forces* e *out of the jump* if there is no finite set F satisfying (D, L) such that $e \in F'$.

2.4.1 Strategies for proving Lemma 2.3.1.

Lemma 2.4.3 (Cholak, Jockusch, and Slaman [5], Lemma 4.4)**.** *Let R be any computable set, (D, L) a computable condition, and $e \in \omega$.*

1. *There is a computable extension of (D, L) which is satisfied only by sets G with $|G| \geq e$, and an index for it can be found effectively from e and an index for (D, L).*

2. *There is a computable extension of (D, L) which is satisfied only by sets G with $G \subseteq^* R$ or $G \subseteq^* \overline{R}$, and an index for it can be found* **d***-effectively from e and an index for (D, L).*

3. *There is a computable extension of (D, L) such that $G'(e)$ is the same for all sets G satisfying it, and an index for it can be found $\emptyset'$-effectively from e and an index for (D, L).*

Lemma 2.4.4. *Let C be any set, (D, L) a condition, and $e \in \omega$. Then there is a finite extension of (D, L) such that, for all sets G satisfying it, either $\Phi_e^G \neq C$, or Φ_e^G is not total, or else Φ_e^G is L-computable. Furthermore, there is a fixed uniform procedure for finding canonical indices of finite sets H_0, H_1 from e, the canonical index of D, and an oracle for $L' \oplus C$ such that the desired finite extension of (D, L) has the form $(D \cup H_0, L - H_1)$.*

Proof. We consider two cases, the distinction between which is uniformly L'-effective in e and the canonical index of D.

Case 1: (D, L) does not force agreement on e. Then we search, L'-effectively from e and the canonical index of D, for $x \in \omega$, and for finite sets F_0, F_1 satisfying (D, L) such that $\Phi_e^{F_0}(x) \downarrow \neq \Phi_e^{F_1}(x) \downarrow$. One of the two computations, say $\Phi_e^{F_j}(x)$, must differ from $C(x)$, and we can, uniformly C-effectively from i, x, F_0, and F_1, find such a j. We then let $\widetilde{D} = F_j$, and let $\widetilde{L}$ consist of the elements of L greater than $\max F_j$ and $\varphi_e^{F_j}(x)$. Any set G satisfying $(\widetilde{D}, \widetilde{L})$ will agree with F_j below $\varphi_e^{F_j}(x)$ and hence will satisfy $\Phi_e^G(x) \downarrow = \Phi_e^{F_j}(x) \downarrow \neq C(x)$.

Case 2: (D, L) forces agreement on e. We then set $(\widetilde{D}, \widetilde{L}) = (D, L)$, and claim that if G satisfies $(\widetilde{D}, \widetilde{L})$ then either Φ_e^G is not total or else it is L-computable. Indeed, if Φ_e^G is total, then to compute its value on input x we have only to L-effectively search through the canonical indices of finite sets satisfying (D, L) until we find one, call it F, such that $\Phi_e^F(x) \downarrow$. Such an F clearly exists since, for example, any sufficiently long initial segment of G will do, and we have $\Phi_e^F(x) = \Phi_e^G(x)$ for any such F, else the previous case would have applied.

In either case, $(\widetilde{D}, \widetilde{L})$ is a finite extension of (D, L) with the desired properties, and the desired uniformity follows easily from the construction. □

2.4.2 Strategies for proving Lemma 2.3.2

Let $C_0, C_1, \ldots$ be a sequence of non-computable sets, and let $C = \oplus_{i \in \omega} C_i$.

Lemma 2.4.5 (Cholak, Jockusch, and Slaman [5], Lemma 4.6)**.** *Let A be any set, (D, L) a condition with $L \cap A$ and $L \cap \overline{A}$ infinite, and $e, i \in \omega$. There is a finite extension $(\widetilde{D}, \widetilde{L})$ of (D, L) such that, for all sets G satisfying this extension, $|G \cap A| \geq e$ and $|G \cap \overline{A}| \geq i$ (so in particular, $(\widetilde{D}, \widetilde{L})$ is low or cone-avoiding if (D, L) is). Moreover, an index for this extension can be found $(\emptyset' \oplus A)$-effectively from e, i, and an index for (D, L).*

The following lemma adds cone avoidance to Lemma 4.6 of [5]:

Lemma 2.4.6. *Let A be any set, (D, L) a cone-avoiding condition, and $e, i \in \omega$. There is a cone-avoiding extension $(\widetilde{D}, \widetilde{L})$ of (D, L) such that either $(G \cap A)'(e)$ is the same for all sets G satisfying this extension, or $(G \cap \overline{A})'(i)$ is the same for all such G. If, additionally, $\mathbf{d}$ is a degree with $\mathbf{d} \gg \mathbf{0}'$, $A, C \leq_T \emptyset'$, and (D, L) is low, then $(\widetilde{D}, \widetilde{L})$ can be chosen low, and its index can be found $\mathbf{d}$-effectively from e, i, and an index for (D, L). Furthermore, $\mathbf{d}$ can compute whether $(\widetilde{D}, \widetilde{L})$ forces the value of $(G \cap A)'(e)$ or $(G \cap \overline{A})'(i)$ and what value is forced.*

Proof. Define the $\Pi_1^{0,L}$ class $\mathscr{P}$ to consist of all (possibly finite) sets $S \subseteq L$ such that $(D \cap A, S)$ forces e out of the jump and $(D \cap \overline{A}, L - S)$ forces i out of the jump. We consider two cases.

Case 1: $\mathscr{P} \neq \emptyset$. By Theorem 2.2.2, relativized to L, there exists an $S \in P$ such that $L \oplus S$ is cone-avoiding, and hence the same is true of S and $L - S$. Fix such an S. At least one of the following subcases must apply since L is infinite:

Subcase 1a: S is infinite. We then set $(\widetilde{D}, \widetilde{L}) = (D, S)$, and observe that if G is any set satisfying $(\widetilde{D}, \widetilde{L})$, then $G \cap A$ satisfies $(D \cap A, S)$. Hence, $e \notin (G \cap A)'$, else any initial segment F of G sufficiently long that $D \cap A \subseteq F \cap A$ and $\Phi_e^{F \cap A}(e) \downarrow$ would contradict that $(D \cap A, S)$ forces e out of the jump.

Subcase 1b: $L - S$ is infinite. We set $(\widetilde{D}, \widetilde{L}) = (D, L - S)$, and it follows by a similar argument that $i \notin (G \cap \overline{A})'$ for all G satisfying $(\widetilde{D}, \widetilde{L})$.

Case 2: $\mathscr{P} = \emptyset$. Then in particular $L \cap A \notin \mathscr{P}$, so at least one of the following subcases must hold:

Subcase 2a: some finite set F with $\Phi_e^F(e)\downarrow$ satisfies $(D \cap A, L \cap A)$. We let $\widetilde{D} = D \cup F$ and let $\widetilde{L}$ consist of those elements of L greater than $\max \widetilde{D}$ and $\varphi_e^F(e)$. Since $\widetilde{L} =^* L$, it follows that $\widetilde{L}$ is cone-avoiding. Furthermore, for any set G satisfying $(\widetilde{D}, \widetilde{L})$, $G \cap A$ and F agree below $\varphi_e^F(e)$. It follows that $\Phi_e^{G\cap A}(e)\downarrow = \Phi_e^F(e)$, and hence $e \in (G \cap A)'$.

Subcase 2b: some finite set F with $\Phi_i^F(i)\downarrow$ satisfies $(D \cap \overline{A}, L - A)$. We let $\widetilde{D} = D \cup F$ and let $\widetilde{L}$ consist of those elements of L greater than $\max \widetilde{D}$ and $\varphi_i^F(i)$. It follows analogously that $i \in (G \cap \overline{A})'$.

In either case, $(\widetilde{D}, \widetilde{L})$ extends (D, L) and clearly has the desired properties.

Now assume that A and C are computable from $\emptyset'$, and that (D, L) is low. Notice that an index for $\mathscr{P}$ as a $\Pi_1^{0,L}$ class can be found L'-effectively, hence $\emptyset'$-effectively, from e, i, and canonical indices for $D \cap A$ and $D \cap \overline{A}$, hence, since A is Δ_2^0, $\emptyset'$-effectively from an index for (D, L). By the effectiveness of Theorem 2.2.2 relativized to L, we can, in Case 1 above, $(\emptyset' \oplus C)$-effectively, hence $\emptyset'$-effectively, from an index for $\mathscr{P}$, find S and a $\Delta_1^{0,\emptyset'}$ index for $(L \oplus S)'$ so as to additionally satisfy

$$(L \oplus S)' \leq_T L' \oplus C \leq_T \emptyset' \oplus C \equiv_T \emptyset'.$$

It follows that S and $L - S$ are both low, and hence that the extension $(\widetilde{D}, \widetilde{L})$ is, in every case considered above, not only cone-avoiding but also low.

The distinction between Cases 1 and 2 is L'-effective from an index for $\mathscr{P}$, and hence $\emptyset'$-effective.

We now argue, as in [5], Lemma 4.6, that one can **d**-effectively choose a true subcase, 1a or 1b, of Case 1. Consider the two statements "S is infinite" and "$L - S$ is infinite". These are $\Pi_2^{0,L\oplus S}$ statements, hence Π_2^0 statements since $L \oplus S$ is low, and indices for them as such can be found effectively from a $\Delta_1^{0,\emptyset'}$ index for $(L \oplus S)'$, an index for (D, L), and an index for $\mathscr{P}$. Since at least one of the statements is true, it follows by Lemma 2.2.8, relative to $\mathbf{0}'$, that we can **d**-effectively choose one which is true.

Since A is Δ_2^0, the distinction between Subcases 2a and 2b amounts to $\emptyset'$-effectively searching through finite sets satisfying $(D \cap A, L \cap A)$ and $(D \cap \overline{A}, L - A)$ until we find one, call it F, such that either $\Phi_e^F(e)\downarrow$ or $\Phi_i^F(i)\downarrow$.

We conclude that, in any case, an index for $(\widetilde{D}, \widetilde{L})$ can be found **d**-effectively from e, i, and an index for (D, L), as desired. Furthermore, we can **d**-effectively find a statement in the list $e \in (G \cap A)', e \notin (G \cap A)', i \in (G \cap \overline{A})', i \notin (G \cap \overline{A})'$ which is true of all G satisfying $(\widetilde{D}, \widetilde{L})$. □

Lemma 2.4.7. *Let A be any set, (D, L) a cone-avoiding condition, and $e, i, s, t \in \omega$. There is a cone-avoiding extension $(\widetilde{D}, \widetilde{L})$ of (D, L) such that, for all sets G satisfying this extension, either $\Phi_e^{G \cap A} \neq C_s$ or $\Phi_i^{G \cap \overline{A}} \neq C_t$. If, additionally, $A, C \leq_T \emptyset'$ and (D, L) is low, then $(\widetilde{D}, \widetilde{L})$ can be chosen low, and its index can be found* **d***-effectively from e, i, and an index for (D, L).*

Proof. Define $\mathscr{P}$ to be the class of all $S \subseteq L$ such that $(D \cap A, S)$ forces agreement on e and $(D \cap \overline{A}, L - S)$ forces agreement on i. Then $\mathscr{P}$ is a $\Pi_1^{0,L}$ class.

Case 1: $\mathscr{P} \neq \emptyset$. By Theorem 2.2.2, relativized to L, there exists $S \in \mathscr{P}$ such that $L \oplus S$ is cone-avoiding. In particular, S and $L - S$ are cone-avoiding as well.

Subcase 1a: S is infinite. We then set $(\widetilde{D}, \widetilde{L}) = (D, S)$, and claim that if G is any set satisfying $(\widetilde{D}, \widetilde{L})$ then $\Phi_e^{G \cap A} \neq C_s$. Indeed, if $\Phi_e^{G \cap A}$ is total then we can, S-effectively from e and the canonical index of $D \cap A$, compute $\Phi_e^{G \cap A}(x)$ for any x, since we have only to search through finite sets F with $D \cap A \subseteq F \subseteq S$ until we find one such that $\Phi_e^F(x) \downarrow$. Since a set satisfies $(\widetilde{D}, \widetilde{L})$ only if its intersection with A satisfies $(D \cap A, S)$, and since the latter forces agreement on e, it follows that we shall then have $\Phi_e^F(x) = \Phi_e^{G \cap A}(x)$. Thus, our claim follows by virtue of S being cone-avoiding.

Subcase 1b: $L - S$ is infinite. We set $(\widetilde{D}, \widetilde{L}) = (D, L - S)$ and can argue as above to conclude that $\Phi_i^{G \cap \overline{A}} \neq C_t$ for any set G satisfying $(\widetilde{D}, \widetilde{L})$.

Case 2: $\mathscr{P} = \emptyset$. Thus $L \cap A \notin \mathscr{P}$, so we have at least one of the following two possibilities:

Subcase 2a: $(D \cap A, L \cap A)$ does not force agreement on e. Then for some $x \in \omega$, and for some finite set F satisfying $(D \cap A, L \cap A)$, $\Phi_e^F(x) \neq C_s(x)$. We set $\widetilde{D} = D \cup F$, and let $\widetilde{L}$ consist of all elements of L greater than $\max \widetilde{D}$ and $\varphi_e^F(x)$. Then $\widetilde{L} =^* L$, hence $\widetilde{L}$ is cone-avoiding, and clearly $\Phi_e^{G \cap A} \neq C_s$ for any set G satisfying $(\widetilde{D}, \widetilde{L})$.

Subcase 2b: $(D \cap \overline{A}, L - A)$ does not force agreement on i. We can then analogously define $(\widetilde{D}, \widetilde{L})$ so that $\Phi_e^{G \cap \overline{A}} \neq C_t$ for any set G satisfying $(\widetilde{D}, \widetilde{L})$.

In either case, $(\widetilde{D}, \widetilde{L})$ extends (D, L) and has the desired properties.

Now if A and C are computable from $\emptyset'$ and (D, L) is low, we can argue as in the proof of the previous lemma. Briefly, an index for $\mathscr{P}$ can, since A is Δ_2^0, be found $\emptyset'$-effectively from e, i, s, t and an index for (D, L), and from this index, S in Case 1 can be found $\emptyset'$-effectively so that S and $L - S$ are both low. The distinction between Cases 1 and 2 is L'-effective, hence $\emptyset'$-effective, from an index for $\mathscr{P}$, while a true subcase of Case 1 (Subcase 1a or 1b) can be chosen **d**-effectively. To distinguish between Subcases 2a and 2b, we can, since $A, C_s, C_t \leq_T \emptyset'$, search $\emptyset'$-effectively through ω and all finite sets until we find some $x \in \omega$ and either a finite set F satisfying $(D \cap A, L \cap A)$ with $\Phi_e^F(x) \downarrow \neq C_s(x)$, or a finite set F satisfying $(D \cap \overline{A}, L - A)$ with $\Phi_i^F(x) \downarrow \neq C_t(x)$. □

Lemma 2.4.8. *Let A be any set, (D, L) a cone-avoiding condition, and $e, i, s, t \in \omega$.*

1. *There is a cone-avoiding extension of (D, L) such that either $(G \cap A)'(e)$ is the same for all sets G satisfying this extension, or $\Phi_i^{G \cap \overline{A}} \neq C_t$ for all such G.*

2. *There is a cone-avoiding extension of (D, L) such that either $(G \cap \overline{A})'(i)$ is the same for all sets G satisfying this extension, or $\Phi_e^{G \cap A} \neq C_s$ for all such G.*

If, additionally, $A, C \leq_T \emptyset'$ and (D, L) is low, then $(\widetilde{D}, \widetilde{L})$ can be chosen low, and its index can be found **d***-effectively from e, i, and an index for (D, L).*

Proof. We prove (1), the proof of (2) being similar. Let $\mathscr{P}$ be the $\Pi_1^{0,L}$ class of all sets $S \subseteq L$ such that $(D \cap A, S)$ forces e out of the jump and $(D \cap \overline{A}, L - S)$ forces agreement on i.

Case 1: $\mathscr{P} \neq \emptyset$. By Theorem 2.2.2, relativized to L, there exists $S \in \mathscr{P}$ such that S and $L - S$ are cone-avoiding.

Subcase 1a: S is infinite. Set $(\widetilde{D}, \widetilde{L}) = (D, S)$. If G satisfies $(\widetilde{D}, \widetilde{L})$, then $e \notin (G \cap A)'$ as in Subcase 1a of the proof of Lemma 2.4.6.

Subcase 1b: $L - S$ is infinite. We set $(\widetilde{D}, \widetilde{L}) = (D, L - S)$, concluding that $\Phi_i^{G \cap \overline{A}} \neq C_t$ for all G satisfying $(\widetilde{D}, \widetilde{L})$, as in Subcase 1b of the proof of Lemma 2.4.7.

Case 2: $\mathscr{P} = \emptyset$. Then $L \cap A \notin \mathscr{P}$, so at least one of the following must obtain:

Subcase 2a: some finite set F with $\Phi_e^F(e) \downarrow$ satisfies $(D \cap A, L \cap A)$. We set $\widetilde{D} = D \cup F$, and let $\widetilde{L}$ consist of all elements of L greater than $\max \widetilde{D}$ and $\varphi_e^F(e)$. As in Subcase 2a of the proof of Lemma 2.4.6, it follows that $e \in (G \cap A)'$ for every set G satisfying $(\widetilde{D}, \widetilde{L})$.

Subcase 2b: $(D \cap \overline{A}, L - A)$ does not force agreement on i. Then for some $x \in \omega$, and some finite set F satisfying $(D \cap \overline{A}, L - A)$, $\Phi_i^F(x) \neq C_t(x)$ for any set G satisfying $(\widetilde{D}, \widetilde{L})$.

Now suppose A and C are computable from $\emptyset'$ and that (D, L) is low. An index for $\mathscr{P}$ can, since A is Δ_2^0, be found $\emptyset'$-effectively from e, i, t and an index for (D, L). In Case 1 then, this index can be used to find S, $\emptyset'$-effectively, so that S and $L - S$ are both low, in addition to being cone-avoiding. The distinction between Cases 1 and 2 is L'-effective, hence $\emptyset'$-effective, from an index for $\mathscr{P}$, and one may **d**-effectively choose a true subcase of Case 1. To distinguish Subcases 2a and 2b, we can, since $A, C_t \leq_T \emptyset'$, search $\emptyset'$-effectively through ω and all finite sets, until we either find a finite set F satisfying $(D \cap A, L \cap A)$ with $\Phi_e^F(e) \downarrow$, or else some $x \in \omega$ and some finite set F satisfying $(D \cap \overline{A}, L - A)$ with $\Phi_i^F(x) \downarrow \neq C_t(x)$. □

2.5 Combining strategies

In this section, we combine the forcing strategies from the previous section to prove Lemmas 2.3.1 and 2.3.2. Again, let $C_0, C_1, \ldots$ be a fixed sequence of non-computable sets with $C = \oplus_{i \in \omega} C_i$.

Lemma 2.5.1. *Fix a degree $\mathbf{d} \gg \mathbf{0}'$. For every uniformly computable sequence $R_0, R_1, \ldots$ of sets, there exists a cone-avoiding set G which is cohesive for this sequence and whose jump is of degree at most $\mathbf{d} \cup \deg(C)$.*

Proof. We force with computable conditions to build a set G which, for all $e, i \in \omega$, meets the following requirements:

$$\begin{array}{lcl} \mathcal{R}_{4e} & : & |G| \geq e; \\ \mathcal{R}_{4e+1} & : & G \subseteq^* R_e \text{ or } G \subseteq^* \overline{R_e}; \\ \mathcal{R}_{4e+2} & : & G'(e) \text{ is decided in the construction;} \\ \mathcal{R}_{4\langle e,s\rangle+3} & : & \Phi_e^G \neq C_s. \end{array}$$

Let a computable condition (D, L) and $n \in \omega$ be given. If $n = 4e$ we apply Lemma 2.4.3 (1), if $n = 4e + 1$ we apply (2) with $R = R_e$, and if $n = 4e + 2$ we apply (3). If $n = 4\langle e, s\rangle+3$ we apply Lemma 2.4.4 with $C = C_s$. By the effectiveness of Lemmas 2.4.3 and 2.4.4, we in any case obtain a computable condition $(\widetilde{D}, \widetilde{L})$ such that any set satisfying this extension meets $\mathcal{R}_n$, and an index for this extension can be found $(\mathbf{d} \cup \deg(C))$-effectively from an index for (D, L).

To complete the proof, let $(D_0, L_0) = (\emptyset, \omega)$, and iterate the preceding density construction to $(\mathbf{d} \cup \deg(C))$-effectively build a chain

$$(D_0, L_0) \succ (D_1, L_1) \succ \cdots$$

such that, for any $n \in \omega$, any set satisfying (D_{n+1}, L_{n+1}) meets $\mathcal{R}_n$. Clearly, $G = \bigcup_i D_i$ has the desired properties. □

Proof of Lemma 2.3.1. In view of the closing remark in Section 2.2, we may fix a cone-avoiding degree $\mathbf{a} \gg \mathbf{0}$. The $\{0, 1\}$-valued partial computable function

$$\tilde{p}(x) = \begin{cases} \Phi_e(x) & \text{if } \Phi_e(x)\downarrow \leq 1, \\ \uparrow & \text{otherwise,} \end{cases}$$

consequently has a total, $\{0, 1\}$-valued extension p with $\deg(p) \leq \mathbf{a}$. Hence, the sequence

$$\vec{R} = \{R_0, R_1, \ldots\},$$

where for $i \in \omega$,

$$R_i = \{x \in \omega : p(e, x) = 1\},$$

has degree $\leq \mathbf{a}$ and contains all the computable sets. Since $\deg(C_i) \not\leq \mathbf{a}$ for all $i \in \omega$, we may relativize Lemma 2.5.1 to $\mathbf{a}$ to consequently obtain a cone-avoiding r-cohesive set X, as desired.

Now suppose $C \leq_T \emptyset'$. We can choose $\mathbf{a}$ so that $\mathbf{a}' \leq \mathbf{0}' \cup \deg(C)$, meaning in fact that $\mathbf{a}$ can be chosen low, and hence such that $\mathbf{d} \gg \mathbf{a}' = \mathbf{0}'$. It follows, by the effectiveness of Lemma 2.5.1, that X can be found so that $\deg(X)' \leq \mathbf{d} \cup \deg(C) = \mathbf{d}$. □

We next present a slightly weaker form of Lemma 2.3.2, from which the latter will follow by an easy induction.

Lemma 2.5.2. *Fix a degree* $\mathbf{d} \gg \mathbf{0}'$.

1. *Any set A either contains or is disjoint from an infinite cone-avoiding set.*
2. *If $A, C \leq_T \emptyset'$, then A contains or is disjoint from an infinite cone-avoiding set whose jump has degree at most* $\mathbf{d}$.

Proof. To prove (1), first note that if there exists a cone-avoiding set S such that either $S \cap A$ is finite or $S \cap \overline{A}$ is finite, then we are done. So assume not. We force with cone-avoiding conditions (D, L) to construct a set G which meets, for all $e, i, s, t \in \omega$ the following requirements:

$$\begin{array}{lcl} \mathcal{R}_{2e} & : & |G \cap A| \geq e \text{ and } |G \cap \overline{A}| \geq e; \\ \mathcal{R}_{2\langle e,i,s,t\rangle+1} & : & \Phi_e^{G\cap A} \neq C_s \text{ or } \Phi_i^{G\cap \overline{A}} \neq C_t. \end{array}$$

We claim that these requirements are dense, in the sense that every cone-avoiding condition can be extended to another, with the property that any set satisfying the extension meets a prescribed requirement. So let (D, L) be a cone-avoiding condition, and fix $n \in \omega$. If $n = 2e$, then by our opening assumption we may apply Lemma 2.4.5. If $n = 2\langle e, i, s, t\rangle + 1$ we apply Lemma 2.4.7. In any case, we obtain a cone-avoiding condition $(\widetilde{D}, \widetilde{L}) \prec (D, L)$ such that any set satisfying this condition meets $\mathcal{R}_n$.

We conclude by setting $(D_0, L_0) = (\emptyset, \omega)$, and iterating the preceding argument to construct a chain

$$(D_0, L_0) \succ (D_1, L_1) \succ \cdots$$

such that, for any n, any set satisfying (D_{n+1}, L_{n+1}) meets $\mathcal{R}_n$. Taking $G = \bigcup_i D_i$, we note that $G \cap A$ and $G \cap \overline{A}$ must both be infinite by virtue of our even-numbered requirements. Now for every $e, i, s, t \in \omega$, either $\Phi_e^{G\cap A} \neq C_s$ or $\Phi_i^{G\cap\overline{A}} \neq C_t$ must hold, by virtue of the odd-numbered requirements. It follows that either $\Phi_e^{G\cap A} \neq C_s$ for every $e, s \in \omega$ or else $\Phi_i^{G\cap\overline{A}} \neq C_t$ for every $i, t \in \omega$, and we let X be $G \cap A$ or $G \cap \overline{A}$, depending on which of these two possibilities is true. Thus, X is an infinite cone-avoiding subset of A or $\overline{A}$.

For (2), without loss of generality assume that neither A nor $\overline{A}$ contains a low cone-avoiding subset, since otherwise the result is trivial. We proceed as in (1), except that we force with conditions that are low as well as cone-avoiding. We renumber the requirements $2e$ and $2\langle e, i, s, t\rangle + 1$ by $5e$ and $5\langle e, i, s, t\rangle + 1$, respectively, and add the following requirements:

$$\begin{array}{lcl} \mathcal{R}_{5\langle e,i\rangle+2} & : & (G \cap A)'(e) \text{ or } (G \cap \overline{A})'(i) \text{ is determined in the construction;} \\ \mathcal{R}_{5\langle e,i,t\rangle+3} & : & (G \cap A)'(e) \text{ is determined in the construction or } \Phi_i^{G\cap\overline{A}} \neq C_t; \\ \mathcal{R}_{5\langle e,i,s\rangle+4} & : & (G \cap \overline{A})'(i) \text{ is determined in the construction or } \Phi_e^{G\cap A} \neq C_s. \end{array}$$

We claim that these expanded requirements are $\mathbf{d}$-effectively dense. Given a low cone-avoiding condition (D, L), and an $n \in \omega$, we proceed as above for $n = 5e$ and $n = 5\langle e, i, s, t\rangle + 1$. If $n = 5\langle e, i\rangle + 2$, we appeal to Lemma 2.4.6, while if $n = 5\langle e, i, t\rangle + 3$ or $n = 5\langle e, i, s\rangle + 4$, we appeal to Lemma 2.4.8 (1) and (2), respectively. In any case, therefore, we pass to an extension $(\widetilde{D}, \widetilde{L})$ which is satisfied only by sets meeting $\mathcal{R}_n$. Moreover, since A and C are computable from $\emptyset'$, the effectiveness of Lemmas 2.4.5, 2.4.6, 2.4.7, and 2.4.8 implies that we can find $(\widetilde{D}, \widetilde{L})$ low and cone-avoiding, and that we can do so $\mathbf{d}$-effectively.

We set $(D_0, L_0) = (\emptyset, \omega)$, and iterate the density construction to $\mathbf{d}$-effectively obtain a chain

$$(D_0, L_0) \succ (D_1, L_1) \succ \cdots$$

with the property that any set satisfying (D_{n+1}, L_{n+1}) meets $\mathcal{R}_n$. Let $G = \bigcup_i D_i$. Then by virtue of the requirements congruent to 0, modulo 5, $G \cap A$ and $G \cap \overline{A}$ are both infinite. One of the following cases must now obtain:

Case 1: $\Phi_e^{G \cap A} \neq C_s$ and $\Phi_i^{G \cap \overline{A}} \neq C_t$ for every $e, i, s, t \in \omega$, so both $G \cap A$ and $G \cap \overline{A}$ are cone-avoiding. By virtue of the requirements congruent to 2, modulo 5, for every $e, i \in \omega$, either $(G \cap A)'(e)$ or $(G \cap \overline{A})'(i)$ is determined during the construction. Hence, either $(G \cap A)'(e)$ is determined for every $e \in \omega$, or else $(G \cap \overline{A})'(i)$ is determined for every $i \in \omega$. In the former case, $(G \cap A)'$ has degree at most $\mathbf{d}$, and so we let $X = G \cap A$. In the latter case, let $X = G \cap \overline{A}$.

Case 2: $\Phi_e^{G \cap A} = C_s$ for some $e, s \in \omega$. The requirements congruent to 1, modulo 5, imply that for every $e, i, s, t \in \omega$, either $\Phi_e^{G \cap A} \neq C_s$ or $\Phi_i^{G \cap \overline{A}} \neq C_t$ holds. Hence $G \cap \overline{A}$ is cone-avoiding. On the other hand, the requirements congruent to 4, modulo 5, imply that for every $e, i, s \in \omega$, either $(G \cap \overline{A})'(i)$ is determined in the construction or else $\Phi_e^{G \cap A} \neq C_s$. It follows, therefore, that $\Phi_i^{G \cap \overline{A}} \neq C_t$ for all $i, t \in \omega$, and that $(G \cap \overline{A})'(i)$ is determined in the construction for all $i \in \omega$, so $(G \cap \overline{A})'$ has degree at most $\mathbf{d}$. We set $X = G \cap \overline{A}$.

Case 3: $\Phi_i^{G \cap \overline{A}} = C_t$ for some $i, t \in \omega$. We can argue as in the previous case, but appeal instead to the requirements congruent to 1 and 3, modulo 5, and set $X = G \cap A$.

In any case, X is an infinite cone-avoiding subset of A or $\overline{A}$ and X' is determined in the construction. Since the construction is $\mathbf{d}$-effective, this implies that $\deg(X)' \leq \mathbf{d}$, which completes the proof. □

Proof of Lemma 2.3.2. We prove part (1) of the lemma in relativized form by induction on k, the case $k = 2$ being simply a statement of Lemma 2.5.2 (1). Since that lemma clearly relativizes, we may assume that the present result, along with all its relativizations, holds for some $k \geq 2$. Let $A_1, \ldots, A_{k+1}$ be a sequence of disjoint sets covering ω. Applying Lemma 2.5.2 (1) to A_{k+1} yields an infinite cone-avoiding set $\widetilde{X}$ either contained in or disjoint from A_{k+1}. In the former case, we can take $\widetilde{X}$ itself for the desired, cone-avoiding subset. In the latter, we have $\widetilde{X} \subseteq A_1 \cup \cdots \cup A_k$, and so $A_1 \cap \widetilde{X}, \cdots, A_k \cap \widetilde{X}$ is a sequence of k disjoint sets whose union is $\widetilde{X}$. Relativizing our inductive hypothesis to $\widetilde{X}$ yields an infinite cone-avoiding subset X of $A_i \cap \widetilde{X}$ for some $i \leq k$, whence since $A_i \cap \widetilde{X} \subseteq A_i$, the induction is complete.

For part (2), we again use induction, with the base case, this time, following from Lemma 2.5.2 (2) by taking $\mathbf{d}$ there low over $\mathbf{0}'$. Thus, assume the result for some $k \geq 2$, and let $A_1, \ldots, A_{k+1}$ be a sequence of disjoint Δ_2^0 sets covering ω. By Lemma 2.5.2 (2), A_{k+1} contains or disjoint from an infinite cone-avoiding set $\widetilde{X}$ whose jump has degree at most $\mathbf{d}$, hence a set which is low$_2$ since $\mathbf{d}' \leq \mathbf{0}''$. If $\widetilde{X} \subseteq A_{k+1}$, the proof is complete. Otherwise, we apply the induction hypothesis relative to $\widetilde{X}$ to the sequence $A_1 \cap \widetilde{X}, \ldots, A_k \cap \widetilde{X}$, finding an infinite cone-avoiding subset X of some $A_i \cap \widetilde{X}$ such that X is low$_2$ over $\widetilde{X}$, and hence low$_2$. □

2.6 A minimal pair of low_2 infinite homogeneous sets

The goal of this section is to prove the following extension of Theorem 2.3.4:

Corollary 2.6.1. *Every computable 2-coloring of pairs has a pair of low_2 infinite homogeneous sets whose degrees form a minimal pair.*

Fix a set B satisfying $B' \leq_T \emptyset''$. The corollary obviously follows from Theorem 1.3.5 and the next theorem, the proof of which will occupy most of the rest of this section.

Theorem 2.6.2. *Every computable 2-coloring of pairs has a low_2 infinite homogeneous set whose degree forms a minimal pair with* $\deg(B)$.

In our proof of Theorem 2.3.3 (2) above, we constructed a low_2 cone-avoiding infinite homogeneous set by controlling its first jump to lie below a fixed degree $\gg\mathbf{0}'$. We forced with low cone-avoiding Mathias conditions and appealed to a variant of the low basis theorem obtained by adding cone avoidance, namely Theorem 2.2.2. It would seem natural to replace the requirement that our conditions (D, L) be cone-avoiding by the requirement that $(\deg(L), \deg(B))$ form a minimal pair. (Indeed, this latter requirement is necessary if L is introreducible, i.e. computable from all of its infinite subsets.) However, it no longer seems possible to require that L be low. (For example, if $B = \emptyset'$ and L is even Δ^0_2, it would follow that L is computable, which is too much to ask, since in our proof we need to apply basis theorems for Π^0_1 classes to obtain L.) The same difficulty persists even if we insist that B be low_2, which would be sufficient to obtain Corollary 2.6.1. For let $\mathscr{P}$ be the Π^0_1 class of all $\{0, 1\}$-valued diagonally non-computable functions. Each member of $\mathscr{P}$ computes a fixed point free function by Lemma 4.1 of Jockusch, Lerman, Soare, and Solovay [37] (see Soare [60], Exercise V.5.9 and Remark VII.1.9). If we take for B any Δ^0_2 element of $\mathscr{P}$, then by Theorem 2 of Kučera [44] no Δ^0_2, let alone low, element of this class can have degree forming a minimal pair with $\deg(B)$.

Thus we instead prove the above theorem by directly forcing the second jump of the constructed set to be computable from $\emptyset''$, which we do by suitably extending the techniques of [5], Section 5. As in that article, one main advantage of presenting the first proof, that is, of controlling only the first jump, is that it uses a considerably simpler notion of forcing. In our case, of course, that proof also lent itself to both a new argument for Seetapun's theorem and an answer to Question 2.1.1, of which the theorem here does only the latter.

Our proof of Theorem 2.6.2 can, as usual, be broken up into an r-cohesive and a stable part, from which the theorem then follows analogously to the way Theorem 2.3.3 did from Lemmas 2.3.1 and 2.3.2. For completeness, we show how this proof goes following the statements of the necessary lemmas.

Lemma 2.6.3. *There exists a low_2 r-cohesive set whose degree makes a minimal pair with B.*

Here we pause to note an extension of Theorem 2.5 of Jockusch and Stephan [40] on the existence of a low_2 cohesive degree. It follows at once from Lemma 2.6.3, the result of Jockusch and Stephan just cited, and the fact mentioned above (cf [40], Corollary 2.4) that the cohesive and r-cohesive degrees coincide.

Corollary 2.6.4. *There exists a minimal pair of low_2 cohesive degrees.*

Lemma 2.6.5. *Any Δ^0_2 set A contains or is disjoint from an infinite low_2 set whose degree forms a minimal pair with B.*

Proof of Theorem 2.6.2. Fix a computable 2-coloring of pairs f, and by Lemma 2.6.3, fix also a low_2 r-cohesive set X with $(\deg(X), \deg(B))$ a minimal pair. Define a stable 2-coloring of pairs g by $g(\{s,t\}) = f(\{p(s), p(t)\})$ for all $\{s,t\} \in [\omega]^2$, where p is the principal function of X. By Lemma 1.3.7 (2), relative to X, there exists a $\Delta^{0,X}_2$ set A such that, if S is any infinite subset of A or $\overline{A}$, $X \oplus S$ computes an infinite homogeneous set for g. We have $B' \leq_T \emptyset'' \leq_T X''$, so we may relativize Lemma 2.6.5 to X to conclude that either A or $\overline{A}$ contains an infinite set S such that $(S \oplus X)'' \leq_T X'' \leq_T \emptyset''$ and every set Turing reducible to both $S \oplus X$ and B is Turing reducible to X. Letting $\widetilde{H}$ be an infinite homogeneous set for g computed by S, we find that $H = p(\widetilde{H})$ is a low_2 infinite homogeneous set for f. Then H is low_2 since $H \leq_T X \oplus S$ and $X \oplus S$ is low_2. Suppose now that Y satisfies $Y \leq_T H$ and $Y \leq_T B$. Then $Y \leq_T S \oplus X$ (as $H \leq_T S \oplus X$). It follows that $Y \leq_T X$ since Y is Turing reducible to both B and $S \oplus X$. Since Y is Turing reducible to both X and B, and the degrees of X and B form a minimal pair, it follows that Y is computable, as needed to complete the proof that $\deg(H)$ and $\deg(B)$ form a minimal pair. □

The proofs of Lemmas 2.6.3 and 2.6.5 are obtained by modifying those in Cholak, Jockusch, and Slaman [5], Section 5, of Theorems 3.3 and 3.6 of the same article. In particular, we change the forcing conditions used, add minimal pair requirements, and modify some versions of "smallness" to take account of minimal pair requirements. The proofs we are modifying are quite involved, and we have not made a large number of changes. Hence, we proceed by assuming the reader is familiar with Section 5 of [5], and instead focus on the differences. Admittedly, this is at the cost of making the proof deceptively short. The interested reader may wish to consult [5], pp. 14–26, for an in-depth introduction with a view towards the underlying motivation, and pp. 16–22 for the technical details.

2.6.1 Proving Lemma 2.6.3

Let $\{\pi_e(G, \vec{x})\}$ be an effective enumeration of the Π^0_1 formulas in the displayed variables, and let $\{R_e\}$ be a listing of the computable sets such that an index of the characteristic function of R_e is $\emptyset''$-computable from e. We seek to construct a set G meeting, for each

$e \in \omega$, the following requirements:

$$\begin{array}{lcl} \mathcal{R}_{4e} & : & |G| \geq e; \\ \mathcal{R}_{4e+1} & : & G \subseteq^* R_e \text{ or } G \subseteq^* \overline{R_e}; \\ \mathcal{R}_{4e+2} & : & (\exists \vec{x})\pi_e(G, \vec{x}) \text{ is decided in the construction;} \\ \mathcal{R}_{4e+3} & : & \Phi_e^G = \Phi_e^B = f \text{ total } \implies f \text{ computable.} \end{array}$$

We proceed as in [5], Section 5.1, but instead of forcing with low conditions we shall force with those in the next definition. Otherwise, our notation and terminology will be exactly as in [5], Definitions 5.1 and 5.2, and the accompanying discussions.

Definition 2.6.6. A condition (D, L) is *low*$_2$ *and minimal pair forming* if L is low$_2$ and of degree forming a minimal pair with $\deg(B)$. An *index* for such a condition is a pair of numbers (d, ℓ) in which d is the canonical index of D and ℓ is a $\Delta_1^{0,\emptyset''}$ index for L''.

Our main task is to adapt Lemmas 5.3, 5.4, and 5.5 of [5] to the above conditions. The first of these carries over easily. For if S is a finite set of Σ_2^0 formulas with at most G free and $\psi(G)$ is a Π_1^0 instance of some formula in S, then the proof of this lemma actually shows that any S-large condition (D, L) can be L'-effectively, finitely extended to a condition forcing $\neg\psi(G)$. This extension will clearly be low$_2$ and minimal pair forming if (D, L) is, in which case it can thus be found $\emptyset''$-effectively from an index for (D, L) and an index for ψ. Lemma 5.4 of [5] is adapted as follows.

Lemma 2.6.7. *Let a low*$_2$*, minimal pair forming condition (D, L) of known index be given, along with the canonical index of a finite set S of Σ_2^0 formulas free in at most G. There is a $\emptyset''$-effective procedure by which to decide, from these indices, whether or not this condition is S-large. If so, there exists an $n \in \omega$ and a sequence $(D_i, L_i, k_i, \vec{w}_i : i < n)$ witnessing this fact such that each L_i is low*$_2$ *and of degree forming a minimal pair with* $\deg(B)$*. Moreover, this sequence, together with indices for the (D_i, L_i), can be found $\emptyset''$-effectively.*

Proof. In the proof of Lemma 5.4 of [5], the definition of (D, L) being S-small is written in the form $(\exists z)(\exists Z)P(z, Z, D, L, S)$, where P is a Π_1^0 predicate with parameters Z and L of known index (we are identifying the finite sets D and S with their canonical indices). As pointed out there, the predicate $(\exists Z)P(z, Z, D, L, S)$ is consequently $\Pi_1^{0,L}$. It follows that, as a predicate of D and S, $(\exists z)(\exists Z)P(z, Z, D, L, S)$ is $\Sigma_2^{0,L}$, and we can consequently find a $\Delta_1^{0,\emptyset''}$ index for it from a $\Delta_1^{0,\emptyset''}$ index for L''. Hence, from the appropriate indices for (D, L) and S, $\emptyset''$ can decide whether or not (D, L) is S-small. If it is, we search $\emptyset''$-effectively for a $z \in \omega$ satisfying $(\exists Z)P(z, Z, D, L, S)$, and then, since the degrees of L and B form a minimal pair, use Theorem 2.2.6 relative to L to find a solution Z to $P(z, Z, D, L, S)$ which is low$_2$ over L, hence low$_2$, and whose degree forms a minimal pair with $\deg(B)$. The proof now concludes much like that of Lemma 5.4 of [5], but appealing to the effectiveness of Theorem 2.2.6 instead of the low basis theorem. □

The statement of Lemma 5.5 of [5] now carries over verbatim to low$_2$, minimal pair forming conditions, the proof being identical except of course that all references to Lemma 5.4 of [5] are replaced by references to Lemma 2.6.7 above.

Under these modifications, the construction of G can go ahead, *mutatis mutandis*, as it did in [5], Section 5.1.3. For each e, the requirement $s = 4e + 3$ is met by applying a modified version of Lemma 2.4.4 in which C is replaced by Φ_e^B to the low$_2$, minimal pair forming condition (D_{s-1}, L_{s-1}) given at stage s. (The proof of this modified lemma goes exactly as that of Lemma 2.4.4, down to the same case division employed there, but since Φ_e^B may not be total we use an oracle for B' to find j in Case 1. Since L_{s-1} is low$_2$ and $B' \leq_T \emptyset''$, a $\emptyset''$-oracle still suffices to carry out the entire construction.)

2.6.2 Proving Lemma 2.6.5

Let A be a Δ_2^0 set, and, in keeping with the notation of Cholak, Jockusch, and Slaman [5], p. 19, define $A_0 = A$ and $A_1 = \overline{A}$. Since we are forcing with conditions that are merely low $_2$ and not necessarily low, it is not sufficient for our needs to assume that neither A_0 nor A_1 has an infinite low subset of degree making a minimal pair with $\deg(B)$. Thus, we assume that neither A_0 nor A_1 has even a low$_2$ such subset, and shall construct a subset $G \subseteq A_i$ for some $i < 2$ to contradict this assumption.

Constructing G inside of A_0.

Following [5], Section 5.2.1, we begin by restricting to low$_2$, minimal pair forming conditions (D, L) with $D \subset A_0$. We define what it means for a such a condition to be S-small$_0$ or S-large$_0$ as in [5], p. 19. However, in order to ensure that the set G we construct has degree forming a minimal pair with $\deg(B)$, we additionally need the following secondary notion of largeness:

Definition 2.6.8. Let

$$S = \{(\exists\vec{x})\psi_0(G, \vec{x}), \ldots, (\exists\vec{x})\psi_k(G, \vec{x})\}$$

be a finite set of Σ_2^0 formulas free in at most G, the ψ_i being Π_1^0, and let $e \in \omega$. A condition (D, L) is (S, e)-*small*$_0$ if there exists an $n \in \omega$ and a sequence $(D_i, L_i, k_i, \vec{w}_i : i < n)$ such that the L_i partition L, each k_i is a number $\leq k$ and each $\vec{w}_i$ a tuple of numbers, each (D_i, L_i) is a precondition extending (D, L) with $D_i \subset A_0$, and for each $i < n$, either $\max L_i \leq \max \vec{w}_i$, or (D_i, L_i) forces $\psi_{k_i}(G, \vec{w}_i)$, or (D_i, L_i) forces agreement on e. A condition is (S, e)-*large*$_0$ if it is not (S, e)-small$_0$. (This definition is the same as the definition of "S-small" except for the final conjunct, which is included for the sake of the eth minimal pair requirement.)

Notice that if a condition is (S, e)-large$_0$ then it is S-large$_0$, and every finite extension of it is also (S, e)-large$_0$. We begin by assuming that whenever a condition (D, L) is S-large$_0$, then $(D, L \cap A_0)$ is (S, e)-large$_0$ for all $e \in \omega$, so that, in particular, whenever (D, L) is an S-large$_0$ condition, so is $(D, L \cap A_0)$. If this assumption holds, we shall construct a suitable $G \subseteq A_0$, and otherwise we shall construct $G \subseteq A_1$. By this assumption, Lemma 5.3 can be adapted to conditions (D, L) with $D \subset A_0$ just as in Section 5.2.1 of [5]. The following gives the necessary adaptation of Lemma 5.4 of [5] for our purposes:

Lemma 2.6.9. *Let an index for a low$_2$, minimal pair forming condition (D, L) with $D \subset A_0$ be given, along with an $e \in \omega$, and the canonical index of a finite set S of Σ^0_2 formulas free in at most G. There are $\emptyset''$-effective procedures by which to decide, from these indices, whether or not this condition is S-small$_0$, and, from e and these indices, whether or not it is (S, e)-small$_0$. If it is S-small$_0$ or (S, e)-small$_0$, there exists an $n \in \omega$ and a sequence $(D_i, L_i, k_i, \vec{w}_i : i < n)$ witnessing this fact such that each $D_i \subset A_0$, and each L_i is low$_2$ and of degree forming a minimal pair with* $\deg(B)$. *Moreover, this sequence, together with indices for the (D_i, L_i), can be found $\emptyset''$-effectively.*

Proof. In Section 5.2.1 of [5] the definition of (D, L) being S-small$_0$ is put in the form $(\exists z)[R^A(z) \wedge (\exists Z)P(z, Z, D, L, S)]$, where R^A is a $\Delta^{0,A}_1$ predicate, and P is the same Π^0_1 predicate defined in the proof of Lemma 2.6.7. The definition of (D, L) being (S, e)-small$_0$ can similarly be put in the form $(\exists z)[\widetilde{R}^A(z) \wedge (\exists Z)\widetilde{P}(z, Z, e, D, L, S)]$, where $\widetilde{R}^A$ is $\Delta^{0,A}_2$ and $\widetilde{P}$ is Π^0_1. Thus, (D, L) being S-small$_0$ and (D, L) being (S, e)-small$_0$ can each be defined by some $\Sigma^{0,A\oplus L'}_1$ predicate, and hence by one computable from $(A \oplus L')'$. But since $A \leq_T \emptyset' \leq_T L'$, and since L is low$_2$, it follows that $(A \oplus L')' \leq_T \emptyset''$, and hence that each of these predicates is still Δ^0_3. The proof can now be completed just as that of Lemma 2.6.7. □

Now Lemma 5.5 of [5] can be adapted in the obvious way to low$_2$, minimal pair forming conditions (D, L) with $D \subset A_0$. Its proof is the same, except that Lemma 2.6.9 above is used in place of Lemma 5.4 of [5].

Finally, we have the following lemma which will ensure that the set we end up constructing is of degree forming a minimal pair with $\deg(B)$:

Lemma 2.6.10. *Let (D, L) be a low$_2$, minimal pair forming condition which is S-large$_0$, and let $e \in \omega$. There exists a low$_2$, minimal, pair forming condition extending (D, L) which is S-large$_0$, and which is satisfied only by sets G such that Φ^G_e is computable whenever it is total and equal to Φ^B_e. Furthermore, an index for this extension can be found $\emptyset''$-effectively from e and an index for (D, L).*

Proof. By Lemma 2.6.9, we can $\emptyset''$-effectively from e and an index for (D, L) determine whether or not this condition is (S, e)-small$_0$.

If it is, we can $\emptyset''$-effectively find an $n \in \omega$ and a sequence $(D_i, L_i, k_i, \vec{w}_i : i < n)$ witnessing this in which each (D_i, L_i) is low$_2$ and minimal pair forming. There must clearly be some $i < n$ so that L_i is infinite and (D_i, L_i) forces agreement on e, else we could easily form a sequence of preconditions to witness that (D, L) is S-small$_0$. Furthermore, if each such (D_i, L_i) was S-small$_0$, then we could replace each one by the preconditions from the sequence witnessing this fact, thereby again obtaining that (D, L) is S-small$_0$. Thus, we may $\emptyset''$-effectively choose i so that L_i is infinite and (D_i, L_i) is S-large$_0$ and forces agreement on e. This condition can serve as the desired extension, since if G satisfies it then Φ^G_e is clearly L-computable whenever it is total.

If (D, L) is (S, e)-large$_0$ then by assumption so is $(D, L \cap A_0)$, meaning in particular that it can not force agreement on e. We can thus $\emptyset''$-effectively find an $x \in \omega$ and a finite set F satisfying $(D, L \cap A_0)$ so that $\Phi^F(x) \downarrow\neq \Phi^B(x)$. Set $\widetilde{D} = D \cup F$ and let $\widetilde{L}$ consist of all elements of F greater than $\max F$ and $\varphi^F(x)$. Clearly, $(\widetilde{D}, \widetilde{L})$ serves as the desired extension. □

We now construct an infinite set $G \subseteq A_0$ which is low$_2$ and of degree forming a minimal pair with $\deg(B)$ analogously to the way we constructed an r-cohesive such set above, only forcing with large$_0$ conditions in place of large ones. Our assumption that A_1 has no low$_2$ subset of degree forming a minimal pair with $\deg(B)$ ensures that we meet requirement $\mathcal{R}_{4e}$ of Section 2.6.1 above, our adaptations of Lemmas 5.3 and 5.5 of [5] ensure that we meet requirement $\mathcal{R}_{4e+2}$, and Lemma 2.6.10 ensures that we meet $\mathcal{R}_{4e+3}$.

Constructing G inside of A_1.

Assume next that there exists a low$_2$, minimal pair forming condition $(\widetilde{D}, \widetilde{L})$, an $\widetilde{e} \in \omega$, and a finite set

$$\widetilde{S} = \{(\exists \vec{x})\widetilde{\psi}_0(G, \vec{x}), \ldots, (\exists \vec{x})\widetilde{\psi}_\ell(G, \vec{x})\}$$

of Σ^0_2 formulas free in at most G, such that $\widetilde{D} \subset A_0$, $(\widetilde{D}, \widetilde{L})$ is $\widetilde{S}$-large$_0$, but $(\widetilde{D}, \widetilde{L} \cap A_0)$ is $(\widetilde{S}, \widetilde{e})$-small$_0$. This is a slightly different assumption from that of [5], Section 5.2.2, resulting from the fact that we needed Definition 2.6.8 above in order to obtain Lemma 2.6.10. Nonetheless, essentially the same argument as there will work here, with only some minor technical alterations along the way. The basic idea is that we switch from trying to build the desired set G inside of A_0 to building it inside its complement, A_1.

We need to suitably modify the notion from [5], p. 20, of a condition being 1-acceptable, and then that from [5], Definition 5.6, of a condition being S-small$_1$. Here we remark in passing that the latter is misstated in [5], and should, for the purposes of Section 5.2.2 of that article, be defined as in part (3) of the next definition, but with (2e) below not obtaining for any $i < n$.

Definition 2.6.11. Let a condition (D, L), an $e \in \omega$, and a finite set

$$S = \{(\exists \vec{x})\psi_0(G, \vec{x}), \ldots, (\exists \vec{x})\psi_k(G, \vec{x})\}$$

of Σ^0_2 formulas free in at most G, be given.

1. We say (D, L) is 1-*acceptable* if $D \subset A_1$, $L \subseteq \widetilde{L}$, and $(\widetilde{D}, L)$ is $(\widetilde{S}, \widetilde{e})$-large$_0$.
2. We say (D, L) is (S, e)-*small*$_1$ if it is 1-acceptable and there exists an $n \in \omega$ and a sequence $(D_i, L_i, k_i, \vec{w}_i : i < n)$ such that the L_i partition L, each k_i is a number $\leq \max\{k, \ell\}$ and each $\vec{w}_i$ a tuple of numbers, each (D_i, L_i) is a precondition, and for each $i < n$, at least one of the following holds:
   - (a) $\max L_i \leq \max \vec{w}_i$;
   - (b) $D_i \subset A_1$, and (D_i, L_i) extends (D, L) and forces $\psi_{k_i}(G, \vec{w}_i)$;
   - (c) $D_i \subset A_1$, and (D_i, L_i) extends (D, L) and forces agreement on e;

(d) $D_i \subset A_0$, and (D_i, L_i) extends $(\widetilde{D}, \widetilde{L})$ and forces $\widetilde{\psi}_{k_i}(G, \vec{w}_i)$;

(e) $D_i \subset A_0$, and (D_i, L_i) extends $(\widetilde{D}, \widetilde{L})$ and forces agreement on $\widetilde{e}$.

3. If (D, L) is (S, e)-small$_1$ via a sequence for which (2c) above does not obtain for any $i < n$, then we say (D, L) is S-small$_1$.

4. We say (D, L) is (S, e)-*large*$_1$, respectively S-*large*$_1$, if it is not (S, e)-small$_1$, respectively if it is not S-small$_1$.

Any condition which is (S, e)-large$_1$ obviously enjoys analogues of the two basic properties mentioned following Definition 2.6.8 for (S, e)-large$_0$ conditions. It is also easily seen that if (D, L) is 1-acceptable and finitely extended by $(\widehat{D}, \widehat{L})$ where $\widehat{D} \subset A_1$, then this extension is also 1-acceptable. That we shall not at any future stage be forced to switch back to constructing G inside of A_0 is ensured by the following analogue of Lemma 5.7 of [5]:

Lemma 2.6.12. *Let S be a finite set of Σ^0_2 formulas free in at most G, and let $e \in \omega$. If (D, L) is S-large$_1$, then so is $(D, L \cap A_1)$. Likewise, if (D, E) is (S, e)-large$_1$, then so is $(D, L \cap A_1)$.*

Proof. By assumption, $(\widetilde{D}, \widetilde{L} \cap A_0)$ is $(\widetilde{S}, \widetilde{e})$-small$_0$, so fix a sequence witnessing this fact. By comparing Definitions 2.6.8 and 2.6.11, it is easy to see that this sequence witnesses that $(D, L \cap A_0)$ is S-small$_1$ for any S, and hence (S, e)-small$_1$ for any S and e. This sequence could consequently be combined with any sequence witnessing that $(D, L \cap A_1)$ is S-small$_1$ or (S, e)-small$_1$ to witness that (D, L) is itself, respectively, S-small$_1$ or (S, e)-small$_1$. □

We can thus adapt Lemma 5.3 of [5] to 1-acceptable, low$_2$, minimal pair forming conditions (D, L) with $D \subset A_1$ almost exactly as we adapted it in Section 2.6.2 to conditions with $D \subset A_0$. Similarly, Lemma 2.6.9 carries over simply by replacing all instances of A_0, small$_0$ and large$_0$ by A_1, small$_1$ and large$_1$, respectively. This leaves Lemma 5.5 of [5] and Lemma 2.6.10, the proofs of which require a minor case analysis in order to work for 1-acceptable conditions. This is essentially what is required to adapt the former lemma in Section 5.2.2 of [5], but since it is not explicitly described there, we provide the details. The latter lemma can be handled similarly.

Lemma 2.6.13. *Let (D, L) be a low$_2$, minimal pair forming condition which is 1-acceptable, S-large$_1$, and $S \cup \{(\exists \vec{x})\psi(G, \vec{x})\}$-small$_1$ for some Π^0_1 formula $\psi(G, \vec{x})$ free in at most G and $\vec{x}$. There exists a low$_2$, minimal pair forming extension of this condition which is 1-acceptable and S-large$_1$, and which forces a Π^0_1 instance of $\psi(G, \vec{x})$. An index of this extension can be found $\emptyset''$-effectively from an index for (D, L), the canonical index for S, and an index for ψ.*

Proof. Fix $n \in \omega$ and a sequence $(D_i, L_i, k_i, \vec{w}_i : i < n)$ with each (D_i, L_i) low$_2$ and minimal pair forming to witness that (D, L) is $S \cup \{(\exists \vec{x})\psi(G, \vec{x})\}$-small$_1$. First, notice that if no (D_i, L_i) with L_i infinite had $D_i \subset A_1$ then our sequence would in fact witness that (D, L)

is $\widehat{S}$-small$_1$ for any set $\widehat{S}$, and hence in particular that it is S-small$_1$. Now let I_0 consist of all $i < n$ such that L_i is infinite and (D_i, L_i) is 1-acceptable, noting that this must be non-empty else each condition (D_i, L_i) with L_i infinite and $D_i \subset A_1$ could be replaced in our sequence by preconditions from a sequence witnessing that $(\widetilde{D}, L_i)$ is $(\widetilde{S}, \widetilde{e})$-small$_0$ to obtain a sequence witnessing that $(\widetilde{D}, L)$ is $(\widetilde{S}, \widetilde{e})$-small$_0$ and hence that (D, L) is not 1-acceptable. Clearly, each (D_i, L_i) with $i \in I_0$ forces a Π^0_1 instance of some formula in $S \cup \{(\exists \vec{x})\psi(G, \vec{x})\}$. Let I_1 consist of all $i \in I_0$ with (D_i, L_i) S-large$_1$. If I_1 was empty, then, depending on whether i is or is not an element of I_0, each (D_i, L_i) with $D_i \subset A_1$ could be replaced in our original sequence either by preconditions from a sequence witnessing that $(\widetilde{D}, L_i)$ is S-small$_0$, or else by preconditions from a sequence witnessing that (D_i, L_i) is S-small$_1$, yielding a sequence witnessing that (D, L) is S-small$_1$. By the effectiveness of Lemma 2.6.9, we may therefore $\emptyset''$-effectively from the appropriate indices find an $i \in I_1$. Since (D_i, L_i) is S-large$_1$, it can not force a Π^0_1 instance of any formula in S, and hence it must force $\psi(G, \vec{w}_i)$. □

All the lemmas are now combined in the usual way to build the desired $G \subseteq A_1$, but forcing with low$_2$, minimal pair forming conditions which are 1-acceptable and large$_1$, starting with the $\emptyset$-large$_1$ condition $(\emptyset, \widetilde{L})$.

2.7 Questions

Stephen Simpson asked the following interesting questions pertaining to Theorem 2.3.4 and Corollary 2.6.1:

Question 2.7.1 (Simpson)**.** Given two non-computable sets C_0 and C_1, does every computable 2-coloring of pairs admit a low$_2$ infinite homogeneous set H that is cone-avoiding for both?

Question 2.7.2 (Simpson)**.** Does every computable 2-coloring of pairs admit a pair of infinite homogeneous sets whose degrees form a minimal pair and whose join is low$_2$?

In response to the first of these, we have the following strengthening of Theorem 2.3.4:

Theorem 2.7.3. *Let $C_0, \ldots, C_n$ be a finite sequence of non-computable sets. Every computable 2-coloring f of pairs has a low$_2$ infinite homogeneous set which is cone-avoiding for this sequence.*

Proof. Write $\{0, \ldots, n\}$ as a disjoint union $I \cup J \cup K$, where $C_i \leq_T \emptyset'$ for all $i \in I$, $C_j \not\leq_T \emptyset'$ and $C_j \leq_T \emptyset''$ for $j \in J$, and $C_k \not\leq_T \emptyset''$ for $k \in K$. By Theorem 2.2.2, relativized to $\emptyset'$ and applied to the $\emptyset'$-computable tree all of whose infinite paths have degree $\gg \mathbf{0}'$, we can find a degree $\mathbf{d} \gg \mathbf{0}'$ such that $\mathbf{d}' \leq \mathbf{0}''$ and $\deg(C_j) \not\leq \mathbf{d}$ for all $j \in J$ (if $J = \emptyset$, let $\mathbf{d}$ be any degree $\gg \mathbf{0}'$ which is low over $\mathbf{0}'$). Now by Theorem 2.3.3 (2), there exists an infinite homogeneous set H for f such that $\deg(H)' \leq \mathbf{d}$ and $C_i \not\leq_T H$ for all $i \in I$ (if $I = \emptyset$, let H be any infinite homogeneous set with jump of degree at most $\mathbf{d}$). Since H is low$_2$ we also have $C_k \not\leq_T H$ for all $k \in K$, as wanted. □

We do not know the answer to Question 2.7.2. A natural attempt at obtaining an affirmative answer is to assume that the set B in Section 6 is low$_2$, rather than merely that $B' \leq_T \emptyset''$, and then to try to show that the homogeneous set asserted to exist by Theorem 2.6.2 can be made to satisfy $(H \oplus B)'' \leq_T \emptyset''$. Our obstacle to doing this is that we do not know how to strengthen Theorem 2.2.6 from asserting the existence of a low$_2$ infinite path to asserting the existence of one whose join with B is low$_2$.

As already mentioned at the end of Section 3, we do not know the answer to the next question.

Question 2.7.4. Given a non-computable set C, does every computable 2-coloring of pairs admit an infinite homogeneous set $H \not\geq_T C$ whose jump is of degree at most $\mathbf{d}$, where $\mathbf{d}$ is a given degree $\gg \mathbf{0}'$?

In a similar vein, we can ask the following:

Question 2.7.5. Given a degree $\mathbf{d} \gg \mathbf{0}'$, does every computable 2-coloring of pairs admit a pair of infinite homogeneous sets whose degrees form a minimal pair, and whose jumps are each of degree at most $\mathbf{d}$?

In Section 2.6 we control the second jump of the constructed set, and so it does not appear that this method would be useful in finding an affirmative answer to either of the above questions. But it does not appear that the methods of Sections 2.4 and 2.5 could be easily adapted to give affirmative answers either. If we tried to do so for the first question, we would certainly need a different case analysis in the proof of Theorem 2.3.4, but this would in turn demand a different proof of Theorem 2.2.2. As for the second question, we would seem to need a version of Theorem 2.2.6 which is effective in a given degree $\gg \mathbf{0}'$, but we do not know of a proof of this theorem which does not make essential use of a $\emptyset''$-oracle.

CHAPTER 3
RAMSEY'S THEOREM AND MEASURE

3.1 Introduction

In this chapter, we concentrate on the stable form of Ramsey's theorem, which has played an important role in the study of Ramsey's theorem proper. We restrict our analysis from the class of all stable colorings to "large" or non-null subclasses of it, using the notion of Δ^0_2 *measure* (see Section 3.2). A previous result in this direction was obtained by Hirschfeldt and Terwijn [29, Theorem 3.1] and appears as Theorem 3.2.5 below. The focus here is to classify properties of infinite homogeneous sets of stable colorings not into those that are and are not universal, as one typically does in computability theory and reverse mathematics, but into those that are and are not typical.

Since computable stable colorings always have Δ^0_2 infinite homogeneous sets, it is natural to ask whether in fact they always have low infinite homogeneous sets. The next result gives a negative answer, and thus, together with with Theorem 1.3.5, it provides sharp bounds with respect to the low$_n$ hierarchy on the complexities of possible infinite homogeneous sets of computable stable colorings.

Theorem 3.1.1 (Downey, Hirschfeldt, Lempp, and Solomon [13])**.** *There exists a computable stable coloring with no low infinite homogeneous set.*

The next result gives an improvement over the Δ^0_2 bound with respect to the arithmetical hierarchy.

Theorem 3.1.2 (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [26], Corollary 4.6)**.** *Every computable stable coloring has an infinite homogeneous set of degree strictly below* $\mathbf{0}'$*.*

The above mentioned result of Hirschfeldt and Terwijn from [29] is a measure-theoretic analysis of Theorem 3.1.1 and shows that this theorem is atypical in that the collection of computable stable colorings that actually do have a low infinite homogeneous set is not null in the sense of Δ^0_2 measure.

Here, we similarly analyze Theorems 1.3.5 and 3.1.2. As both theorems are positive, we turn our attention to uniformity. Mileti [46, Theorem 5.3.7 and Corollary 5.4.6] showed that neither of these theorems admits a uniform proof, in the sense that there is no single low$_2$ degree, respectively no single degree strictly below $\mathbf{0}'$, bounding the degree of an infinite homogeneous set for every computable stable coloring. In Section 3.3, we extend one of his results by showing the following:

Theorem 3.1.3. *For each* $\mathbf{d} < \emptyset'$*, the class of computable stable colorings having an infinite homogeneous set of degree at most* $\mathbf{d}$ *is* Δ^0_2 *null.*

We then prove the following theorem showing that uniformity results can differ between the class of all computable stable colorings and more general subclasses of it that are not Δ^0_2 null:

Theorem 3.1.4. *There is a degree* $\mathbf{d} \leq \emptyset''$ *such that the class of computable stable colorings having an infinite homogeneous set of degree at most* $\mathbf{d}$ *is not* Δ^0_2 *null but is not equal to the class of all such colorings.*

The Δ^0_3 bound also gives a partial result in the direction of showing that $< \mathbf{0}'$ in the preceding theorem can not be replaced by low_2.

In Section 3.4, we turn briefly to studying the Muchnik degrees of infinite homogeneous sets of computable stable colorings. And in Section 3.5, we introduce several combinatorial principles related to SRT^2_2 from a measure-theoretic viewpoint, and study these in the context of reverse mathematics. In particular, we introduce the principle ASRT^2_2 which asserts that "non-negligibly many", rather than all, computable stable colorings admit an infinite homogeneous set, and show that it lies strictly in between SRT^2_2 and the axiom DNR, and that it does not imply WKL_0.

3.2 Δ^0_2 measure

Martin-Löf introduced the definition of 1-randomness as a constructive notion of nullity. A stricter approach is that of Schnorr [55], which we now briefly recall.

Definition 3.2.1. A *martingale* is a function $M : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ that satisfies, for every $\sigma \in 2^{<\omega}$, the averaging condition

$$2M(\sigma) = M(\sigma 0) + M(\sigma 1). \tag{3.1}$$

We say that M *succeeds* on a set A if $\limsup_{n\to\infty} M(A \upharpoonright n) = \infty$, and we let the *success set* of M, $S[M]$, be the class of all sets on which M succeeds.

Unless otherwise noted, we shall assume that all our martingales are rational-valued, so that it makes sense to speak of martingales being computable. A class of $\mathscr{C} \subseteq 2^\omega$ is said to be *computably null* if there is a computable martingale M which succeeds on each $A \in \mathscr{C}$, and *Schnorr null* if in fact there is a computable non-decreasing unbounded function h with $\limsup_{n\to\infty} \frac{M(A\upharpoonright n)}{h(n)} = \infty$ (i.e., the martingale succeeds sufficiently fast). The motivation here comes from the following classical result of Ville (the interested reader may wish to consult [65], Section 1.5, for a thorough treatment of effective measure, and [12] for background on algorithmic complexity):

Theorem 3.2.2 (Ville)**.** *A class* $\mathscr{C} \subseteq 2^\omega$ *has Lebesgue measure* 0 *if and only if there is martingale* M *such that* $\mathscr{C} \subseteq S[M]$.

By relativizing computable nullity to $\emptyset'$, we thus obtain a notion of nullity for the class of Δ^0_2 sets.

Definition 3.2.3. A class $\mathscr{C} \subseteq 2^\omega$ is Δ^0_2 *null* (or has Δ^0_2 *measure* 0) if there exists a Δ^0_2 martingale M such that $\mathscr{C} \subseteq S[M]$.

The study of this measure has been principally by Terwijn [65, 66] and by Terwijn and Hirschfeldt [29], although in more general contexts it goes back to Schnorr (see [55], p. 55). It is a reasonable notion of nullity in that many of the basic properties one would expect to hold, do.

Proposition 3.2.4 (Lutz, see [65], Section 1.5)**.**

1. *The class of all Δ^0_2 sets is not Δ^0_2 null.*
2. *For every Δ^0_2 set A, $\{A\}$ is Δ^0_2 null.*
3. *If $\mathscr{C}_0, \mathscr{C}_1, \ldots$ is a sequence of subsets of 2^ω and $M_0, M_1, \ldots$ a uniformly Δ^0_2 sequence of martingales such that $\mathscr{C}_e \subseteq S[M_e]$ for every $e \in \omega$, then $\bigcup_{e\in\omega} \mathscr{C}_e$ is Δ^0_2 null.*

Additionally, Lutz and Terwijn (see [65], Theorem 6.2.1) have shown that for every $A >_T \emptyset$, the upper cone $\{B : B \geq_T A\}$ is Δ^0_2 null, thereby effectivizing the corresponding classical result of Sacks for Lebesgue measure.

In view of Lemma 1.3.7, we can use Δ^0_2 measure as a reasonable notion of size for computable stable colorings. It is easy to show that the class of Δ^0_2 sets having a computable infinite subset or co-subset is Δ^0_2 null, meaning that most stable colorings do not have a computable infinite homogeneous set (it is equally easy to extend this from computable to c.e. or even co-c.e.). The following result is an instance where the measure-theoretic approach differs from the classical computability-theoretic one:

Theorem 3.2.5 (Hirschfeldt and Terwijn [29], Theorem 3.1)**.** *The class of low sets is not Δ^0_2 null.*

In fact, the proof of the above theorem gives the stronger result that the class of Δ^0_2 sets not having an infinite low subset or co-subset is Δ^0_2 null. It follows that most computable stable colorings do not satisfy Theorem 3.1.1.

We shall need a more uniform version of the above theorem, which we present in the form of the next proposition, in our proof of Theorem 3.1.4 below. It will rely on the following three facts:

1. the first is the existence of a universal oracle c.e. martingale, i.e., of a real-valued martingale U such that for all sets X, $\{x \in \mathbb{Q} : x < U^X(\sigma)\}$ is uniformly X-c.e. in σ, and $S[U^X] = \{B \in 2^\omega : B \text{ is not 1-random relative to } X\}$ (see, e.g., [12], Corollary 5.3.5);
2. the second, which we shall use repeatedly in the sequel, is van Lambalgen's theorem (see [12], Theorem 5.9.1), which states that a set is 1-random if and only if its odd and even halves are relatively 1-random;

3. and the third, due to Nies and Stephan (see [12], Theorem 13.2.6), is the following theorem (recall that if $\{C_s\}_{s\in\omega}$ is a computable approximation of a Δ^0_2 set, its *modulus of convergence* of is the function $m(x) = (\mu s)(\forall t \geq s)[C_s(x) = C_t(x)]$):

Theorem 3.2.6 (Nies and Stephan)**.** *Let C and B be sets such that C is Δ^0_2 and 1-random relative to B. If m is the modulus of convergence of a computable approximation of C, then $\varphi^B_x(x) \leq m(x)$ for all large enough x such that $\Phi^B_x(x)\downarrow$. In particular, since $m \leq_T \emptyset'$, B is GL_1 (i.e., $B' \leq_T B \oplus \emptyset'$).*

Regarding fact (1), we note that by the proof of Proposition 1.5.5 in [65], we can fix a $u \in \omega$ so that for all X, $\Phi^{X'}_u$ is a (rational-valued) martingale with $S[\Phi^{X'}_u] \supseteq S[U^X]$. Regarding fact (3), we note that its proof actually shows that, given a Δ^0_2 approximation of C with modulus of convergence m_C, if x is sufficiently large then $\Phi^B_x(x)\downarrow$ only if its use is bounded by $m_C(x)$.

We draw attention to our use below of $\Phi^X_{e,s}(x)$ to indicate a computation with oracle X run for s steps on input x, versus our use of $\Phi^X_e(x)[s]$ to indicatee the computation $\Phi^{X_s}_{e,s}(x)$ under the assumption of a fixed computable approximation (or enumeration) $\{X_s\}_{s\in\omega}$ of X. In particular, determining whether $\Phi^X_{e,s}(x)$ converges is X-computable, while for $\Phi^{X_s}_{e,s}(x)$ it is computable. We fix a computable enumeration $\{\emptyset'_s\}_{s\in\omega}$ of $\emptyset'$.

Proposition 3.2.7. *There exists a $\emptyset''$-computable function f such that for every $e, i \in \omega$, if $\Phi^{\emptyset'}_e$ is total and a martingale, and if i is a $\Delta^{0,\emptyset'}_1$ index for the jump of some set L, then there is a set $B \notin S[\Phi^{\emptyset'}_e]$ such that $f(e,i)$ is a $\Delta^{0,\emptyset'}_1$ index for the jump of $L \oplus B$.*

Proof. Fix $e, i \in \omega$ and let $u \in \omega$ be as described above. We define a partial $\emptyset'$-computable function $M : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$. Given $\sigma \in 2^{<\omega}$, let $\widetilde{\sigma}$ be either $\emptyset$ if $\sigma = \emptyset$, or $\sigma(0)\sigma(2)\cdots\sigma(2m)$ if σ has length $2m+1$ or $2m+2$ for some $m \geq 0$. If there exist $q, r \in \mathbb{Q}^{\geq 0}$ and $\tau \in 2^{<\omega}$ such that

1. $\Phi^{\emptyset'}_e(\widetilde{\sigma})\downarrow= q$,
2. $\Phi^{\emptyset'}_i(x)\downarrow= \tau(x)$ for all $x < |\tau|$ and $\Phi^{\tau}_u(\sigma)\downarrow= r$,

then let $M(\sigma) = \frac{1}{2}(q+r)$, and otherwise let $M(\sigma)$ be undefined. It is not difficult to see that M satisfies the averaging condition (3.1) where defined.

We next define $\{0,1\}$-valued partial $\emptyset'$-computable functions A, B, and C. Given x, let

$$A(x) = \begin{cases} 0 & \text{if } M((A\restriction x)\,0)\downarrow \leq M(A\restriction x)\downarrow, \\ 1 & \text{if } M((A\restriction x)\,0)\downarrow > M(A\restriction x)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then let $B(x) = A(2x)$ and $C(x) = A(2x+1)$ for all x, and let c be a $\Delta^{0,\emptyset'}_1$ index for C. Finally, define also $m_C(x) = (\mu s)(\forall t \geq s)[\Phi^{\emptyset'}_c(x)[t]\downarrow= \Phi^{\emptyset'}_c(x)[s]\downarrow]$.

Notice that if $\Phi^{\emptyset'}_e$ is a total martingale and $\Phi^{\emptyset'}_i$ is (the characteristic function of) the jump of some set L, then M is a Δ^0_2 martingale whose success set includes that of $\Phi^{L'}_u$, and A is a Δ^0_2 set on which M does not succeed. We then also have that $A = B \oplus C$, and it is readily seen from the definition of M that $B \notin S[\Phi_e^{\emptyset'}]$. Now because $A \notin S[M]$, A must be 1-random relative to L, and so by van Lambalgen's theorem relative to L, C must be 1-random relative to $L \oplus B$. Moreover, m_C is in this case the modulus of converge for the Δ_2^0 approximation $C(x,s)$ of C defined by $C(x,s) = i$ if $\Phi_c^{\emptyset'}(x)[s] \downarrow = i$ and $C(x,s) = 0$ otherwise. Hence, as discussed above, there must be an n so that for all $x \geq n$, whenever $\varphi_x^{L \oplus B}(x)$ is defined it is bounded by $m_C(x)$.

Now to define $f(e,i)$, choose $j \in \omega$ so that $\Phi_j^{X'} = X$ for all sets X, and let h be a computable function so that for all $x \in \omega$, $x \in X$ if and only if $h(x) \in X'$. Using a $\emptyset''$ oracle, we search for the first of the following to occur:

1. $\Phi_e^{\emptyset'}$ is undefined or does not satisfy the averaging condition (3.1) on some string;
2. $\Phi_i^{\emptyset'}$ is undefined on some number;
3. there exist a $\sigma \in 2^{<\omega}$ and an $x < |\sigma|$ such that $\Phi_i^{\emptyset'}(h(y)) \downarrow = \sigma(y)$ for all $y < |\sigma|$, and either $\Phi_x^{\sigma}(x) \downarrow$ and $\Phi_i^{\emptyset'}(x) \downarrow = 0$, or else $\Phi_x^{\tau}(x) \uparrow$ for all $\tau \succeq \sigma$ and $\Phi_i^{\emptyset'}(x) \downarrow = 1$;
4. there is an $n \in \omega$ so that for all σ, τ of the same length and all $x \geq n$, if
   (a) $\Phi_i^{\emptyset'}(h(y)) \downarrow = \sigma(y)$ for all $y < |\sigma|$,
   (b) $B(y) \downarrow = \tau(y)$ for all $y < |\tau|$,
   (c) $\Phi_x^{\sigma \oplus \tau}(x) \downarrow$ and $m_C(x) \downarrow$,

   then $\varphi_x^{\sigma \oplus \tau}(x) \leq m_C(x)$.

This search necessarily terminates, for if (1), (2), and (3) above do not obtain, then we are precisely in the situation of the preceding paragraph, so (4) must obtain as discussed there. If (1), (2), or (3) occur, let $f(e,i) = 0$. Otherwise, choose the least n witnessing the occurrence of (4) and let $f(e,i)$ be a $\Delta_1^{0,\emptyset'}$ index, found according to some fixed effective procedure, for the following function: on input x, the function waits for $m_C(x)$ to converge, then chooses the smallest $y \geq n$ such that $\Phi_x^X = \Phi_y^X$ for all sets X and searches for the first σ, τ of the same positive length so that (a) and (b) in (4) above hold; it then outputs 1 or 0 depending as $\Phi_y^{\sigma \oplus \tau}(x) \downarrow$ with use bounded by $m_C(x)$ or not. □

3.3 Analysis of uniformity results

3.3.1 s-Ramsey and almost s-Ramsey degrees

In [5, Sections 4 and 5], Cholak, Jockusch, and Slaman give two proofs of Theorem 1.3.5 for the stable case, but neither of them is uniform over the stable colorings (see the discussion at the beginning of Section 12.3 of [5]), and similarly in the case of the proof of Theorem 3.1.2. To address whether such non-uniformities are essential, Mileti introduced the following class of degrees:

Definition 3.3.1 (Mileti [46], Definition 5.1.2)**.** A Turing degree $\mathbf{d}$ is *s-Ramsey* if every Δ_2^0 set has an infinite subset or co-subset of degree at most $\mathbf{d}$.

Obviously, an s-Ramsey degree can also be defined as one which bounds the degree of an infinite homogeneous set for every computable stable coloring. (Mileti also defined a degree $\mathbf{d}$ to be *Ramsey* if every computable coloring of pairs has an infinite homogeneous set of degree at most $\mathbf{d}$.) Thus, the following results imply that Theorems 1.3.5 and 3.1.2 do not have uniform proofs:

Theorem 3.3.2 (Mileti [46], Theorem 5.3.7 and Corollary 5.4.6)**.**

1. *The only Δ^0_2 s-Ramsey degree is $\mathbf{0}'$.*
2. *There is no low$_2$ s-Ramsey degree.*

With the definition of Δ^0_2 measure in hand, we can generalize s-Ramsey degrees by passing from the class of all Δ^0_2 sets to subclasses of it which are not Δ^0_2 null.

Definition 3.3.3. A Turing degree $\mathbf{d}$ is *almost s-Ramsey* if the collection of Δ^0_2 sets with an infinite subset or co-subset of degree at most $\mathbf{d}$ is not Δ^0_2 null.

We obtain the same class of degrees in the above definition whether we insist on considering co-subsets or not. For if a martingale M succeeds on the class of all Δ^0_2 sets having an infinite subset of degree at most $\mathbf{d}$, then the martingale $M + N$, where $N(\sigma) = M((1 - \sigma(0))(1 - \sigma(1)) \cdots (1 - \sigma(|\sigma| - 1)))$ for all σ, succeeds on the class of all Δ^0_2 sets having an infinite such subset or co-subset. This is in stark contrast to Definition 3.3.1 even if we deal only with Δ^0_2 infinite, coinfinite sets, as it is easy to construct such a set so that all of its infinite subsets compute $\emptyset'$.[1]

The preceding definition was suggested by D. Hirschfeldt, who asked whether Mileti's results still hold if s-Ramsey degrees are replaced by the weaker almost s-Ramsey degrees, and more generally, whether the two classes of degrees are the same. Theorem 3.1.3, stated in Section 3.1 and restated in terms of almost s-Ramsey degrees below, is an affirmative answer with respect to the analogue of Theorem 3.3.2 (1). We discuss the the other questions in the next section.

Theorem 3.1.3. *The only Δ^0_2 almost s-Ramsey degree is $\mathbf{0}'$.*

Proof. Fix a set $D <_T \emptyset'$. For each $e \in \omega$, we construct uniformly in $\emptyset'$ a martingale M_e so as to satisfy the requirement:

$$\mathcal{R}_e \;\;:\;\; (\exists^\infty x)(\forall y \leq x)[\Phi^D_e(y) \downarrow\in \{0,1\} \wedge \Phi^D_e(x) = 1] \rightarrow (\forall A \supseteq \Phi^D_e)[A \in S[M_e]].$$

By Theorem 3.2.4 (3), this will ensure that the collection of sets containing an infinite subset computable from D is Δ^0_2 null, and hence, by the remarks following Definition 3.3.3, that $\deg(D)$ is not almost s-Ramsey.

Fix a total increasing function $f \leq_T \emptyset'$ not dominated by any function of degree strictly below $\mathbf{0}'$. We define M_e by stages, at stage s defining M_e on all strings of length t for some $t \geq s$.

[1] Let $A(x) = 1$ if $x = 2\langle e, s\rangle$ with $s = (\mu t)[\emptyset' \upharpoonright e = \emptyset'_t \upharpoonright e]$, and let $A(x) = 0$ otherwise.

Construction.

Stage $s = 0$. Let $M_e(\emptyset) = 1$.

Stage $s + 1$. Assume M_e has been defined on all strings of length t for some $t \geq s$. Search $\emptyset'$-computably for a string $\tau \preceq D$ and a number $x \geq t$ such that $|\tau|, x \leq f(t)$ and $\Phi^{\tau}_{e,|\tau|}(x) \downarrow= 1$. If the search succeeds, choose the least x for which it does so. Then for each $\sigma \in 2^{<\omega}$ of length t, and for all $\tau \succ \sigma$ with $|\tau| \leq x + 1$, define

$$M_e(\tau) = \begin{cases} M_e(\sigma) & \text{if } |\tau| \leq x, \\ 2M_e(\sigma) & \text{if } |\tau| = x + 1 \wedge \tau(x) = 1, \\ 0 & \text{if } |\tau| = x + 1 \wedge \tau(x) = 0. \end{cases}$$

Otherwise, set $M_e(\sigma 0) = M_e(\sigma 1) = M_e(\sigma)$ for all σ of length t.

End construction.

Verification. It is clear that the construction succeeds in defining M_e on all of $2^{<\omega}$. To verify that $\mathcal{R}_e$ is met, suppose that Φ^D_e is the characteristic function of an infinite set. Then the function

$$g(y) = (\mu s)(\exists x \geq y)(\forall z < x)[\Phi^D_{e,s}(x) \downarrow= 1 \wedge (y \leq z \rightarrow \Phi^D_{e,s}(z) \downarrow= 0)]$$

is total and computable from D, so by choice of f there must exist infinitely many y such that $g(y) \leq f(y)$. Fix $A \supseteq \Phi^D_e$ and suppose that at the end of some stage s' of the construction, $M_e(A \upharpoonright t)$ for some $t \geq 0$ is defined and positive, while $M_e(A \upharpoonright t + 1)$ is not yet defined. Choose the least $y \geq t$ such that $g(y) \leq f(y)$. If f is replaced by g in the search performed at each stage of the construction, then the search always succeeds, so it must necessarily succeed at some stage s with $s' < s \leq s' + (y + 1 - t)$. For the least such s, we then have $M_e(A \upharpoonright t + (s - s') - 1) = M_e(A \upharpoonright t)$ and so $M_e(A \upharpoonright t + (s - s')) = 2M_e(A \upharpoonright t)$. It follows that $\limsup_n M_e(A \upharpoonright n) = \infty$. □

3.3.2 An almost s-Ramsey degree that is not s-Ramsey

In this subsection, we give a proof of Theorem 3.1.4, restated equivalently below, thereby showing that the s-Ramsey degrees are a proper subclass of the almost s-Ramsey degrees. We do not know whether the analogue of Theorem 3.3.2 (2) holds for almost s-Ramsey degrees, but as every low$_2$ degree is Δ^0_3, our result is a partial step towards a negative answer.

Theorem 3.1.4. *There is a Δ^0_3 almost s-Ramsey degree that is not s-Ramsey.*

Proof. Fix a Δ^0_2 set A with no low infinite subset or co-subset. Computably in $\emptyset''$, we construct a set D and infinite low sets $L_0, L_1, \ldots$ that satisfy, for every $e \in \omega$ and $i < 2$, the following requirements:

$\mathcal{R}_{2e}$: $L_e \times \{e\} =^* D^{[e]} \wedge (\Phi^{\emptyset'}_e$ is a total martingale $\rightarrow L_e \notin S[\Phi^{\emptyset'}_e]$;

$\mathcal{S}_{2\langle e,i\rangle+1}$: Φ^D_e is total, $\{0,1\}$-valued and infinite $\rightarrow (\exists x)[\Phi^D_e(x) = 1 \wedge A(x) = i]$.

The first set of requirements ensures that $\{L_e : e \in \omega\}$ is not Δ^0_2 null and that $L_e \leq_T D$ for all e, and the second that no infinite subset or co-subset of A is computable from D. Hence, $\deg(D)$ will be the desired degree.

We let $D = \bigcup_s D_s$, where $D_0, D_1, \ldots$ are constructed in stages. At stage s, we define D_s, a function f_s with domain ω, and for each e a restraint $r_{e,s}$. We also declare each requirement either *active* or *inactive*. Let h be a computable function such that for all sets X and all $x \in \omega$, $x \in X$ if and only if $h(x) \in X'$.

Construction.

Stage $s = 0$. Set $D_0 = \emptyset$, and $f_0(e) = r_{e,s} = 0$ for all e. Declare all requirements active.

Stage $s+1$. Let D_s, f_s, and $r_{0,s}, r_{1,s}, \ldots$ be given. Assume inductively that cofinitely many requirements are still active, and that the value of f_s is 0 on cofinitely many arguments.

Case 1: $s + 1 \equiv 0 \bmod 3$ *or* $s + 1 \equiv 1 \bmod 3$. Suppose $s + 1 = 3\langle e, j\rangle + i$, where $e, j \in \omega$ and $i < 2$. If $\mathcal{R}_{2\langle e,i\rangle+1}$ is active, ask whether there exists an $x \in \omega$ and a finite set F satisfying the following:

1. $D_s \subseteq F \subseteq D_s \cup \{\langle y, e'\rangle \geq r_{e,s} : e' \leq e \to \mathcal{R}_{2e'} \text{ active}\}$;
2. $\Phi^F_e(x)\downarrow = 1$ and $A(x) = i$;
3. for $e' \leq e$ with $\mathcal{R}_{2e'}$ active and all $\langle y, e'\rangle \leq \max F \cup \{x : x \leq \varphi^F_e(x)\}$, $\Phi^{\emptyset'}_{f_s(e')}(h(2y+1))\downarrow$, and if $\langle y, e'\rangle \in F - D_s$ then $\Phi^{\emptyset'}_{f_s(e')}(h(2y+1)) = 1$;
4. for $e' \leq e$ with $\mathcal{R}_{2e'}$ active and all $\langle y, e'\rangle \leq \varphi^F_e(x)$, if $\langle y, e'\rangle \notin F - D_s$ then $\Phi^{\emptyset'}_{f_s(e')}(h(2y+1)) = 0$.

If so, we find the first such F and x in some fixed enumeration, set $D_{s+1} = F$, let $r_{e',s+1} = r_{e',s}$ for $e' < e$, and let $r_{e',s+1}$ be the least number greater than $\max\{r_{e'',s} : e \leq e'' \leq e'\}$ and $\varphi^F_e(x)$ for $e' \geq e$. We say that $\mathcal{R}_{2\langle e,i\rangle+1}$ *acts* at stage $s+1$, declare it inactive, and declare all $\mathcal{R}_{2\langle e',i\rangle+1}$ with $e' > e$ currently inactive active again. Otherwise, or if $\mathcal{R}_{2\langle e,i\rangle+1}$ is already inactive, we set $D_{s+1} = D_s$ and $r_{e',s+1} = r_{e',s}$ for all e'. Either way, we let $f_{s+1} = f_s$. Notice that the question of whether or not x and F in Case 1 exist is $\Sigma^{0,\emptyset'}_1$, and hence can be answered by $\emptyset''$.

Case 2: $s + 1 \equiv 2 \bmod 3$. We begin by choosing the least e such that $\mathcal{R}_{2e}$ is active and $f_s(e') = 0$ for all $e' \geq e$, which must exist by inductive hypothesis. Set $r_{e',s+1} = r_{e',s}$ for all e'. Fix $e' \in \omega$ and assume we have defined f_{s+1} on all $e'' < e'$. If $e' > e$ or if $\mathcal{R}_{2e'}$ is inactive, set $f_{s+1}(e') = 0$. Otherwise, let i be either a fixed $\Delta^{0,\emptyset'}_1$ index for $\emptyset'$ if there is no $e'' < e'$ such that $\mathcal{R}_{2e''}$ is active, or else $f_{s+1}(e'')$ for the greatest such e''. Then let $f_{s+1}(e')$ be the result of applying to e' and i the $\emptyset''$-computable function asserted to exist by Proposition 3.2.7.

To define D_{s+1}, begin by letting $D^{[e']}_{s+1} = D^{[e']}_s$ for all e' such that at least one of the following holds:

1. $e' > e$;

2. $\mathcal{R}_{2e'}$ is inactive;

3. $\Phi^{\emptyset'}_{f_{s+1}(e')}$ is not defined or not $\{0,1\}$-valued on $h(2x+1)$ for some $x \leq s$;

4. $\Phi^{\emptyset'}_{e'}$ is not defined or does not satisfy the averaging condition (3.1) on some string of length $\leq s$.

For all e' for which (4) obtains, declare $\mathcal{R}_{2e'}$ inactive, and declare all inactive $\mathcal{R}_{2\langle e'',i\rangle+1}$ requirements for $e'' \geq e'$ active. For all e' such that none of the above obtain, let $D^{[e']}_{s+1} = D^{[e']}_s \cup \{\langle x, e'\rangle > r_{e',s+1} : x \leq s \wedge \Phi^{\emptyset'}_{f_{s+1}(e')}(h(2x+1)) \downarrow= 1\}$.

In either case above only finitely many requirements are declared inactive, and f_{s+1} is defined to be positive on only finitely many elements. Thus, the induction can continue.

End construction.

Verification. The entire construction can be performed using a $\emptyset''$ oracle, hence $D \leq_T \emptyset''$. We now verify that all requirements are satisfied. To begin, note that each $\mathcal{R}_{2e}$ requirement can only switch from being active to being inactive but not back, and each $\mathcal{R}_{2\langle e,i\rangle+1}$ requirement, once inactive, can only become active again because some $\mathcal{R}_{2e'}$ requirement with $e' \leq e$ has become inactive. In particular, each $\mathcal{R}_{2\langle e,i\rangle+1}$ requirement acts at most finitely many times. Since for every e, $r_{e,s}$ is a non-decreasing function in s that increases only when some $\mathcal{R}_{2\langle e',i\rangle+1}$ with $e' \leq e$ acts, $\lim_s r_{e,s}$ exists.

Claim 3.3.4. *For every $e \in \omega$, $f(e) = \lim_s f_s(e)$ exists. Moreover, if $\mathcal{R}_{2e}$ is permanently active then $f(e)$ is a $\Delta^{0,\emptyset'}_1$ index for some jump, and if $\mathcal{R}_{2e}$ is not permanently active then $f(e) = 0$ and $D^{[e]}$ is finite.*

Proof. Fix $e \in \omega$ and assume the claim holds for all $e' < e$. Fix a stage $s \geq 0$ such that for all $e' \leq e$ and all $i < 2$, the following hold:

1. if $e' < e$ then $f(e') \downarrow= f_t(e')$ for all $t > s$;

2. if $\mathcal{R}_{2e'}$ is cofinitely often inactive, then it is inactive at all stages $t \geq s$;

3. if $\mathcal{R}_{2\langle e',i\rangle+1}$ is cofinitely often inactive, then it is inactive at all stages $t \geq s$.

First suppose $\mathcal{R}_{2e}$ is active at stage s, and hence permanently thereafter. Since 0 is not a $\Delta^{0,\emptyset'}_1$ index for any jump, the inductive hypothesis implies that at any stage $t \geq s$ that is congruent to 2 modulo 3, the number chosen at the beginning of Case 2 of the construction is at least as big as e. Hence, we see from the construction that the value of $f_t(e)$ at any stage $t \geq s$ depends only on e and, if there is an $\mathcal{R}_{2e'}$ with $e' < e$ which is active at stage s, on $f_t(e') = f(e')$ for the largest such e'. Thus $f_t(e)$ has the same value for all $t \geq s$, so $f(e) = f_s(e)$.

As $\mathcal{R}_{2e}$ is never declared inactive, it must be that condition (4) in Case 2 of the construction never occurs, and hence that $\Phi^{\emptyset'}_e$ is a total martingale. Let L be either $\emptyset$ or, if there exists an $e' < e$ with $\mathcal{R}_{2e'}$ permanently active, $\Phi^{\emptyset'}_{f(e')}$ for the greatest such e'. Then it follows by construction and by Proposition 3.2.7 that $f(e)$ is a $\Delta_1^{0,\emptyset'}$ index for $(L \oplus B)'$, where B is a set not in $S[\Phi_e^{\emptyset'}]$. In particular, $f(e)$ is a $\Delta_1^{0,\emptyset'}$ index for a jump, as desired.

Now suppose $\mathcal{R}_{2e}$ is inactive at stage s. Then $f_t(e)$ is defined to be 0 at all stages $t \geq s$, so $f(e) = 0$. Now no elements can be put into $D_t^{[e]}$ at any stage $t > s$ under Case 1 of the construction, because by condition (1) in that case this can only be done because of the action of some requirement $\mathcal{R}_{2\langle e',i\rangle+1}$ with $e' \leq e$, and all such requirements have stopped acting by stage s. Moreover, no elements can be put into $D_t^{[e]}$ under Case 2, because condition (2) in that case allows this only when $\mathcal{R}_{2e}$ is still active. Hence, $D_t^{[e]} = D_s^{[e]}$ for all $t \geq s$, and so $D^{[e]}$ is finite. □

Claim 3.3.5. *For every $e \in \omega$, requirement $\mathcal{R}_{2e}$ is satisfied via a set L_e such that $\bigoplus_{e' \leq e} L_{e'}$ is low.*

Proof. First suppose that $\Phi_e^{\emptyset'}$ is a total martingale. Then condition (4) in Case 2 of the construction never occurs and $\mathcal{R}_{2e}$ is active at all stages. Let L be as in the proof of the preceding claim, and let L_e be the set B from there, so that $f(e)$ is a $\Delta_1^{0,\emptyset'}$ index for $(L \oplus L_e)'$ and $L_e \notin S[\Phi_e^{\emptyset'}]$.

It then remains only to show that $L_e \times \{e\} =^* D^{[e]}$. Let s be a stage as in the proof of the preceding claim. Since no $\mathcal{R}_{2\langle e',i\rangle+1}$ requirement with $e' \leq e$ can act at any stage $t \geq s$, it follows by condition (3) in Case 1 of the construction, as well as the fact that $L_e = \{x : \Phi_{f(e)}^{\emptyset'}(h(2x+1)) \downarrow= 1\}$, that any element put into $D_t^{[e]}$ for the sake of an odd-indexed requirement must belong to $L_e \times \{e\}$. For the same reason we must have that $r_e = r_{e,t}$ for any stage $t \geq s$, and, as mentioned in the previous claim, the number chosen at the beginning of Case 2 of the construction at any such stage t can not be smaller than e. Hence, at the end of every stage $t \geq s$ that is congruent to 2 modulo 3, all elements x in $L_e \times \{e\}$ with $r_e < x \leq t$ are put into $D_t^{[e]}$. It follows that $\{x \in D^{[e]} : x > \max D_s^{[e]}\} \subseteq L_e \times \{e\}$ and $\{x \in L_e \times \{e\} : x > r_e\} \subseteq D^{[e]}$, which yields the desired result.

Next suppose that $\Phi_e^{\emptyset'}$ is not a total martingale. Then at some stage, condition (4) in Case 2 of the construction occurs and $\mathcal{R}_{2e}$ is declared inactive. By the previous claim, $D^{[e]}$ is finite, so if we let $L_e = \emptyset$ then L_e is low and requirement $\mathcal{R}_{2e}$ is met.

Finally, given e let $e_0 < e_1 < \cdots < e_n$ be a listing of all $e' \leq e$ such that $\mathcal{R}_{2e'}$ is active at stage s. Then $\bigoplus_{j \leq n} L_{e_j}$ is low, for $f(e_0)$ is a $\Delta_1^{0,\emptyset'}$ index for $(\emptyset \oplus L_{e_0})'$, $f(e_1)$ is a $\Delta_1^{0,\emptyset'}$ index for the jump of $(\emptyset \oplus L_{e_0}) \oplus L_{e_1}$, and so on. Hence $\bigoplus_{e' \leq e} L_{e'}$ is low since $L_{e'} = \emptyset$ for all $e' \neq e_j$ for any j, and this completes the proof. □

Claim 3.3.6. *For every $e \in \omega$ and $i < 2$, $\mathcal{R}_{2\langle e,i\rangle+1}$ is satisfied.*

Proof. Fix e and i and assume inductively that the claim holds for all $e' < e$. Fix a stage $s \geq 0$ congruent to i modulo 3 such that for all $e' \leq e$, $f_s(e') = f(e)$ and $D_s^{[e']} = D^{[e']}$ if $\mathcal{R}_{2e'}$ is not permanently active, and for all $e' < e$, $r_{e',s} = r_e$ and no $\mathcal{R}_{2\langle e',i\rangle+1}$ requirement with $e' < e$ acts at or after stage s. Assume further that Φ_e^D is total, $\{0,1\}$-valued, and infinitely often takes the value 1, as otherwise $\mathcal{R}_{2\langle e,i\rangle+1}$ is satisfied trivially. Since $L_{e'} \times \{e'\} =^* D^{[e']}$ for all $e' \leq e$, it follows by the previous claim that $\bigcup_{e' \leq e} D^{[e']}$ is low, and since D_s is finite, also that $\bigcup_{e' \leq e} D^{[e']} \cup D_s$ is low.

Now there must exist an $x \in \omega$ and a finite set F such that $A(x) = i$ and such that the following conditions hold:

1. $D_s \subseteq F \subseteq D_s \cup \{\langle y, e' \rangle \geq r_{e,s} : e' \leq e \to \mathcal{R}_{2e'} \text{ active}\}$;

2. $\Phi_e^F(x) \downarrow = 1$;

3. for all $e' \leq e$, $F^{[e']} \subseteq D^{[e']}$;

4. for all $e' \leq e$, $F^{[e']} \restriction \varphi_e^F(x)) = D^{[e']} \restriction \varphi_e^F(x)$.

Indeed, from our assumptions about Φ_e^D it follows that there exist arbitrarily large numbers x and corresponding finite sets F satisfying (1)–(4) above, for example all sufficiently long initial segments of D. And we can clearly find such x and F computably in $\bigcup_{e' \leq e} D^{[e']} \cup D_s$. Hence, if $A(x)$ was equal to $1-i$ for all such x, then depending as i is 0 or 1, $\bigcup_{e' \leq e} D^{[e']} \cup D_s$ could compute an infinite subset or infinite co-subset of A, contradicting that A has no low infinite subset or co-subset.

By choice of s, it is easily seen that for all $e' \leq e$, all elements in $D^{[e']} - D_s$ belong to $L_{e'} \times \{e'\}$. It follows that the question about an $x \in \omega$ and a finite set F asked at stage s of the construction is precisely the question of whether there exist x and F satisfying the conditions above, and as such must have an affirmative answer. Hence $\mathcal{R}_{2\langle e,i \rangle+1}$ acts, meaning that for some such x and F, $D_{s+1} = F$ and $r_{e',t}$ is greater than $\varphi_e^F(x)$ for all $t > s$ and all $e' \geq e$. No requirements can then ever put into D_t any elements below $\varphi_e^F(x)$ at any stage $t > s$, meaning that the $\Phi_e^F(x)$ computation is preserved and so $\Phi_e^D(x) = 1$. Consequently, requirement $\mathcal{R}_{2\langle e,i \rangle+1}$ is satisfied. □

□

3.4 Muchnik degrees of infinite homogeneous sets

We illustrate some applications of the preceding section by looking at the Muchnik degrees of classes of infinite subsets and co-subsets of Δ_2^0 sets.

Definition 3.4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be classes of sets.

1. $\mathscr{A}$ is *Muchnik* (or *weakly*) reducible to $\mathscr{B}$, written $\mathscr{A} \leq_w \mathscr{B}$, if every element of $\mathscr{B}$ computes an element of $\mathscr{A}$.

2. If $\mathscr{A} \leq_w \mathscr{B}$ and $\mathscr{B} \leq_w \mathscr{A}$, we write $\mathscr{A} \equiv_w \mathscr{B}$.

3. The *Muchnik degree* of $\mathscr{A}$, denoted $\deg_w(\mathscr{A})$, is the $\equiv_w$-equivalence class of $\mathscr{A}$. For Muchnik degrees $\mathbf{a}$ and $\mathbf{b}$, write $\mathbf{a} \leq \mathbf{b}$ if $\mathscr{A} \leq_w \mathscr{B}$ for every $\mathscr{A} \in \mathbf{a}$ and every $\mathscr{B} \in \mathbf{b}$.

We refer the reader to Binns and Simpson [2], Section 1, for additional background.

Definition 3.4.2. Given a Δ^0_2 set A, let $\mathbf{h}(A)$ denote the Muchnik degree of the class of all infinite subsets or co-subsets of A, and for a class $\mathscr{C}$ of Δ^0_2 sets let $\mathbf{h}(\mathscr{C})$ denote the structure $\{\mathbf{h}(A) : A \in \mathscr{C}\}$ under Muchnik degree reducibility, $\leq$. Given a computable stable coloring f, let $\mathbf{h}(f)$ be the Muchnik degree of the class of all infinite homogeneous sets of f.

Clearly, for each Δ^0_2 set A there is a computable stable f with $\mathbf{h}(A) = \mathbf{h}(f)$, and conversely. Thus, we may use the two notions interchangeably here. It is also clear that $\mathbf{h}(\Delta^0_2)$ contains a least element, equal to $\mathbf{h}(A)$ for any non-bi-immune Δ^0_2 set A. (Recall that a set is *immune* if it has no computable infinite subset, and *bi-immune* if it and its complement are immune.) By contrast, we have:

Corollary 3.4.3. *There is no largest element in* $\mathbf{h}(\Delta^0_2)$*.*

Proof. Immediate by Theorem 3.1.2 and part (1) of Theorem 3.3.2. (It also follows from Corollary 5.4.8 (2) in [46], which is established by a different argument.) □

Using Theorem 3.1.3, we can extend this result as follows.

Corollary 3.4.4. *If* $\mathscr{C}$ *is a class of* Δ^0_2 *sets that is not* Δ^0_2 *null, then there is no largest element in* $\mathbf{h}(\mathscr{C})$*.*

A central question concerning the precise logical strength of the stable Ramsey's theorem is whether there is a computable stable coloring all of whose infinite homogeneous sets have PA degree. (This is essentially Question 13.12 in [5].) In reverse mathematics, this relates to the long-standing question of whether SRT^2_2 implies WKL_0 over RCA_0 (see also Section 3.5). In terms of the present discussion this can be stated as follows: does there exist a Δ^0_2 set A satisfying $\mathbf{p} \leq \mathbf{h}(A)$, where $\mathbf{p}$ is the Muchnik degree of the class of all sets of PA degree? The following proposition shows that the collection of sets that do not satisfy this property is at least not negligible:

Proposition 3.4.5. *The class of* Δ^0_2 *sets* A *with* $\mathbf{p} \nleq \mathbf{h}(A)$ *is not* Δ^0_2 *null.*

Proof. Given a total Δ^0_2 martingale $\Phi^{\emptyset'}_e$, let M, A, B and C be as in the proof of Proposition 3.2.7 with i a $\Delta^{0,\emptyset'}_1$ index for $\emptyset'$. Then $A = B \oplus C$, the set B is low, and $B \notin S[\Phi^{\emptyset'}_e]$. Since A is not in $S[M]$ it is 1-random, and so by van Lambalgen's theorem, B is 1-random too. If we denote this B by B_e, then the class $\{B_e : \Phi^{\emptyset'}_e \text{ is total martingale}\}$ is consequently not Δ^0_2 null. Furthermore, since each B_e is low and 1-random, it can not have PA degree. This is because every 1-random PA degree bounds $\mathbf{0}'$ by the main result of Stephan [62]. □

We can also ask whether there exists a Δ^0_2 set A satisfying the reverse of the above inequality, that is, $\mathbf{h}(A) \leq \mathbf{p}$. Of course, this will be the case for any non-bi-immune set, but the following proposition gives a more interesting example:

Proposition 3.4.6. *There exists a bi-immune* Δ^0_2 *set* A *with* $\mathbf{h}(A) \leq \mathbf{p}$*.*

Proof. We seek to satisfy, for every e, the standard bi-immunity requirement:

$$\mathcal{R}_e : W_e \text{ infinite} \implies W_e \cap A \neq \emptyset \wedge W_e \cap \overline{A} \neq \emptyset.$$

We let $A = \lim_s A_s$, where $A_0, A_1, \ldots$ are constructed as follows.

Construction. At stage $s \geq 0$, we build A_s by considering successive substages for each $e \leq s$. At substage e, let $t \leq s$ be least such that $W_{e,t}$ contains two or more elements $\geq e(e+2)$ not claimed at stage s by $\mathcal{R}_i$ for any $i < e$. Then let $x < y$ be the least two such elements in $W_{e,t}$, say that $\mathcal{R}_e$ *claims* x and y at stage s, and define $A_s(x) = 0$ and $A_s(y) = 1$. If t does not exist, do nothing. Either way, if $e < s$ go to substage $e+1$, and if $e = s$ set $A_s(x) = 0$ for all x not claimed at stage s by $\mathcal{R}_e$ for any $e < s$.

End Construction.

Verification. A standard argument establishes that $\lim_s A_s(x)$ exists for all x, whence it follows that A is Δ^0_2 and satisfies all the requirements $\mathcal{R}_e$. Fixing $\mathbf{d} \gg \mathbf{0}$, we exhibit an infinite subset S of $\overline{A}$ with $\deg(S) \leq \mathbf{d}$. For each $e \in \omega$ and each x with $e(e+2) \leq x < (e+1)(e+3)$, let $\psi_{e,x}$ be the statement $(\forall s)[A_s(x) = 0]$. Notice that if $\psi_{e,x}$ is false for some x, it is so because x is claimed at some stage by some $\mathcal{R}_i$ for $i \leq e$, in which case it will also be claimed at each subsequent stage. Since no requirement ever claims more than two elements, and since there are $2(e+1)+1$ many x with $e(e+2) \leq x < (e+1)(e+3)$, there is at least one such x which is never claimed by any requirement. Thus, there is at least one such x such that $\psi_{e,x}$ is true. We can, effectively from e, determine Π^0_1 indices for each of $\psi_{e,e(e+2)}, \psi_{e,e(e+2)+1}, \ldots, \psi_{e,(e+1)(e+3)-1}$, and from these, by Lemma 2.2.8, $\mathbf{d}$-effectively find an x such that $\psi_{e,x}$ is true and hence $x \in \overline{A}$. By carrying out this procedure for every e, we consequently obtain the desired infinite subset of $\overline{A}$. □

Note that, for any incomplete Δ^0_2 set D of PA degree, the class of sets satisfying the preceding proposition is contained in the class of Δ^0_2 sets with a D-computable infinite subset or co-subset. Thus, by Theorem 3.1.3, the former class is Δ^0_2 null.

For general interest, we conclude this section by briefly looking at the algebraic properties of the structure $\mathbf{h}(\Delta^0_2)$, which has not previously been studied in the literature.

Proposition 3.4.7. *Let f_0 and f_1 be computable stable colorings, and for each $i < 2$, let $\mathscr{C}_i$ be the class of all infinite homogeneous sets of f_i. Then there exists a computable stable coloring f such that $\mathbf{h}(f) = \deg_w(\mathscr{C}_0 \cup \mathscr{C}_1)$. Hence, $\mathbf{h}(\Delta^0_2)$ is a lower semilattice.*

Proof. For $x, y \in \omega$, let

$$f(x,y) = \begin{cases} f_0(\frac{x}{2}, \frac{z}{2}) & \text{if } x \text{ is even and } z \text{ is the least even number} \geq y, \\ f_1(\frac{x-1}{2}, \frac{z-1}{2}) & \text{if } x \text{ is odd and } z \text{ is the least odd number} \geq y. \end{cases}$$

It is easy to see that f is stable. Now if H is an infinite homogeneous set for f_0, respectively for f_1, then the set $\{2x : x \in H\}$, respectively $\{2x+1 : x \in H\}$, is homogeneous for f. Thus, the class of infinite homogeneous sets of f is Muchnik reducible to $\mathscr{C}_0 \cup \mathscr{C}_1$. Conversely, let H be any infinite homogeneous set for f and let $H_0 = \{\frac{x}{2} : x \in H \wedge x \text{ even}\}$ and $H_1 = \{\frac{x-1}{2} : x \in H \wedge x \text{ odd}\}$. One of H_0 and H_1, say H_i, must be infinite, and this set is homogeneous for f_i. Thus, $\mathscr{C}_0 \cup \mathscr{C}_1$ is Muchnik reducible to the class of infinite homogeneous sets of f. It is now easily seen that $\mathbf{h}(f) = \mathbf{h}(h_0) \wedge \mathbf{h}(h_1)$. □

Proposition 3.4.8. *There are no minimal elements in $\mathbf{h}(\Delta^0_2)$ other than the least element.*

We shall use the following results of Kent and Lewis [41]:

Theorem 3.4.9 (Kent and Lewis [41], Theorem 7.1 and proof of Theorem 4.5)**.**

1. *If a computable tree has an infinite path of every low degree then it has an infinite path of every degree.*
2. *If a computable tree has an infinite path of every degree then it contains a computable perfect subtree.*

Proof of Proposition 3.4.8. Fix any computable stable coloring f with no computable infinite homogeneous set, so that $\mathbf{h}(f)$ is not the least element of $\mathbf{h}(\Delta^0_2)$. First, we claim that f can not have an infinite homogeneous set of every non-zero Δ^0_2 degree. If not, then the Π^0_1 class $\mathscr{P}$ of all homogeneous sets of f (finite or infinite) would have a member of every Δ^0_2 degree, and so, in particular, a member of every low degree. By Theorem 3.4.9, it would have a member of every degree and would thus contain the set of infinite paths through some computable perfect subtree of $2^{<\omega}$. But every such tree has a computable infinite path that is the characteristic function of an infinite set. Thus, $\mathscr{P}$ would have to contain a computable infinite member, contrary to assumption. This proves the claim, so we can fix a Δ^0_2 set A such that f has no infinite homogeneous set of degree $\deg(A)$. By Theorem 1.3.3 (5), this means f also has no A-computable infinite homogeneous set, and since every non-zero Δ^0_2 degree is bi-immune, there is a bi-immune set $B \equiv_T A$. Now by Proposition 3.4.7, $\mathbf{h}(f) \wedge \mathbf{h}(A)$ must lie strictly between the minimum of $\mathbf{h}(\Delta^0_2)$ and $\mathbf{h}(f)$. □

Proposition 3.4.10. *For every finite collection of elements in $\mathbf{h}(\Delta^0_2)$ not equal to the minimum element, there exists an element in $\mathbf{h}(\Delta^0_2)$ incomparable with each of them.*

Proof. Let $A_0, \ldots, A_n$ be given bi-immune Δ^0_2 sets. By iterating Theorem 1.3.5, fix an infinite subset or co-subset L_i of each A_i such that $L = \bigoplus_{i \leq n} A_i$ is low$_2$. More precisely, let L_0 be a low$_2$ infinite subset or co-subset of A_0, and having found $L_0, \ldots, L_j$ for some $j < n$ such that $\bigoplus_{i \leq j} L_i$ is low$_2$, apply the theorem in relativized form to find an infinite subset of co-subset L_{j+1} of A_{j+1} such that $\bigoplus_{i \leq j} L_i \oplus L_{j+1}$ is low$_2$ relative to $\bigoplus_{i \leq j} L_i$ and hence low$_2$. We build an infinite Δ^0_2 set A with no L-computable, and hence no L_i-computable, infinite subset or co-subset, and computing no infinite subset or co-subset of any A_i. Thus, $\mathbf{h}(A)$ will be incomparable with $\mathbf{h}(A_i)$ for every i.

We obtain A as $\bigcup_s \sigma_s$, where $\sigma_0 \prec \sigma_1 \prec \cdots$ are constructed by stages. To ensure that L computes no infinite subset or co-subset of A, we use the technique in the proof of Theorem 5.4.2 of Mileti [46]. As in that proof, we fix a low$_2$ function $f : \omega^2 \to \omega$ such that every

L-computable set is equal to $F_e = \{x : f(e,x) = 1\}$ for some e. We wish to satisfy, for all $e \in \omega$ and $i \leq n$, the following requirements:

$$\begin{array}{lcl} \mathcal{R}_{2e} & : & F_e \text{ infinite} \implies F_e \not\subseteq A; \\ \mathcal{R}_{2e+1} & : & F_e \text{ infinite} \implies F_e \not\subseteq \overline{A}; \\ \mathcal{S}_{2\langle e,i\rangle} & : & \Phi_e^A = S \subseteq A_i \implies S \text{ finite}; \\ \mathcal{S}_{2\langle e,i\rangle+1} & : & \Phi_e^A = S \subseteq \overline{A_i} \implies S \text{ finite}. \end{array}$$

By using the recursion theorem relative to $\emptyset'$ to fix a $\Delta_1^{0,\emptyset'}$ index for A ahead of time, we can give a $\Sigma_2^{0,f}$ definition for the set of all i such that requirement $\mathcal{R}_i$ is satisfied. (See [46], p. 58). Since f is low_2, it follows that there exists a $\emptyset'$-computable function $g : \omega^2 \to 2$ such that for every i, $\lim_s g(i,s)$ exists and $\mathcal{R}_i$ is satisfied if and only if this limit is 1.

Construction. Let $\sigma_0 = \emptyset$. At stage $s \geq 0$, assume σ_s is defined. If $s > 0$ and the least $i \leq s-1$ with $g(i, s-1) = 0$ equals the least $j \leq s$ with $g(j,s) = 0$, assuming both exist, let σ_{s+1} be $\sigma_s 0$ or $\sigma_s 1$ depending as this i is even or odd. Otherwise, let $2\langle e,i\rangle + j$, $e \in \omega$, $i \leq n$, $j < 2$, be least such that requirement $\mathcal{S}_{2\langle e,i\rangle+j}$ has not been satisfied. Let σ_{s+1} be the least $\sigma \succ \sigma_s$ such that for some $x \in \omega$, either $\Phi_e^\sigma(x) \downarrow= 1$ and $A(x) = j$, or for all $y \geq x$ it is not the case that $\Phi_e^\tau(y) \downarrow= 1$. This search must succeed, since otherwise we could computably find infinitely many x such that $\Phi_e^\tau(x) \downarrow= 1$ for some $\tau \succeq \sigma_s$, and then the set of these x would be either an infinite subset of A if $j = 0$, or an infinite co-subset of A of $j = 1$. Note that this action causes $\mathcal{S}_{2\langle e,i\rangle+j}$ to be satisfied.

End construction.

Verification. To verify that each of the $\mathcal{R}_i$ requirements is met, suppose not and let i be least such that $\mathcal{R}_i$ is not met. Let $e = i/2$ if i is even, and let $e = (i-1)/2$ if i is odd. Fix a stage $s \geq i$ so that for all $t \geq s$, $g(j,t) = 1$ for all $j < i$ and $g(i,s) = 0$. Then by construction, if i is even then $\sigma_t = \sigma_{s+1} 0^{t-s-1}$ for all $t > s$, and if i is odd then $\sigma_t = \sigma_{s+1} 1^{t-s-1}$ for all $t > s$. Since $\mathcal{R}_i$ is not met, F_e must be infinite, and so it must contain some $x > |\sigma_{s+1}|$. But then if i is even we have $F_e(x) = 1 \neq A(x)$, whereas if i is odd we have $F_e(x) = 1 \neq \overline{A}(x)$. Hence, $\mathcal{R}_i$ is satisfied after all. It follows that $\lim_s g(i,s) = 1$ for all i, and hence that there are infinitely many stages $s > 0$ such that the least $i \leq s-1$ with $g(i,s-1) = 0$ and the least $j \leq s$ with $g(j,s) = 0$ are not both defined and equal. Since at each such stage we get to act for, and thereby satisfy, a new $\mathcal{S}_{2\langle e,i\rangle+j}$ requirement, we conclude that all such requirements are eventually satisfied as well. □

3.5 Almost stable Ramsey's theorem

The proof-theoretic strength of SRT^2_2, as a principle of second-order arithmetic, was first studied by Cholak, Jockusch, and Slaman in [5], Sections 7 and 10. As discussed above, one major open problem is whether SRT^2_2 implies WKL_0 over RCA_0 (see [5], p. 53), the closest related result being by Hirschfeldt, et al. [26, Theorem 2.4] that SRT^2_2 implies the weaker axiom DNR. (That WKL_0 does not imply SRT^2_2 is by [5], Theorems 11.1 and 11.4; it can also be seen by Theorem 3.1.1 and the fact that WKL_0 has a model consisting entirely of low sets). Another question is whether SRT^2_2 implies COH, which is equivalent by Theorem 1.3 of [5] and the correction given in section A.1 of [46] to the question of whether SRT^2_2 implies RT^2_2. For completeness, we recall the definitions of DNR and COH, which were introduced by Giusto and Simpson [24, p. 1478] and Cholak, Jockusch, and Slaman [5, Statement 7.7], respectively.

Cohesive principle (COH)**.** *For every sequence $\langle X_i : i \in \mathbb{N}\rangle$ of sets, there is an infinite set X such that for every $i \in \mathbb{N}$, either $X \subseteq^* X_i$ or $X \subseteq^* \overline{X_i}$.*

Diagonally non-recursive principle (DNR)**.** *For every set X there exists a function f that is diagonally non-computable relative to X, i.e., such that for all $e \in \mathbb{N}$, $f(e) \neq \Phi^X_e(e)$.*

In this section, we study several principles inspired by our investigations above and related to SRT^2_2 by means of a formal notion of Δ^0_2 measure.

Definition 3.5.1. The following definitions are made in RCA_0:

1. a *martingale approximation* is a function $M : 2^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}^{\geq 0}$ such that $\lim_s M(\sigma, s)$ exists for every $\sigma \in 2^{<\mathbb{N}}$ (i.e., $M(\sigma, s) = M(\sigma, t)$ for all sufficiently large s and t), and for all $s \in \mathbb{N}$,
$$2M(\sigma, s) = M(\sigma 0, s) + M(\sigma 1, s);$$
2. we say M *succeeds on* a stable coloring $f : [\mathbb{N}]^2 \to 2$ if
$$(\forall n)(\exists \sigma)(\exists s)(\forall t \geq s)(\forall x < |\sigma|)[\sigma(x) = f(x, t) \wedge M(\sigma, t) = M(\sigma, s) > n]. \qquad (3.2)$$

We can now state an "almost stable Ramsey's theorem", along with principles asserting the existence of s-Ramsey and almost s-Ramsey degrees. The following statements are defined in RCA_0:

Almost stable Ramsey's theorem (ASRT^2_2)**.** *For every martingale approximation M, there is a stable coloring $f \leq_T M$ on which M does not succeed and which has an infinite homogeneous set.*

Existence of s-Ramsey degrees principle (SRAM)**.** *For every set X, there is a set Y as follows: every stable coloring $f \leq_T X$ has an infinite homogeneous set $H \leq_T Y$.*

Existence of almost s-Ramsey degrees principle (ASRAM)**.** *For every set X, there is a set Y as follows: for every martingale approximation $M \leq_T X$ there is a stable coloring $f \leq_T X$ on which M does not succeed and which has an infinite homogeneous set $H \leq_T Y$.*

Notice that the class of Δ^0_2 sets having an infinite subset or co-subset in a given ω-model of ASRT^2_2 is not Δ^0_2 null.

We begin with a formalization of Proposition 3.2.4 (1). We do not know if the use of $\mathsf{B}\Sigma^0_2$ below can be avoided.

Lemma 3.5.2. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ *proves that for every martingale approximation M, there is a stable coloring $f \leq_T M$ on which M does not succeed.*

Proof. Let M be a martingale approximation, say with $\lim_s M(\emptyset, s) = 1$. Then by Definition 3.5.1, if $M(\sigma, s) \leq 1$ for some s, either $M(\sigma 0, s) \leq 1$ or $M(\sigma 1, s) \leq 1$. Choose s_0 so that $M(\emptyset, s) \leq 1$ for all $s \geq s_0$. For every x and $s \geq s_0$, a simple Σ^0_0 induction then shows that there exists $\sigma \in 2^{<\mathbb{N}}$ of length $x+1$ and satisfying

$$(\forall y \leq x+1)[M(\sigma \upharpoonright y, s) \leq 1] \wedge (\forall y \leq x)[\sigma(y) = 1 \rightarrow M((\sigma \upharpoonright y)\, 0, s) > 1],$$

and that this string is unique. Define $f : [\mathbb{N}]^2 \rightarrow 2$ by letting $f(x, s)$ for $x < s$ be 0 or $\sigma(x)$ for the above σ depending as $s < s_0$ or $s \geq s_0$. Clearly, f has a Σ^0_0 definition with M as parameter, so $f \leq_T M$. We claim that f is stable and that M does not succeed on it. Fix x in $\mathbb{N}$ and using $\mathsf{B}\Pi^0_1$ (which is equivalent to $\mathsf{B}\Sigma^0_2$) choose an $s \geq s_0$ with $M(\sigma, t) = M(\sigma, s)$ for all $t \geq s$ and $\sigma \in 2^{<\mathbb{N}}$ of length $\leq x+1$. Then the σ used to define $f(x, s)$ will be same as that used to define $f(x, t)$ for all $t \geq s$. Hence, $f(x, t) = \sigma(x)$ for all $t \geq s$, and as $M(\sigma, t) \leq 1$ we have the negation of (3.2) holding with $n = 1$. $\square$

Basic relations of implication and non-implication between SRT^2_2 and the principles introduced above are established in the next proposition.

Proposition 3.5.3.

1. *Over* RCA_0*,* $\mathsf{ACA}_0 \rightarrow \mathsf{SRAM} \rightarrow \mathsf{SRT}^2_2 \rightarrow \mathsf{ASRT}^2_2$ *and* $\mathsf{SRAM} \rightarrow \mathsf{ASRAM} \rightarrow \mathsf{ASRT}^2_2$.

2. SRAM *does not imply* ACA_0 *over* RCA_0*, and* SRT^2_2 *does not imply* SRAM.

Proof. Clearly, $\mathsf{SRAM} \rightarrow \mathsf{SRT}^2_2$ and $\mathsf{ASRAM} \rightarrow \mathsf{ASRT}^2_2$. As for the implications $\mathsf{SRT}^2_2 \rightarrow \mathsf{ASRT}^2_2$ and $\mathsf{SRAM} \rightarrow \mathsf{ASRAM}$, these follow from the preceding lemma and the fact that SRT^2_2, and hence also SRAM, implies $\mathsf{B}\Sigma^0_2$. That $\mathsf{ACA}_0 \rightarrow \mathsf{SRAM}$ amounts to a formalization of the fact that $\mathbf{0}'$ is an s-Ramsey degree, and is straightforward.

We now prove (2). By relativizing Corollary 5.1.7 of Mileti [46], we get that for any set $X \not\geq_T \emptyset'$ there is set $Y \geq_T X$ such that $Y \not\geq_T \emptyset'$ and Y is s-Ramsey relative to X (i.e., computes an infinite homogeneous set for every X-computable stable coloring). Iterating, we thus obtain a sequence $Y_0 \leq_T Y_1 \leq_T \cdots$ such that $Y_e \not\geq_T \emptyset'$ and Y_{e+1} is s-Ramsey relative to Y_e for every e. Then the ideal $\{S : (\exists e)[S \leq_T Y_e]\}$ is clearly an ω-model of SRAM containing no set of degree $\mathbf{0}'$, and hence not a model of ACA_0. That SRT^2_2 does not imply SRAM is because the former has an ω-model consisting entirely of low_2 sets by relativizing and iterating Theorem 1.3.5, whereas the latter does not by Theorem 3.3.2 (2). $\square$

The next result establishes a certain degree of similarity between ASRT^2_2 and SRT^2_2. In particular, we see that ASRT^2_2 is not overly weak by comparison with at least some of the principles studied in conjunction with SRT^2_2. The proof resembles that of Theorem 2.4 of [26] in that it uses the result that every effectively immune set computes a diagonally non-computable function (see [36], p. 199)). Here we also need the fact, due to Kučera, that every 1-random set is effectively bi-immune ([43], Theorem 6).

Proposition 3.5.4. *Over* RCA_0*,* ASRT^2_2 *implies* DNR *but is not implied by* WKL_0*.*

Proof. For the implication, we give only an argument for ω-models, as it, and all the results it employs, admit straightforward formalization in RCA_0. So let $\mathscr{M}$ be an ω-model of ASRT^2_2 and fix $X \in \mathscr{M}$. Fix u as in the proof of Proposition 3.2.7, let $\widetilde{M} = \Phi^{X'}_u$, and let M be a $\Delta^{0,X}_2$ approximation of $\widetilde{M}$, sped up to ensure that $2M(\sigma, s) = M(\sigma 0, s) + M(\sigma 1, s)$ for all σ and s. Then $M \in \mathscr{M}$ and is a martingale approximation, so there exists a stable X-computable coloring $f \in \mathscr{M}$ and an infinite set $H \in \mathscr{M}$ such that M does not succeed on f and H is homogeneous for f. If we let $A = \{x : \lim_s f(x, s) = 1\}$ then $\widetilde{M}$ does not succeed on A, so A is 1-random relative to X and hence effectively bi-immune relative to X. Then H, being an infinite subset or co-subset of A, is effectively immune relative to X, and so computes a function $g \in \mathscr{M}$ that is diagonally non-computable relative to X.

For the non-implication, recall that for every incomplete Δ^0_2 PA degree $\mathbf{d}$ there exists a model of WKL_0 consisting only of sets of degree below $\mathbf{d}$. Let $\mathscr{M}$ be any such model. By Theorem 3.1.3, $\mathbf{d}$ is not almost s-Ramsey, and so there is a Δ^0_2 martingale $\widetilde{M}$ which succeeds on every Δ^0_2 set containing an infinite subset or co-subset of degree at most $\mathbf{d}$. Let M be a (suitably sped up) Δ^0_2 approximation to $\widetilde{M}$, so that $M \in \mathscr{M}$ and is a martingale approximation. Since all stable colorings in $\mathscr{M}$ that have an infinite homogeneous set in $\mathscr{M}$ have one of degree below $\mathbf{d}$, it follows that M succeeds on them all. Thus, $\mathscr{M}$ is not a model of ASRT^2_2. □

It follows that neither DNR nor COH imply ASRT^2_2 either, the latter because COH does not imply DNR by Theorem 3.7 of [26].

In view of the remarks made at the beginning of the section, it is natural to ask whether ASRT^2_2 implies WKL_0 or COH (the preceding proposition makes the first of these at least plausible). We conclude this section by giving negative answers to both questions.

Proposition 3.5.5. *Over* RCA_0, ASRT^2_2 *does not imply* WKL_0.

Proof. We modify the argument from the proof of Proposition 3.4.5 above. Let L be a given low 1-random set, and let $e \in \omega$ be given. If $\Phi^{\emptyset'}_e$ is a total martingale, let M, A, B and C be as in the proof of Proposition 3.2.7 with i a $\Delta^{0,\emptyset'}_1$ index for L'. Then $A = B \oplus C$, the set $L \oplus B$ is low, and $B \notin S[\Phi^{\emptyset'}_e]$. Furthermore, A is not in $S[M]$ and is therefore 1-random relative to L, so, by van Lambalgen's theorem relative to L, B is 1-random relative to L too. Since L is 1-random, another application of van Lambalgen's theorem yields that $L \oplus B$ is 1-random. By iterating, we can thus obtain an increasing sequence of sets $L_0 \leq_T L_1 \leq_T \cdots$ such that each L_e is low, 1-random, and computes a set $B \notin S[\Phi^{\emptyset'}_e]$ when $\Phi^{\emptyset'}_e$ is a total martingale.

Let $\mathscr{M}$ be the ideal $\{S : (\exists e)[S \leq_T L_e]\}$. We claim that this is a model of ASRT^2_2. Indeed, suppose that $M \in \mathscr{M}$ is a martingale approximation. Then $\widetilde{M} : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$ defined by $\widetilde{M}(\sigma) = \lim_s M(\sigma, s)$ for all σ is a $\Delta^{0,M}_2$ martingale and hence a Δ^0_2 martingale since every element in $\mathscr{M}$ is low. We can thus fix an e so that $\widetilde{M} = \Phi^{\emptyset'}_e$. Then by construction, L_e computes a Δ^0_2 infinite set $B \notin S[\widetilde{M}]$, so if we let f be a Δ^0_2 approximation of B then f is a computable stable coloring, and hence $f \in \mathscr{M}$ and $f \leq_T M$. Clearly, M does not succeed on f in the sense of Definition 3.5.1, but B computes an infinite homogeneous set H for f, which, since $H \leq_T B \leq_T L_e$, belongs to $\mathscr{M}$.

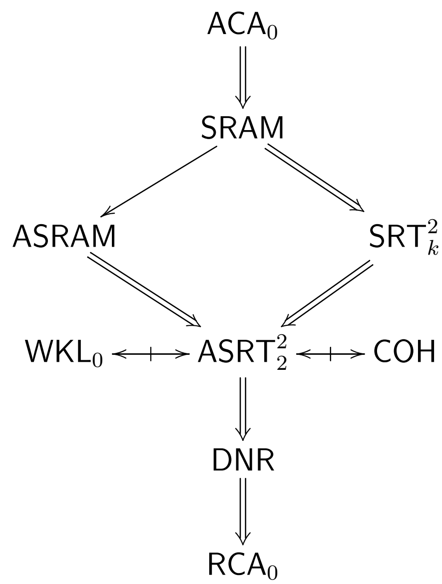

Figure 3.1: Relationship of SRAM, ASRAM, and ASRT^2_2 to other weak principles, with $k \geq 2$ being arbitrary. Arrows denote implications provable in RCA_0, double arrows denote implications that are known to be strict, and negated arrows indicate non-implications.

Now recall that every ω-model of WKL_0 contains a set of PA degree. As noted above, there are no low 1-random degrees, and so in particular none of the sets L_e has PA degree. Since the PA degrees are closed upwards, it follows that no element of $\mathscr{M}$ has PA degree either. Thus, WKL_0 does not hold in $\mathscr{M}$. $\square$

By Theorems 3.1.1 and 3.1.3 respectively, neither SRT^2_2 nor ASRAM has an ω-model consisting entirely of low sets. The same is true of COH because each of its ω-models must contain a p-cohesive set (see [5], p. 27), and each p-cohesive set has jump of degree strictly greater than $\mathbf{0}'$ by Theorem 2.1 of [40]. (A set is *p-cohesive* is if it is $\vec{R}$-cohesive for the sequence $\vec{R} = \{R_0, R_1, \ldots\}$ containing precisely the primitive recursive sets.) Hence, we immediately get the following:

Corollary 3.5.6. *Over* RCA_0*,* ASRT^2_2 *does not imply* SRT^2_2*,* ASRAM*, or* COH*.*

All the relations between the principles studied above are recapitulated in Figure 3.1.

3.6 Questions

We end this chapter with a few questions. From Sections 3.3 and 3.4 we have the following:

Question 3.6.1. Does there exist a low$_2$ almost s-Ramsey degree?

Question 3.6.2. Is $\mathbf{h}(\Delta^0_2)$ elementarily equivalent to $\mathbf{h}(\mathscr{C})$ for every class $\mathscr{C}$ of Δ^0_2 sets that is not Δ^0_2 null?

As noted above, Theorem 3.1.4 is a partial step towards an affirmative answer to Question 3.6.1.

From Section 3.5, our questions are about the remaining implications between ASRAM, SRAM, and SRT^2_2.

Question 3.6.3. Over RCA_0, does ASRAM imply SRAM? Does SRT^2_2 imply ASRAM or conversely?

Since SRT^2_2 has an ω-model consisting entirely of low_2 sets while SRAM does not, one of the two would likely be answered by a solution to Question 3.6.1.

Our final question concerns the system WWKL_0, introduced in Simpson and Yu [68], whose statement we recall:

Weak weak König's lemma (WWKL_0)**.** *Every tree $T \subseteq 2^{<\omega}$ such that*

$$\lim_n \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n} \neq 0$$

has an infinite path.

Question 3.6.4. Does ASRT^2_2 imply WWKL_0?

WWKL_0 follows from WKL_0, and so can not imply ASRT^2_2 by Proposition 3.5.4. Since the ω-models of WWKL_0 are precisely those that for every set X in them contain also an 1-random relative to X ([1], Lemma 1.3 (2)), a negative solution to the last question may follow from showing that the collection of Δ^0_2 sets having an infinite subset or co-subset not computing any 1-randoms is not Δ^0_2 null. It is worth remarking that Kjos-Hanssen [42] (see also [4], Theorem 7.4) has proved the non-effective version of this, showing that almost every infinite subset of ω has an infinite subset not computing any 1-randoms.

CHAPTER 4
POLARIZED AND TREE THEOREMS

4.1 Introduction

Call a tuple $\langle x_0, \dots, x_{n-1} \rangle \in \omega^n$ (by which we always we mean one with $x_i \neq x_j$ whenever $i \neq j$) *increasing* if $x_0 < \cdots < x_{n-1}$. As discussed in Chapter 1, when dealing with a coloring $f : [\omega]^n \to k$ it is convenient to write $f(x_0, \dots, x_{n-1})$ in place of $f(\{x_0, \dots, x_{n-1}\})$ whenever $\langle x_0, \dots, x_{n-1} \rangle$ is an increasing tuple. In fact, for the purposes of studying the computability-theoretic properties of infinite homogeneous sets, it would make no difference if we regarded f as being defined on increasing tuples only. Our investigation here, however, will be more sensitive to this distinction. We shall investigate several variants of Ramsey's theorem obtained by relaxing the definition of homogeneous set as follows:

Definition 4.1.1. Fix $n, k \geq 1$ and $f : [\omega]^n \to k$.

1. A *p-homogeneous* set for f is a sequence $\langle H_0, \dots, H_{n-1} \rangle$ of infinite sets such that for some $c < k$, called the *color* of this sequence, $f(\{x_0, \dots, x_{n-1}\}) = c$ for every $\langle x_0, \dots, x_{n-1} \rangle \in H_1 \times \cdots \times H_n$.

2. If (1) holds just for increasing tuples, we call $\langle H_0, \dots, H_{n-1} \rangle$ an *increasing* p-homogeneous set.

Our focus will be on the following "polarized" versions of Ramsey's theorem (the name comes from a similar combinatorial principle first studied by Erdős and Rado in [20]):

Polarized Ramsey's theorem (PT). *For all $n, k \geq 1$, every coloring $f : [\omega]^n \to k$ has a p-homogeneous set.*

Increasing polarized Ramsey's theorem (IPT). *For all $n, k \geq 1$, every coloring $f : [\omega]^n \to k$ has an increasing p-homogeneous set.*

Note that we could analogously define what it means for a tuple to be *decreasing* rather than increasing, and call a tuple *monotone* if it is either increasing or decreasing. A *monotone* p-homogeneous set for $f : [\omega]^n \to k$ could then be defined in the obvious way. But since every 2-tuple is monotone, it follows that for all $k \in \omega$, PT restricted to k-colorings of pairs coincides with the statement that every $f : [\mathbb{N}]^2 \to k$ has a monotone p-homogeneous set. We shall see in Theorem 4.3.7 that, modulo provability in RCA_0, the same is true in higher exponents, so we do not consider this principle separately.

Remark 4.1.2. Every infinite homogeneous set computes a p-homogeneous one. For if $f : [\omega]^n \to k$ is a coloring and H is infinite and homogeneous for f, then clearly $\langle H_0, \dots, H_{n-1} \rangle$, where $H_0 = \cdots = H_{n-1} = H$, is p-homogeneous for f. Note also that arithmetical complexity of $\langle H_0, \dots, H_{n-1} \rangle$ is the same as that of H.

The question of whether PT is weaker than RT, at least for pairs, was asked originally by Schmerl [54] while working on an application of the polarized theorem. It can be motivated by the following striking point of dissimilarity between the two principles: define $f : [\omega]^2 \to 2$ by letting $f(\{x, y\})$ equal 0 if x and y have like parity, and 1 otherwise, so that if we let H_0 consist of the even numbers and H_1 of the odd, then $\langle H_0, H_1 \rangle$ is p-homogeneous for f with color 1, yet f obviously has no infinite homogeneous set with this color.

Analyzing the computability-theoretic strength of PT and IPT will be the focus of Section 4.2 below. We show there that the major complexity bounds established by Jockusch [34] for infinite homogeneous sets of computable colorings hold also for p-homogeneous and increasing p-homogeneous sets. In Section 4.3, we study the proof-theoretic strength of PT, IPT, and related principles. Our main result is that for all n and k, PT restricted to k-coloring of n-tuples is equivalent to RT^n_k over RCA_0, which, at least for $n = 2$, may be surprising given the above example. This is also the first, and so far only, example of a classical principle other than RT^2_2 whose logical strength is equivalent to that of RT^2_2.[1] This is joint work with Jeffry L. Hirst.

In Sections 4.4 and 4.5, which is joint work with Jeffry L. Hirst and Tamara J. Lakins, we turn to a variant of Ramsey's theorem first considered by Chubb, Hirst, and McNicholl in [7].

Definition 4.1.3 ([7], p. 201)**.** Fix $S \subseteq 2^{<\omega}$ and $n, k \geq 1$.

1. We denote by $[S]^n$ the collection of linearly ordered subsets of $2^{<\omega}$ of cardinality n; i.e., if $\sigma, \tau \in S$ then either $\sigma \preceq \tau$ or $\tau \preceq \sigma$.

2. S is *isomorphic* to $2^{<\omega}$, written $S \cong 2^{<\omega}$, if there is a bijection $g : 2^{<\omega} \to S$ such that for all $\sigma, \tau \in 2^{<\omega}$, $\sigma \preceq \tau$ if and only if $g(\sigma) \preceq g(\tau)$.

3. A *k-coloring of S of exponent n* is a map $[S]^n \to k$. If $S = \omega$ and $n = 2$, we call f a k-coloring of *pairs*.

4. A subset $H \subseteq S$ is *homogeneous* for a k-coloring $f : [S]^n \to k$ if f is constant on $[H]^n$.

Given $f : [2^{<\omega}]^n \to k$ and $\{\sigma_0, \ldots, \sigma_{n-1}\} \in [2^{<\omega}]^n$, we shall write $f(\sigma_0, \ldots, \sigma_{n-1})$ instead of $f(\{\sigma_0, \ldots, \sigma_{n-1}\})$ when $\sigma_0 \preceq \cdots \preceq \sigma_{n-1}$.

Tree theorem (TT)**.** *For all $n, k \geq 1$, every coloring $f : [2^{<\omega}]^n \to k$ has a homogeneous set isomorphic to $2^{<\omega}$.*

---

[1] Montalbán [48, Section 1] calls a subsystem of second-order arithmetic *robust* if it is equivalent to small 'perturbations' of itself. All the big five subsystems are robust in this informal sense, but most of the weak combinatorial principles lying below ACA_0 are not. The equivalence of PT^2_2 with RT^2_2 suggests that RT^2_2 may be, at least to a small degree, an exception.

We introduce and study polarized and increasing polarized versions of the tree theorem, and formulate several possible notions of stability for colorings on $2^{<\omega}$. While we are able to replicate many of the arguments from the linear setting here, our failure to do so in some cases exposes how essential linearity is to many arguments about Ramsey's theorem. Identifying further examples of this phenomenon, as we do below, could thus lead to a better understanding of the key differences between Ramsey's theorem and the tree theorem.

4.2 Computability-theoretic aspects of PT

It is clear from Remark 4.1.2 that many theorems about the complexity of infinite homogeneous sets carry over trivially to p-homogeneous sets. We list some of these.

Theorem 4.2.1. *Fix $n, k \geq 2$.*

1. *Every computable $f : [\omega]^n \to k$ has a Π^0_n p-homogeneous set.*
2. *Every computable $f : [\omega]^n \to k$ has a p-homogeneous set with Δ^0_{n+1} jump.*
3. *Every computable stable $f : [\omega]^2 \to k$ has a Δ^0_2 p-homogeneous set.*
4. *Given a sequence of non-computable sets, every computable $f : [\omega]^2 \to k$ has a p-homogeneous set not computing any member of this sequence.*
5. *Every computable $f : [\omega]^2 \to k$ has a low$_2$ p-homogeneous set.*

Proof. This follows at once from Remark 4.1.2 and, for (1) and (2), Theorem 1.3.3; for (3), Lemma 1.3.7; for (4), Seetapun's theorem (Theorem 1.3.4); and for (5), Theorem 1.3.5. □

The next proposition shows that for stable colorings, the converse to Remark 4.1.2 holds. Thus, up to degree, infinite homogeneous sets for stable colorings are the same as p-homogeneous sets, which in turn are the same as increasing p-homogeneous sets.

Proposition 4.2.2. *For all $k \geq 2$ and all stable $f : [\omega]^2 \to k$, every increasing p-homogeneous set for f computes an infinite homogeneous set.*

Proof. Let $f : [\omega]^2 \to k$ be a stable coloring and assume that $\langle H_0, H_1 \rangle$ is an increasing p-homogeneous set for f, say with color $c < k$. We construct an infinite homogeneous set for f computably from $\langle H_0, H_1 \rangle$. Let $a_0 = \min H_0$, and suppose that for some $n \geq 0$, we have defined a sequence $a_0 < \cdots < a_n$ of elements of H_0. Since H_1 is infinite and, for each $i \leq n$, $f(a_i, x) = c$ for every $x > a_i$ in H_1, it follows by stability of f that $\lim_s f(a_i, s) = c$. Hence, there exists $x > a_n$ in H_0 such that $f(a_i, x) = c$ for all $i \leq n$, and we let a_{n+1} be the least such x. By induction we get an infinite set $\{a_0, a_1, \ldots\} \subseteq H_0$ such that $f(a_m, a_n) = c$ whenever $m < n$, and so this set is homogeneous for f. □

It follows, for example, that there exists a computable stable coloring with no low increasing p-homogeneous set (see Theorem 3.1.1).

The degrees of infinite homogeneous and p-homogeneous sets for general colorings are harder to compare. One point of similarity is the next theorem, the analogue of Theorem 1.3.3 (4), which establishes that non-trivial information can be coded into increasing p-homogeneous sets by coloring 3-tuples instead of 2-tuples. The proof closely follows that of the linear version, and we include it here only to highlight a minor necessary adjustment.

Theorem 4.2.3. *For every $n \geq 1$, there exists a computable $f : [\omega]^{n+1} \to 2$ every increasing p-homogeneous set of which computes $\emptyset^{(n-1)}$.*

We first need a lemma.

Lemma 4.2.4. *Fix $n \geq 1$. If $f : [\omega]^n \to 2$ is Δ^0_2, there exists a computable $g : [\omega]^{n+1} \to 2$ such that whenever $\langle H_0, \ldots, H_n \rangle$ is an increasing p-homogeneous set for g, $\langle H_0, \ldots, H_{n-1} \rangle$ is an increasing p-homogeneous set for f.*

Proof. We let g be any computable function such that $f(\overline{x}) = \lim_s g(\overline{x}, s)$ for all increasing tuples $\overline{x} \in \omega^n$, which exists since f is Δ^0_2. Let $\langle H_0, \ldots, H_n \rangle$ be any increasing p-homogeneous set for g, say with color $c < 2$, and suppose $\langle x_0, \ldots, x_{n-1} \rangle$ is an increasing tuple in $H_0 \times \cdots \times H_{n-1}$. We have that $g(x_0, \ldots, x_{n-1}, x) = c$ for all sufficiently large $x \in H_n$, which implies, since H_n is infinite and $\lim_s g(\overline{x}, s)$ exists for all $\overline{x}$, that $\lim_s g(x_0, \ldots, x_{n-1}, s) = c$. It follows that $f(x_0, \ldots, x_{n-1}) = c$ by definition of g, and hence that $\langle H_0, \ldots, H_{n-1} \rangle$ is an increasing p-homogeneous set for f, as claimed. □

Proof of Theorem 4.2.3. Jockusch and McLaughlin [38, Theorem 4.13] proved the existence of an increasing function from ω to ω Turing equivalent to $\emptyset^{(n-1)}$ and computable from every function which dominates it. Let g be such a function and let $f_0 : [\omega]^2 \to 2$ be the Δ^0_n coloring defined by

$$f_0(x, y) = \begin{cases} 0 & \text{if } y > g(x), \\ 1 & \text{otherwise,} \end{cases}$$

for all numbers $x < y$.

Let $\langle H_0, H_1 \rangle$ be an increasing p-homogeneous set for f_0, noting that it must have color 0 since for any $x \in H_0$ there is certainly an element $y \in H_1$ with $y > g(x)$ and hence $f_0(x, y) = 0$. Define a sequence $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ inductively by letting a_n be the least element of H_0 greater than a_i and b_i for all $i < n$, and letting b_n be the least element of H_1 greater than a_n. The function $m(n) = b_n$ is then computable from $\langle H_0, H_1 \rangle$ and, since g is increasing and $f_0(a_n, b_n) = 0$ and $a_n \geq n$ for all n, we have $m(n) > g(a_n) \geq g(n)$. It follows that m dominates g and hence that $\emptyset^{(n-1)} \equiv_T g \leq_T m \leq_T \langle H_0, H_1 \rangle$.

Thus f_0 can serve as the base case of a finite induction with which we complete the proof. Assume that for some $m \geq 0$ there exists a Δ^0_{n-m} coloring $f_m : [\omega]^{m+2} \to 2$ every increasing p-homogeneous set of which computes $\emptyset^{(n-1)}$. Applying Lemma 4.2.4, relativized to $\emptyset^{(n-m-2)}$, to f_m yields a Δ^0_{n-m-1} coloring $f_{m+1} : [\omega]^{m+3} \to 2$ every increasing p-homogeneous set of which computes an increasing p-homogeneous set for f_m and so computes $\emptyset^{(n-1)}$. By induction, $f = f_{n-1}$ is the desired computable 2-coloring of $[\omega]^{n+1}$. □

Part (1) of Theorem 4.2.1 gives an upper bound on the complexity of p-homogeneous sets with respect to the arithmetical hierarchy. A lower bound on increasing p-homogeneous sets can be obtained from the proof of Theorem 1.3.3 (2), thus establishing that the arithmetical bounds on infinite homogeneous, p-homogeneous, and increasing p-homogeneous sets agree.

Theorem 4.2.5. *For every $n \geq 2$, there exists a computable $f : [\omega]^n \to 2$ with no Δ^0_n increasing p-homogeneous set, and hence also no Σ^0_n increasing p-homogeneous set.*

Proof. The second part follows from the first because every Σ^0_n infinite set has a Δ^0_n infinite subset. Thus, if $\langle H_0, \ldots, H_{n-1} \rangle$ is a Σ^0_n p-homogeneous set for some computable coloring, then letting $\widetilde{H}_i$ be a Δ^0_2 infinite subset of H_i for every i, yields a Δ^0_n p-homogeneous set $\langle \widetilde{H}_0, \ldots, \widetilde{H}_{n-1} \rangle$.

Now to prove the first part, we proceed by induction on n. First, suppose $n = 2$. The proof of Theorem 1.3.3 (2) for $n = 2$ (Lemma 5.9 of [34]) constructs a computable $f : [\omega]^2 \to 2$ such that for every e, if $\lim_s \Phi_e(x, s)$ exists for every x and the limit is the characteristic function of an infinite set H, then for all sufficiently large s there exist $x < y < s$ in H with $f(x, s) \neq f(y, s)$. Fix this f, and suppose $\langle H_0, H_1 \rangle$ is a Δ^0_2 pair of infinite sets. Since H_0 is Δ^0_2, there is an s_0 such that for all $s \geq s_0$ there exist $x < y < s$ in H_1 with $f(x, s) \neq f(y, s)$. In particular, since H_1 is infinite, we can choose such an s in H_1 to witness that $\langle H_0, H_1 \rangle$ is not p-homogeneous. This establishes the theorem for $n = 2$.

Since the above argument obviously relativizes, we now assume the theorem and all its relativizations hold for some $n \geq 2$. We prove it in relativized form for $n + 1$. Fixing an arbitrary set X and relativizing the induction hypothesis to X' yields a $\Delta^{0,X}_2$ coloring $f : [\omega]^n \to 2$ with no $\Delta^{0,X'}_n = \Delta^{0,X}_{n+1}$ increasing p-homogeneous set. By Lemma 4.2.4 relative to X, there exists an X-computable coloring $g : [\omega]^{n+1} \to 2$ such that whenever $\langle H_0, \ldots, H_n \rangle$ is increasing p-homogeneous for g, $\langle H_0, \ldots, H_{n-1} \rangle$ is increasing p-homogeneous for f. So in particular, any $\Delta^{0,X}_{n+1}$ increasing p-homogeneous set for g would yield such a set for f, which can not be. This completes the proof. □

4.3 Proof-theoretic content of PT

By analogy with Ramsey's theorem, for $n, k \geq 1$, we use PT^n_k and IPT^n_k to denote the restrictions of PT and IPT to k-colorings of exponent n. Similarly, we let SPT^2_k and SIPT^2_k be the restrictions of PT and IPT to stable k-colorings of pairs. We define PT^n, IPT^n, SPT^2, and SIPT^2 in the obvious fashion. It is straightforward to formalize the statements of all these principles in RCA_0. The following proposition summarizes the most obvious relationships between the principles introduced above (he proofs are immediate from Remark 4.1.2 and the relevant definitions):

Proposition 4.3.1. *For all $n, k \geq 2$, the following implications are provable in RCA_0:*

1. $\mathsf{RT}^n_k \to \mathsf{PT}^n_k \to \mathsf{IPT}^n_k$;
2. $\mathsf{PT}^2_k \to \mathsf{SPT}^2_k$;

3. $\mathsf{IPT}^2_k \to \mathsf{SIPT}^2_k$*;*

4. $\mathsf{SRT}^2_k \to \mathsf{SPT}^2_k \to \mathsf{SIPT}^2_k$*.*

The same implications hold between RT^n*,* PT^n*,* IPT^n*,* SRT^2*,* SPT^2*, and* SIPT^2*.*

4.3.1 Colorings of pairs

We begin by considering stable colorings. As noted above, for all $k \geq 1$, SRT^2_k implies SPT^2_k, which in turn implies SIPT^2_k, whence it is natural to ask whether either of the reverse implications holds. Proposition 4.2.2 above makes it seem likely that in fact SIPT^2_k reverses to SRT^2_k. We shall show that this is indeed the case, but our proof will be somewhat indirect. Merely formalizing the proof of Proposition 4.2.2 in RCA_0 yields the following:

Proposition 4.3.2. *Over* $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$*,* SIPT^2_2 *implies* SRT^2_2*.*

Proof. The only non-trivial point in the proof of Proposition 4.2.2 was the inductive step in the construction of the sequence $\langle a_n : n \in \mathbb{N}\rangle$. Note that if $a_0 < \cdots < a_n$ are defined then for every $i \leq n$ there exists $s \in \mathbb{N}$ such that for all $x > s$, $f(a_i, x) = c$. By $\mathsf{B}\Pi^0_1$ there exists $t \in \mathbb{N}$ such that for every $i \leq n$, there exists $s \leq t$ such that for all $x > s$, $f(a_i, x) = c$. Hence, for every $i \leq n$ and any $x > t$, $f(a_i, x) = c$, and we can let a_{n+1} be the least element of H_0 that is greater than t. Thus the construction of the sequence can be carried out using $\mathsf{B}\Sigma^0_2$, as desired. □

Since SRT^2_2 implies $\mathsf{B}\Sigma^0_2$, it follows that for every $k \geq 1$, SPT^2_k, SIPT^2_k, and SRT^2_k are all equivalent over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. A similar situation exists between SRT^2_k and the following principle, introduced by Cholak, Jockusch, and Slaman [5, Statement 7.9]:

Δ^0_2 **subset or co-subset principle** (D^2)**.** *For all* $k \geq 1$ *and all stable* $f : [\omega]^2 \to k$*, there exists an infinite set* X *and* $c < k$ *such that* $\lim_s f(x, s) = c$ *for all* $x \in X$*.*

(For $k \geq 1$, D^2_k is the obvious restriction of D^2.) It is clear that, over RCA_0, SRT^2_k implies D^2_k for every $k \geq 1$. In Lemma 7.10 of [5], it is claimed that D^2_2 implies SRT^2_2, but their proof appears to require the use of $\mathsf{B}\Sigma^0_2$, as pointed out in Section 2.2 of [26]. The question of whether this use can be avoided was not answered until the work of Chong, Lempp, and Yang in [6]:

Theorem 4.3.3 (Chong, Lempp, and Yang [6], Theorem 1.4)**.** *Over* RCA_0*,* D^2_2 *implies* $\mathsf{B}\Sigma^0_2$*.*

While we do not know of a direct proof that SIPT^2_2 implies SRT^2_2, we can show the following:

Lemma 4.3.4. *For all* $k \geq 1$*,* SIPT^2_k *implies* D^2_k *over* RCA_0*.*

Proof. Let a stable $f : [\mathbb{N}]^2 \to k$ be given, and by SIPT^2_k choose an increasing p-homogeneous set $\langle H_0, H_1\rangle$ for f, say with color $c < k$. Fix $x \in H_0$. Since $f(x, y) = c$ for every $y \in H_1$, and since H_1 is infinite and $\lim_s f(x, s)$ exists, it must be that $\lim_s f(x, s) = c$. □

The following now easily follows:

Proposition 4.3.5. *For all $k \geq 1$, the following are equivalent over RCA_0:*

1. SRT^2_k;
2. SPT^2_k;
3. SIPT^2_k.

As a straightforward consequence, SRT^2, SPT^2, and SIPT^2 are equivalent as well. For general interest, we note that this equivalence can be established without having to go through Theorem 4.3.3. This is because SIPT^2 implies RT^1, and hence $\mathsf{B}\Sigma^0_2$: given $f : \mathbb{N} \to k$ define $g : [\mathbb{N}]^2 \to k$ by $g(x, y) = f(x)$, noting that g is stable, and that if $\langle H_0, H_1\rangle$ is increasing p-homogeneous for g then H_0 is infinite and homogeneous for f.

We can extend the equivalence of SRT^2_k with SPT^2_k from stable to general colorings of pairs. For this, we recall the following principle first studied in the context of reverse mathematics by Hirschfeldt and Shore [27, p. 180]:

Ascending or descending sequence principle (ADS)**.** *For every linear order $\leq_L$ on $\mathbb{N}$ there is an infinite set X that is either an ascending sequence under this order, i.e., $x \leq_L y$ for all $x \leq y$ in X, or a descending sequence, i.e., $y \leq_L x$ for all $x \leq y$ in X.*

Theorem 4.3.6. *For every $k \geq 2$, RT^2_k is equivalent to PT^2_k over RCA_0, and RT^2 to PT^2.*

Proof. Fix $k \geq 2$. That RT^2_k implies PT^2_k over RCA_0 is by (1) of Proposition 4.3.1 above, so we need only establish the converse. By Proposition 1.3.9, RT^2_2 is equivalent to $\mathsf{SRT}^2_2 + \mathsf{COH}$, and this result easily generalizes to k colors. Since PT^2_k implies SPT^2_k over RCA_0, and hence, by Proposition 4.3.5, SRT^2_k, it suffices to show that PT^2_k implies COH. By Proposition 2.10 of [27], COH follows from ADS, so it is enough if we prove that PT^2_k implies ADS.

In RCA_0, let $\leq_L$ be a linear ordering on $\mathbb{N}$, and define $f : [\mathbb{N}]^2 \to 2$ by

$$f(x, y) = \begin{cases} 0 & \text{if } x \leq_L y, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x < y$ in $\mathbb{N}$. Let $\langle H_0, H_1\rangle$ be a p-homogeneous set for f obtained by applying PT^2_k. Set $a_0 = \min H_0$ and for $n \geq 0$, let a_{n+1} to be the least element $> a_n$ of H_1 if n is even and of H_0 if n is odd. By Δ^0_1 comprehension we obtain an infinite set $\{a_0, a_1, \ldots\}$ such that for all n, $a_n < a_{n+1}$, $a_{2n} \in H_0$, and $a_{2n+1} \in H_1$. Since for all n, $f(a_n, a_{n+1}) = c$ where c is the color of $\langle H_0, H_1\rangle$, it follows that either $a_n \leq_L a_{n+1}$ for all n or $a_{n+1} \leq_L a_n$ for all n. The set $\{a_0, a_1, \ldots\}$ is therefore an ascending or descending sequence under $\leq_L$. □

We do not know if it is possible to extend the theorem to show also that RT^2_k is equivalent to IPT^2_k. This suggests that the difference between parts (1) and (2) of Definition 4.1.1 may be more significant than it seems.

We conclude this section by summarizing our results in Figure 4.1.

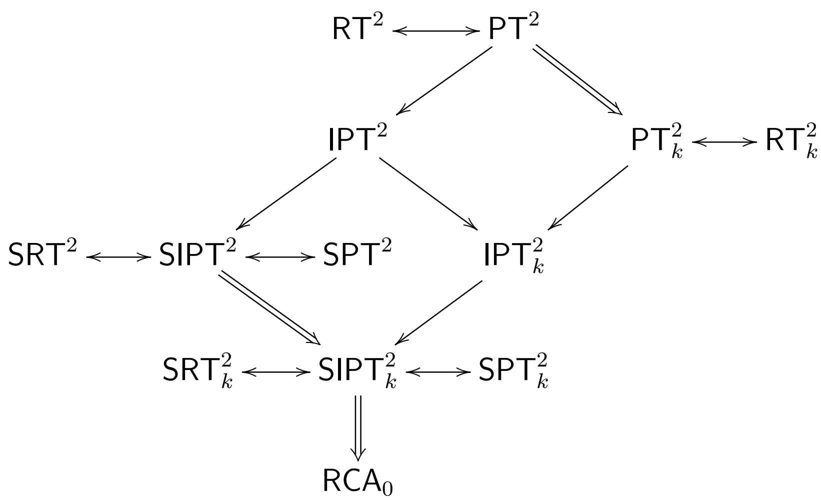

Figure 4.1: Relationship of the polarized principles to other weak principles, with $k \geq 2$ being arbitrary. Arrows denote implications provable in RCA_0, and double arrows denote implications that are known to be strict. All the implications there been explained above. That SRT^2 is strictly stronger than SRT^2_k and RT^2 than RT^2_k holds for $k = 2$ by a remark following Corollary 11.5 of [5], and thus in general because SRT^2_k is equivalent to SRT^2_2 and RT^2_k to RT^2_2 for any standard k.

4.3.2 Colorings of higher exponents

In this section, we show that in higher exponents the polarized and increasing polarized theorems are provably equivalent to Ramsey's theorem. Drawing on results of Jockusch [34], Simpson (see [59, Theorem III.7.6]) was able to show that for $n \geq 3$, RT^n is equivalent over RCA_0 to ACA_0. We can similarly draw on Theorem 4.2.3 above to obtain the next result. Here we recall that a monotone p-homogeneous set, as discussed at the end of Section 4.1, is defined as in Definition 4.1.1, but restricting to only monotone tuples.

Theorem 4.3.7. *For all $n \geq 3$ and $k \geq 2$, the following are equivalent over RCA_0:*

1. ACA_0;
2. PT^n;
3. PT^n_k;
4. *for all $j \in \mathbb{N}$, every $f : [\mathbb{N}]^n \to j$ has a monotone p-homogeneous set;*
5. *every $f : [\mathbb{N}]^n \to k$ has a monotone p-homogeneous set;*
6. IPT^n;
7. IPT^n_k.

Proof. We fix $n \geq 3, k \geq 2$, and argue in RCA_0. That (1) implies (2) follows from Remark 4.1.2 and the fact, mentioned above, that ACA_0 proves RT^n.

The implications from (2) to (3) to (5) to (7) and from (2) to (4) to (6) to (7) are trivial.

It remains to show that (7) implies (1). To this end, first notice that IPT^n_k implies IPT^3_2. Indeed, given $f : [\mathbb{N}]^3 \to 2$, define $g : [\mathbb{N}]^n \to 2$ by $g(x_0, x_1, x_2, \ldots, x_{n-1}) = f(x_0, x_1, x_2)$, let $\langle H_0, \ldots, H_{n-1}\rangle$ be any p-homogeneous set for g given by IPT^n_k, and notice that $\langle H_0, H_1, H_2\rangle$ is necessarily p-homogeneous for f. Thus to complete the proof it suffices to show that IPT^3_2 implies arithmetical comprehension, or equivalently, that IPT^3_2 implies the existence of the range of a given injective function $F : \mathbb{N} \to \mathbb{N}$. Define $f : [\mathbb{N}]^3 \to 2$ by

$$f(x_0, x_1, x_2) = \begin{cases} 1 & \text{if } (\forall x)[x_1 \leq x \leq x_2 \to F(x) > x_0], \\ 0 & \text{otherwise,} \end{cases}$$

and let $\langle H_0, H_1, H_2\rangle$ be p-homogeneous for f. Notice that the color of $\langle H_0, H_1, H_2\rangle$ must be 1, since otherwise we could fix an $x_0 \in H_0$ and pick $x_0 + 2$ disjoint increasing pairs of elements $> x_0$ from $H_1 \times H_2$ to find two different numbers mapping to the same value below x_0 and thereby witnessing that F is not injective. Now if $y \in \mathbb{N}$ is given, choose any increasing sequence $\langle x_0, x_1, x_2\rangle \in H_0 \times H_1 \times H_2$ with $y < x_0$. Then y is in the range of F if and only if there is an $x < x_1$ with $F(x) = y$, since if $y = F(x)$ for some $x \geq x_1$ we can choose $x_3 > x$ in H_2 to witness that $f(x_0, x_1, x_3) = 0$, a contradiction. □

The preceding theorem and Theorem 4.3.6 now immediately yield the following new characterization of Ramsey's theorem in a fixed exponent:

Corollary 4.3.8. *For every $n, k \geq 1$, RT^n_k is equivalent to PT^n_k over RCA_0, and RT^n to PT^n.*

In the linear version of Ramsey's theorem, every subset of an homogeneous set is also homogeneous. Furthermore, repeated applications of Ramsey's theorem can easily yield infinite sets that are simultaneously homogeneous for a number of colorings. With a little effort, a similar result can be proved in the increasing polarized case. This is formulated in the next lemma and applied in the subsequent result.

Lemma 4.3.9. *For all $n, k \geq 1$, it is provable in RCA_0 that if $\langle S_0, \ldots, S_{n-1}\rangle$ is a sequence of infinite sets, then IPT^n_k implies that for any $f : [\mathbb{N}]^n \to k$ there is an increasing p-homogeneous set $\langle H_0, \ldots, H_{n-1}\rangle$ for f such that $H_i \subseteq S_i$ for all i.*

Proof. Working in RCA_0, assume IPT^n_k and fix $\langle S_0, \ldots, S_{n-1}\rangle$ as above. We define infinite subsets of each S_i inductively. Let $s_{n,0}$ be the minimum element of S_{n-1}. Having defined $s_{i+1,j}$ for some $i < n$, let $s_{i,j}$ be the least element of S_i greater than $s_{i+1,j}$. Finally, for $j \geq 0$, let $s_{n,j+1}$ be the least element of S_{n-1} greater than $s_{1,j}$. Then for all $i < n$, the sequence $\langle s_{i,j} : j \in \mathbb{N}\rangle$ exists by primitive recursion and is an infinite increasing subsequence of S_i. Furthermore, if $\langle s_{0,j_0}, \ldots, s_{n-1,j_{n-1}}\rangle \in S_0 \times \cdots \times S_{n-1}$ is an increasing tuple, then so is $\langle j_0, \ldots, j_{n-1}\rangle \in \mathbb{N} \times \cdots \times \mathbb{N}$.

Now fix $f : [\omega]^n \to k$, and define the coloring $g : [\omega]^n \to k$ by $g(j_0, \ldots, j_{n-1}) = f(s_{1,j_0}, \ldots, s_{n,j_{n-1}})$. Apply IPT^n_k to g to obtain a fixed color c and a c-colored increasing p-homogeneous set $\langle G_0, \ldots, G_{n-1}\rangle$. Define $\langle H_0, \ldots, H_{n-1}\rangle$ by setting $H_i = \{s_{i,j} : j \in G_i\}$. Then for each $i < n$, H_i is an infinite subset of S_i.

We claim that $\langle H_0, \ldots, H_{n-1}\rangle$ is a c-colored increasing p-homogeneous set for f. Choose any increasing tuple in $H_0 \times \cdots \times H_{n-1}$. By definition of the H_i, we can write this tuple as $\langle s_{0,j_0}, \ldots, s_{n-1,j_{n-1}}\rangle$ where for each i we have $j_i \in G_i$. Hence, as remarked above, $\langle j_0, \ldots, j_{n-1}\rangle$ is an increasing tuple in $G_0 \times \cdots \times G_{n-1}$. By p-homogeneity of $\langle G_0, \ldots, G_{n-1}\rangle$, $g(j_0, \ldots, j_{n-1}) = c$, and so by construction $f(s_{0,j_0}, \ldots, s_{n-1,j_{n-1}}) = c$ also. □

For our last result in this section, we turn to ACA_0', the subsystem of second-order arithmetic obtained by adding to the axioms of ACA_0 the statement that for all sets X and all $n \in \mathbb{N}$, the nth Turing jump of X exists. (It is known that ACA_0' is strictly stronger than ACA_0 and strictly weaker than ACA_0^+, the system consisting of ACA_0 together with the assertion of the existence for every set X of its ωth Turing jump.) Since the set universe of any ω-model of ACA_0 is closed under any standard number of jumps, it follows from Lemma 5.9 of Jockusch [34] that every ω-model of $\mathsf{RCA}_0 + \mathsf{RT}$ is also a model of ACA_0'. Mileti [46, Proposition 7.1.4] established the stronger result that over RCA_0, ACA_0' is equivalent to RT, and we now show that this equivalence extends to IPT. Our argument is different from that sketched by Mileti in that the reversal from IPT to ACA_0' is obtained not by formalizing the proof of Lemma 5.9 of [34] but by directly appealing to the definition of the jump.

Thus we begin by looking at precisely how the jump is formalized. In the language of second-order arithmetic, we define some convenient abbreviations. Given any set X, we write $Y = X'$ precisely when

$$(\forall \langle m, e\rangle)[\langle m, e\rangle \in Y \leftrightarrow (\exists t)\, \Phi^X_{e,t}(m)\downarrow].$$

Here $\Phi^X_{e,t}(m)\downarrow$ is a fixed formalization of the assertion that the Turing machine with code number e, using an oracle for X, halts on input m with the entire computation (including the use) bounded by t. For the nth jump, $n \geq 1$, we write $Y = X^{(n)}$ if there is a finite sequence $\langle X_0, \ldots, X_n\rangle$ such that $X_0 = X$, $X_n = Y$, and for every $i < n$, $X_{i+1} = X_i'$. Thus $Y = X'$ if and only if $Y = X^{(1)}$. For a given $n \in \mathbb{N}$, we say that $X^{(n)}$ *exists* provided there exists a set Y with $Y = X^{(n)}$. In what follows, we shall also need a notation for finite approximations to jumps. For any set X and integer s define

$$X_s' = \{\langle m, e\rangle : (\exists t < s)\ \Phi^X_{e,t}(m)\downarrow\},$$

and for integers $u_1, \ldots, u_n$ define

$$X^{(n+1)}_{u_n,\ldots,u_1,s} = (X^{(n)}_{u_n,\ldots,u_1})_s'.$$

Using this notation, we can state and prove our proposition addressing polarized versions of Ramsey's theorem and ACA_0'. As the current literature contains very few examples of detailed proofs involving ACA_0', we include the following somewhat technical proof of this final implication:

Proposition 4.3.10. *The following are equivalent over* RCA_0*:*

1. ACA_0';
2. RT;
3. PT;
4. IPT.

Proof. In Proposition 7.1.4 of [46], Mileti gives a proof of the implication from (1) to (2). Remark 4.1.2 can be used to prove that (2) implies (3) implies (4), so it only remains to prove that (4) implies (1).

Assume RCA_0 and (4). Given X and n, we wish to show that $X^{(n)}$ exists. Define the function $f : [\mathbb{N}]^{2n+1} \to n+1$ by setting $f(s_0, s_1, \ldots, s_n, u_1, \ldots, u_n)$ equal to the least positive $i \leq n$ such that

$$\exists \langle m, e\rangle < s_{n-i}\ [\langle m, e\rangle \in X^{(i)}_{s_n,\ldots,s_{n-i+1}} \not\leftrightarrow \langle m, e\rangle \in X^{(i)}_{u_n,\ldots,u_{n-i+1}}]$$

if such an i exists, and 0 otherwise. Apply IPT to f to get an increasing p-homogeneous set $\langle H_0, \ldots, H_{2n}\rangle$ of color c. The argument below shows that $c = 0$.

Seeking a contradiction, suppose $c = i > 0$. By removing elements from $H_1, \ldots, H_{2n}$ if necessary, we can arrange for $\min H_0 < \min H_1 < \cdots < \min H_{2n}$. Define a coloring g by letting $g(s_0, s_1, \ldots, s_n, u_1, \ldots, u_n)$ be the least $\langle m, e\rangle < \min H_{n-i}$ with $\langle m, e\rangle \in X^{(i)}_{s_n,\ldots,s_{n-i+1}} \not\leftrightarrow \langle m, e\rangle \in X^{(i)}_{u_n,\ldots,u_{n-i+1}}$ if such exists, and $\min H_{n-i}$ otherwise. Applying Lemma 4.3.9 to g, we can find a p-homogeneous set for g contained in $H_0, \ldots, H_{2n}$, and clearly its color must be less than $\min H_{n-i}$. Consequently, without loss of generality, we may assume that there is a fixed $\langle m_0, e_0\rangle < \min H_{n-i}$ such that for all increasing tuples $\langle s_0, \ldots, s_n, u_1, \ldots, u_n\rangle$ in $H_0 \times \cdots \times H_{2n}$,

$$\langle m_0, e_0\rangle \in X^{(i)}_{s_n,\ldots,s_{n-i+1}} \not\leftrightarrow \langle m_0, e_0\rangle \in X^{(i)}_{u_n,\ldots,u_{n-i+1}}. \tag{4.1}$$

Fixing any such increasing tuple, notice that the minimality of i forces $X^{(i-1)}_{s_n,\ldots,s_{n-i+2}}$ to agree with $X^{(i-1)}_{u_n,\ldots,u_{n-i+2}}$ on all values less than s_{n-i+1}. Thus, we have that

$$(\exists t < s_{n-i+1})\Phi^{X^{(i-1)}_{s_n,\ldots,s_{n-i+2}}}_{e_0,t}(m_0)\downarrow \text{ implies } (\exists t < u_{n-i+1})\Phi^{X^{(i-1)}_{u_n,\ldots,u_{n-i+2}}}_{e_0,t}(m_0)\downarrow.$$

By (4.1) and the definition of approximations to the jump, the converse of this implication must fail, so it must be that

$$(\exists t < u_{n-i+1})\Phi^{X^{(i-1)}_{u_n,\ldots,u_{n-i+2}}}_{e_0,t}(m_0)\downarrow.$$

Now choose an increasing tuple $\langle s^*_0, \ldots, s^*_n, u^*_1, \ldots, u^*_n\rangle$ in $H_0 \times \cdots \times H_{2n}$ with $s^*_{n-i+1} > u_{n-i+1}$ and $u^*_{n-i+2} \geq u_{n-i+2}$. Since the argument just given applies to any increasing tuple, and in particular to

$$\langle s_0, \ldots, s_n, u_1, \ldots, u_{n-i+1}, u^*_{n-i+2}, \ldots, u^*_n\rangle,$$

we have

$$(\exists t < u_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{u^*_n,\dots,u^*_{n-i+2}}}(m_0)\downarrow. \tag{4.2}$$

But $X^{(i-1)}_{s^*_n\dots s^*_{n-i+2}}$ and $X^{(i-1)}_{u^*_n\dots u^*_{n-i+2}}$ must agree on all elements below s^*_{n-i+1} and hence below u_{n-i+1}. And since u_{n-i+1} bounds the use of the computation in (4.2) and $u_{n-i+1} < s^*_{n+i-1}$, we have

$$(\exists t < s^*_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{s^*_n,\dots,s^*_{n-i+2}}}(m_0)\downarrow \wedge (\exists t < u^*_{n-i+1})\Phi_{e_0,t}^{X^{(i-1)}_{u^*_n,\dots,u^*_{n-i+2}}}(m_0)\downarrow,$$

which contradicts (4.1). This completes the proof of our claim that $c = 0$.

Next, we use $\langle H_0,\dots,H_{2n}\rangle$ to define a new finite sequence of sets $\langle X_0,\dots,X_n\rangle$. Let $X_0 = X$ and for each i with $1 \le i \le n$, let $\langle m,e\rangle \in X_i$ if and only if $\langle m,e\rangle \in X^{(i)}_{s_n,\dots,s_{n-i+1}}$, where $\langle s_0,\dots s_n, u_1\dots u_n\rangle$ is the lexicographically least increasing tuple in $H_0\times\cdots\times H_{2n}$ such that $\langle m,e\rangle < s_{n-i}$. Note that the X_i are defined simultaneously rather than inductively, so by recursive comprehension the entire sequence $\langle X_0,\dots,X_n\rangle$ exists.

We claim that for each $i < n$, $X_{i+1} = X_i'$. Fixing i, we prove containment in both directions. This will obviously complete the proof, since then $\langle X_0,\dots,X_n\rangle$ will be a sequence witnessing that $X_n = X^{(n)}$ and hence that $X^{(n)}$ exists.

First, suppose $\langle m,e\rangle \in X_{i+1}$, and let $\langle s_0,\dots s_n,u_1\dots u_n\rangle$ be the lexicographically least increasing tuple with $\langle m,e\rangle < s_{n-i-1}$, so we have $\langle m,e\rangle \in X^{(i+1)}_{s_n,\dots,s_{n-i}}$. Applying the definition of approximations of jumps, $\langle m,e\rangle \in (X^{(i)}_{s_n,\dots,s_{n-i+1}})'_{s_{n-i}}$, and so

$$(\exists t < s_{n-i})\Phi_{e,t}^{X^{(i)}_{s_n,\dots,s_{n-i+1}}}(m)\downarrow.$$

Since s_{n-i} bounds the use of this computation, homogeneity of $\langle H_0,\dots,H_{2n}\rangle$ implies that $X^{(i)}_{s_n,\dots,s_{n-i+1}}$ agrees with X_i below this use. It consequently follows that $(\exists t < s_{n-i})\Phi^{X_i}_{e,t}(m)\downarrow$, so $\langle m,e\rangle \in X_i'$, as wanted.

Now suppose $\langle m,e\rangle \in X_i'$. By definition of the jump, we can find a t such that $\Phi^{X_i}_{e,t}(m)\downarrow$. Let $\langle s_0,\dots,s_n,u_1,\dots,u_n\rangle$ be the lexicographically least increasing tuple in $H_0\times\cdots\times H_{2n}$ such that $\langle m,e\rangle < s_{n-i-1}$, and choose $v_{n-i}\in H_{n-i}$ such that $v_{n-i} > \max\{t, s_{n-i-1}\}$. Choose an increasing tuple $\langle v_{n-i+1},\dots,v_n\rangle$ in $H_{n-i+1}\times\cdots\times H_n$ with $v_{n-i} < v_{n-i+1}$. By homogeneity of $\langle H_0,\dots,H_{2n}\rangle$ and the definition of X_i, the sets X_i and $X^{(i)}_{v_n,\dots,v_{n-i+1}}$ agree on elements below v_{n-i}. Thus

$$(\exists w < v_{n-i})\Phi_{e,w}^{X^{(i)}_{v_n,\dots,v_{n-i+1}}}(m)\downarrow,$$

or more succinctly, $\langle m,e\rangle \in (X^{(i)}_{v_n,\dots,v_{n-i+1}})'_{v_{n-i}} = X^{(i+1)}_{v_n,\dots,v_{n-i}}$. Homogeneity of $\langle H_0,\dots,H_{2n}\rangle$ now implies that $\langle m,e\rangle \in X^{(i+1)}_{s_n,\dots,s_{n-i}}$ and hence that $\langle m,e\rangle \in X_{i+1}$, which is what was to be shown. □

4.4 A polarized version of TT

In this section and the next, we turn to studying the tree theorem, TT. Chubb, Hirst, and McNicholl showed in [7, Theorems 2.1 and 2.2] that the main arithmetical bounds of Theorem 1.3.3, with the exception of part (3), hold also for TT, and in [7, Theorem 1.5] that for $n \geq 3$, TT restricted to colorings of exponent n is equivalent over RCA_0 to ACA_0. A number of these and other points of similarity between the tree theorem and Ramsey's theorem can be established using the following observation:

Remark 4.4.1 ([7], proof of Theorem 1.5))**.** For all $n, k \geq 1$, TT for k-colorings of exponent n implies RT for k-colorings of exponent n. Given $f : [\omega]^n \to k$, define $g : [2^{<\omega}]^n \to k$ by setting $g(\sigma, \tau) = f(|\sigma|, |\tau|)$ for all strings $\sigma \prec \tau$, and notice that if $H \subseteq 2^{<\omega}$ is isomorphic to $2^{<\omega}$ and homogeneous for g then $\{|\sigma| : \sigma \in H\}$ is infinite and homogeneous for f. We call g the *level coloring* of f.

Part (3) of Theorem 1.3.3 for the tree setting is not specifically mentioned in [7], but it is implicit in Lemma 2.6 of that work. We fill in the details by way of Lemma 4.4.8 below.

We shall make a number of references to the following result:

Lemma 4.4.2 (essentially Lemma 1.1 of [7])**.** *For all $k \geq 1$ and all partitions of $2^{<\omega}$ into disjoint pieces $A_0, \ldots, A_{k-1}$, there exists $c < k$ such that A_c computes a subset of itself isomorphic to $2^{<\omega}$.*

Proof. We proceed by induction on k. The base case $k = 1$ is clear, and it clearly relativizes from $2^{<\omega}$ to any subset S of $2^{<\omega}$ isomorphic to $2^{<\omega}$, with the desired subset of A_c then being $(S \oplus A_c)$-computable. Now assume the result, and all its relativizations, for some $k \geq 1$. Let $S \cong 2^{<\omega}$ be given, and suppose $A_0, \ldots, A_k$ is a partition of S. There are two cases. If there exists $c < k$ and $\sigma \in S$ such that no $\tau \succ \sigma$ in S belongs to A_c, apply the inductive hypothesis to the set S_σ of extensions of σ in S and the partition of it given by the sets $S_\sigma \cap A_i$ for $i \neq c$. The desired subset of A_i, $i \neq c$, will then be $(S_\sigma \oplus (S_\sigma \cap A_i))$-computable, and hence $(S \oplus A_i)$-computable since $S_\sigma \leq_T S$ and $A_i \cap S_\sigma \leq_T A_i \oplus S_\sigma \leq_T S$. If this case does not hold, we build an $(S \oplus A_0)$-computable subset $T = \{\sigma_\alpha : \alpha \in 2^{<\omega}\} \cong 2^{<\omega}$ of A_0 as follows: let $\sigma_\emptyset = \emptyset$, and given σ_α, let $\sigma_{\alpha i}$ for each $i < 2$ be the least extension of $\sigma_\alpha i$ in $S \cap A_0$. □

In this section, we formulate a polarized version of the tree theorem. To this end, we must describe how to interweave sequences of subsets of $2^{<\omega}$ isomorphic to $2^{<\omega}$. One method of interweaving a sequence of n such sets, say $T_0, \ldots, T_{n-1}$, with $T_i \cong 2^{<\omega}$ via $g_i : 2^{<\omega} \to T_i$, is to take a copy of $2^{<\omega}$, and replace each node σ by the chain $g_1(\sigma), \ldots, g_n(\sigma)$. We choose instead the following notationally simpler approach:

Definition 4.4.3. Suppose S is a subset of $2^{<\omega}$, isomorphic to $2^{<\omega}$ via $g : 2^{<\omega} \to S$, and let $n \geq 1$. The *mod n stratification* of S is the sequence $\langle S_0, \ldots, S_{n-1} \rangle$ where

$$S_i = \{\sigma \in S \ : |g^{-1}(\sigma)| \equiv i \bmod n\}$$

for each $i < n$. We write $S = \langle S_0, \ldots, S_{n-1} \rangle$.

Henceforth, when writing $S \cong 2^{<\omega}$ or $S = \langle S_0, \ldots, S_1\rangle$ without further qualification, we shall always mean that S is a subset of $2^{<\omega}$.

We can now define the analogue of p-homogeneity in the tree setting, emulating Definition 4.1.1.

Definition 4.4.4. Fix $n, k \geq 1$ and $f : [2^{<\omega}]^n \to k$.

1. A *p-homogeneous* set for f is a set $H = \langle H_0, \ldots, H_{n-1}\rangle \cong 2^{<\omega}$ such that for some $c < k$, called the *color* of H, $f(\sigma_0, \ldots, \sigma_{n-1}) = c$ for every $\{\sigma_0, \ldots, \sigma_{n-1}\} \in [2^{<\omega}]^n$ with $\sigma_i \in H_i$ for all $i < n$.
2. If (1) holds just for subsets $\{\sigma_0, \ldots, \sigma_{n-1}\}$ satisfying $\sigma_0 \prec \cdots \prec \sigma_{n-1}$, we call H an *increasing* p-homogeneous set.

We can now formulate polarized and increasing polarized versions of the tree theorem.

Polarized tree theorem (PTT). *For all $n, k \geq 1$, every coloring $f : [2^{<\omega}]^n \to k$ has a p-homogeneous set.*

Increasing polarized tree theorem (IPTT). *For all $n, k \geq 1$, every coloring $f : [2^{<\omega}]^n \to k$ has an increasing p-homogeneous set.*

We abbreviate the usual restrictions of TT, PTT, and IPTT by TT^n_k, TT^n, PT^n_k, etc.

Remark 4.4.5. Given a coloring $f : [\omega]^n \to k$, every (increasing) p-homogeneous set $H = \langle H_0, \ldots, H_{n-1}\rangle$ for the induced coloring of f computes a (increasing) p-homogeneous set for f, namely

$$\langle \{|\sigma| : \sigma \in H_0\}, \ldots, \{|\sigma| : \sigma \in H_{n-1}\}\rangle.$$

From this it follows that for all $n, k \geq 1$, PTT and IPTT for k-colorings of exponent n imply PT and IPT, respectively.

The following result are now clear (part (1) and the first implication in part (2) follow by translating Remark 4.1.2 from the linear to the tree setting):

Proposition 4.4.6. *Fix $n, k \geq 1$.*

1. *For every $f : [2^{<\omega}]^n \to k$, every homogeneous set for f isomorphic to $2^{<\omega}$ computes a p-homogeneous set.*
2. *RCA_0 proves that $\mathsf{TT}^n_k \to \mathsf{PTT}^n_k \to \mathsf{IPTT}^n_k$.*

Hence, for $n \geq 3$, PTT^n_k and IPTT^n_k are equivalent to ACA_0 over RCA_0. The same implications hold between TT^n, PTT^n, and IPT^n.

We can also easily obtain the analogue of Theorem 1.3.3 for PTT and IPTT.

Theorem 4.4.7. *Fix $n, k \geq 2$.*

1. *Every computable $f : [2^{<\omega}]^n \to k$ has a Π^0_n p-homogeneous set.*
2. *There exists a computable $f : [2^{<\omega}]^n \to 2$ with no Δ^0_n increasing p-homogeneous set, and hence also no Σ^0_n increasing p-homogeneous set.*
3. *Every computable $f : [2^{<\omega}]^n \to k$ has a p-homogeneous set with Δ^0_{n+1} jump.*
4. *There exists a computable $f : [\omega]^{n+1} \to 2$ every increasing p-homogeneous set of which computes $\emptyset^{(n-1)}$.*

Parts (1), (2), and (4) follow immediately by the preceding proposition and remark. Part (3) follows by applying the preceding proposition to the following lemma:

Lemma 4.4.8. *Every computable $f : [2^{<\omega}]^n \to k$ has a homogeneous set isomorphic to $2^{<\omega}$ with Δ^0_{n+1} jump.*

Proof. We prove the result by induction on n, beginning with $n = 1$. The base case follows by applying Lemma 4.4.2 to the partition $A_0, \ldots, A_{k-1}$ of $2^{<\omega}$ where $A_c = \{\sigma \in 2^{<\omega} : f(\sigma) = c\}$ for all $c < k$. Since each A_c is computable, it follows that $f : 2^{<\omega} \to k$ has a computable, and hence Δ^0_2, homogeneous set isomorphic to $2^{<\omega}$. This argument clearly relativizes from $2^{<\omega}$ to any $T \cong 2^{<\omega}$, with the jump of the homogeneous set constructed subsequently being $\Delta^{0,T}_2$.

Next, assume that the result and all its relativizations hold for some $n \geq 1$. Let $T_0 \cong 2^{<\omega}$ be arbitrary, and suppose $f : [T_0]^{n+1} \to k$ is T_0-computable. Chubb, Hirst, and McNicholl [7, Lemma 2.6] proved that there exists $T \cong 2^{<\omega}$ such that $T \subseteq T_0$, $(T_0 \oplus T)' \leq_T T''_\emptyset$, and for all $\sigma_0 \prec \cdots \prec \sigma_{n-1} \in T$ and all extensions τ_0 and τ_1 of σ_n in T, $f(\sigma_0, \ldots, \sigma_{n-1}, \tau_0) = f(\sigma_0, \ldots, \sigma_{n-1}, \tau_1)$. Define $\widehat{f} : [T]^n \to k$ as follows: given a sequence $\sigma_0 \preceq \cdots \preceq \sigma_{n-1}$ of comparable elements in T, let σ_n be the least extension of σ_{n-1} in T, and let $\widehat{f}(\sigma_0, \ldots, \sigma_{n-1}) = f(\sigma_0, \ldots, \sigma_{n-1}, \sigma_n)$. Note that $\widehat{f} \leq_T f \oplus T \leq_T T_0 \oplus T$. By the inductive hypothesis, relativized to $T_0 \oplus T$, choose a homogeneous set $H \subseteq T$ for $\widehat{f}$ isomorphic to $2^{<\omega}$ such that $H' \leq_T (T_0 \oplus T)^{(n)}$. By choice of T, H is then homogeneous for f and $H' \leq_T T_0^{(n+1)}$, as desired. □

4.5 Notions of stability

There are several ways to translate the notion of stability for colorings from the linear setting to the tree setting. In the definition below, the weakest version, 6-stability, corresponds to the most obvious rephrasing of the notion of stability. The strongest, in turn, 1-stability, is motivated by properties of level colorings, in the sense of Remark 4.4.1, of stable coloring of pairs of numbers.

Definition 4.5.1. Fix $k \geq 1$ and $f : [2^{<\omega}]^2 \to k$. We say that f is

1. *1-stable* if for every $\sigma \in 2^{<\omega}$ there exists $c < k$ and $n > |\sigma|$ such that for every $\tau \succ \sigma$ of length $\geq n$, $f(\sigma, \tau) = c$;

2. *2-stable* if for every $\sigma \in 2^{<\omega}$ there is an $n > |\sigma|$ such that for every $\tau \succ \sigma$ of length n and every $\rho \succeq \tau$, $f(\sigma, \rho) = f(\sigma, \tau)$;

3. *3-stable* if for every $\sigma \in 2^{<\omega}$ there exists $c < k$ such that for every $\sigma' \succeq \sigma$ there exists $\tau \succ \sigma'$ such that for every $\rho \succeq \tau$, $f(\sigma, \rho) = c$;

4. *4-stable* if for every $\sigma \in 2^{<\omega}$ and every $\sigma' \succeq \sigma$ there exists $\tau \succ \sigma'$ such that for every $\rho \succeq \tau$, $f(\sigma, \rho) = f(\sigma, \tau)$;

5. *5-stable* if for every $\sigma \in 2^{<\omega}$ there exists $\tau \succ \sigma$ such that for every $\rho \succeq \tau$, $f(\sigma, \rho) = f(\sigma, \tau)$.

Intuitively, we can think of the various notions of stability as follows: given $f : [2^{<\omega}]^2 \to k$, any fixed $\sigma \in 2^{<\omega}$ induces a coloring f_σ of the singleton nodes of the set of extensions of σ given by $f_\sigma(\tau) = f(\sigma, \tau)$, so that if f is 5-stable, then every such induced coloring has at least one monochromatic *cone* (i.e., set of all extensions of some extension of σ); if f is 4-stable, then monochromatic cones are dense above σ under $\preceq$; f f is 3-stable, then for each σ there is a single color such that the monochromatic cones of that color are dense above σ; if f is 2-stable, then there is a level of $2^{<\omega}$ such that each cone rooted at (i.e., having $\prec$-least element of length) that level is colored the same as its root, and if f is 1-stable then the color of each of these cones is the same.

4.5.1 Basic relations between notions

In this section, we collect some results about the combinatorial relationships between the various notions of stability. The proof of the next proposition is obvious.

Proposition 4.5.2. *For $k \geq 1$ and $f : [2^{<\omega}]^2 \to k$. Then*

$$f \text{ is 1-stable} \implies f \text{ is 2-stable} \implies f \text{ is 4-stable} \implies f \text{ is 5-stable}$$

and

$$f \text{ is 1-stable} \implies f \text{ is 3-stable} \implies f \text{ is 4-stable.}$$

The next result gives a characterization of 5-stability in terms of a seemingly weaker property.

Proposition 4.5.3. *Fix $k \geq 1$ and $f : [2^{<\omega}]^2 \to k$. Then f is 5-stable if and only if for every $\sigma \in 2^{<\omega}$ there exists $\tau \succ \sigma$ and $c < k$ such that for every subset $T \cong 2^{<\omega}$ of extensions of τ, there exists $\rho \in T$ such that $f(\sigma, \rho) = c$.*

Proof. If f is 5-stable then for every σ there is a $\tau \succ \sigma$ and a $c < k$ such that for every $\rho \succeq \tau$, $f(\sigma, \rho) = c$. Hence, in particular, for every set $T \cong 2^{<\omega}$ of extensions of τ, and any $\rho \in T$, we have $f(\sigma, \rho) = c$. Conversely, suppose f is not 5-stable and fix σ such that for every $\tau \succ \sigma$ there is a $\rho \succ \tau$ with $f(\sigma, \tau) \neq f(\sigma, \rho)$. Fix $\tau \succ \sigma$ and $c < k$, and for every $\alpha \in 2^{<\omega}$, define $\tau_\alpha \in 2^{<\omega}$ inductively as follows: let $\tau_\emptyset$ be the least $\rho \succeq \tau$ such that $f(\sigma, \rho) \neq c$, and given τ_α, let $\tau_{\alpha i}$ be the least $\rho \succeq \tau_\alpha i$ such that $f(\sigma, \rho) \neq c$. Then τ_α exists for every α by assumption, so $T = \{\tau_\alpha : \alpha \in 2^{<\omega}\}$ is a set of extensions of τ isomorphic to $2^{<\omega}$ such that $f(\sigma, \rho) \neq c$ for all $\rho \in T$. □

We obtain a partial converse to Proposition 4.5.2 by combining 2-stability and 3-stability.

Proposition 4.5.4. *For all $k \geq 1$, a coloring $f : [2^{<\omega}]^2 \to k$ is 1-stable if and only if it is both 2-stable and 3-stable.*

Proof. Fix $f : [2^{<\omega}]^2 \to k$. It is clear that if f is 1-stable, then f is both 2-stable and 3-stable. So assume that f is both 2-stable and 3-stable. Let $\sigma \in 2^{<\omega}$ be given. Since f is 2-stable, we may fix $n > |\sigma|$ such that if $\tau_0, \ldots, \tau_{2^{n-|\sigma|}-1}$ are all the extensions of σ of length n, then for every $i < 2^{n-|\sigma|}$ and every $\tau \succeq \tau_i$, $f(\sigma, \tau) = f(\sigma, \tau_i)$. But since f is 3-stable, there exists a $c < k$ such that for every $\sigma' \succeq \sigma$, and so in particular for $\sigma' = \tau_i$ for $i < 2^{n-|\sigma|}$, there exists $\tau \succ \sigma$ such that for every $\rho \succeq \tau$, $f(\sigma, \rho) = c$. It follows that for each i there exists $\rho \succ \tau_i$ with $f(\sigma, \rho) = c$, meaning $f(\sigma, \rho) = c$ for every $\rho \succeq \tau_i$. Hence, f is 1-stable. □

While the property of being 1-stable or 2-stable is preserved under subsets isomorphic to $2^{<\omega}$, colorings which are 3-stable do not necessarily have this property. This seems to be a barrier to extending our proof of Corollary 4.5.9, below, beyond 2-stability.

Proposition 4.5.5.

1. *For all $k \geq 1$, if $f : [2^{<\omega}]^2 \to k$ is 1-stable or 2-stable and $T \cong 2^{<\omega}$, then $f \upharpoonright [T]^2$ is also 1-stable or 2-stable, respectively.*

2. *There exists a 3-stable coloring $f : [2^{<\omega}]^2 \to 3$ and a $T \cong 2^{<\omega}$ such that $f \upharpoonright [T]^2$ is not 3-stable (or even 5-stable).*

Proof. Part (1) is immediate from the definitions of 1-stable and 2-stable. In each case, for each σ, the choice of n for the full tree works for any subtree.

For (2), the coloring f is defined in terms of a coloring $h : 2^{<\omega} \to 3$. Given a string $\sigma \in 2^{<\omega}$ of even length, say $2n$, let $\widehat{\sigma} \in 4^{<\omega}$ be the string of length n defined by $\widehat{\sigma}(i) = j < 4$ where $\sigma(2i)\sigma(2i+1)$ is the binary representation of j. Then define

$$h(\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset, \text{ or if } |\sigma| \text{ is even, } \sigma(|\sigma|-1) = 0, \text{ and } (\forall i < |\widehat{\sigma}|)[\widehat{\sigma}(i) \neq 2, 3], \\ 1 & \text{if } |\sigma| \text{ is odd, or if } |\sigma| \text{ is even and } (\exists i < |\widehat{\sigma}|)(\widehat{\sigma}(i) = 2 \vee \widehat{\sigma}(i) = 3], \\ 2 & \text{otherwise.} \end{cases}$$

For $\sigma \prec \tau$, define $f(\sigma, \tau) = h(\tau)$. It is not hard to show that for every $\sigma' \succ \sigma$, there is a $\tau \succeq \sigma'$ such that $f(\sigma, \rho) = 1$ for all $\rho \succeq \tau$, so f is 3-stable. Also, the subset of $2^{<\omega}$ consisting of those nodes not colored 1 is isomorphic to $2^{<\omega}$, and the restriction of f to this set is not 5-stable. $\square$

4.5.2 Reverse mathematics of stable tree theorems

Definition 4.5.1, and the preceding section, can be formalized in RCA_0 in a straightforward manner. We can thus define stable versions of the tree theorem, the polarized tree theorem, and the increasing polarized tree theorem, in RCA_0. For $k \geq 1$ and $i \in \{1, 2, 3, 4, 5\}$, we let $\mathsf{S}^i\mathsf{TT}^2_k$, $\mathsf{S}^i\mathsf{PTT}^2_k$, and $\mathsf{S}^i\mathsf{IPTT}^2_k$ denote the restrictions of, respectively, TT^2_k, PTT^2_k, and IPTT^2_k to i-stable colorings. $\mathsf{S}^i\mathsf{TT}^2$, $\mathsf{S}^i\mathsf{PTT}^2$, and $\mathsf{S}^i\mathsf{IPTT}^2$ are then defined as usual. The following are immediate corollaries to Proposition 4.5.2:

Corollary 4.5.6. RCA_0 *proves that for all* $k \geq 1$,

$$\mathsf{S}^5\mathsf{TT}^2_k \to \mathsf{S}^4\mathsf{TT}^2_k \to \mathsf{S}^2\mathsf{TT}^2_k \to \mathsf{S}^1\mathsf{TT}^2_k,$$

and

$$\mathsf{S}^4\mathsf{TT}^2_k \to \mathsf{S}^3\mathsf{TT}^2_k \to \mathsf{S}^1\mathsf{TT}^2_k.$$

The same implications hold between the $\mathsf{S}^i\mathsf{TT}^2$ *principles, as well as between the polarized and increasing polarized versions.*

Proposition 4.5.7. *For* $i \in \{1, 2, 3, 4, 5\}$, RCA_0 *proves that for all* $k \geq 1$:

1. $\mathsf{PTT}^2_k \to \mathsf{S}^i\mathsf{PTT}^2_k$;
2. $\mathsf{IPTT}^2_k \to \mathsf{S}^i\mathsf{IPTT}^2_k$;
3. $\mathsf{S}^i\mathsf{TT}^2_k \to \mathsf{S}^i\mathsf{PTT}^2_k \to \mathsf{S}^i\mathsf{IPTT}^2_k$;
4. $\mathsf{S}^i\mathsf{PTT}^2_k \to \mathsf{SPT}^2_k$;
5. $\mathsf{S}^i\mathsf{IPTT}^2_k \to \mathsf{SIPT}^2_k$.

Proof. Parts (1)–(3) are clear. Parts (4) and (5) follow by noting that when $f : [\mathbb{N}]^2 \to k$ is stable, its level coloring is 1-stable, and hence i-stable every i. $\square$

The next theorem shows that Proposition 4.2.2 holds for trees and leads to a proof that the 2-stable tree theorem and the 2-stable increasing polarized tree theorem are equivalent.

Theorem 4.5.8. *For all* $k \geq 1$ *and all 2-stable* $f : [2^{<\omega}]^2 \to k$, *every increasing p-homogeneous set for* f *computes a homogeneous set isomorphic to* $2^{<\omega}$.

Proof. Let $f : [2^{<\omega}]^2 \to k$ be 2-stable, and let $S = \langle S_0, S_1 \rangle \cong 2^{<\omega}$ be an increasing p-homogeneous set for f with color $c < k$. We construct a homogeneous subset $H = \{\sigma_\alpha : \alpha \in 2^{<\omega}\} \subseteq S_0$ isomorphic to $2^{<\omega}$ inductively, using S as an oracle. Let $\sigma_\emptyset$ be the root node of S_0, and assume that for some $s \geq 0$, we have defined $\sigma_\alpha \in S_0$ for all α of length s, and that for all $\alpha_0 \prec \alpha_1$ of length $\leq s$, $f(\sigma_{\alpha_0}, \sigma_{\alpha_1}) = c$. By 2-stability of f, for every α of length $\leq s$, there exists $n_\alpha > |\sigma_\alpha|$ such that for every $\tau \succ \sigma_\alpha$ of length n_α and every $\rho \succeq \tau$, $f(\sigma_\alpha, \tau) = f(\sigma_\alpha, \rho)$. In particular, for any extension $\tau \in S_0$ of σ_α of length $\geq n_\alpha$, and any extension $\rho \in S_1$ of τ, we thus have $f(\sigma_\alpha, \tau) = f(\sigma_\alpha, \rho)$, which must equal c by p-homogeneity of S. In other words, $f(\sigma_\alpha, \tau) = c$ for all sufficiently long extensions $\tau \in S_0$ of σ_α. For α of length s, we can thus let $\sigma_{\alpha 0}$ and $\sigma_{\alpha 1}$ be the least incomparable extensions of σ_α in S_0 such that $f(\sigma_\beta, \sigma_{\alpha i}) = c$ for all $\beta \preceq \alpha$ and $i < 2$. It follows that $H \cong 2^{<\omega}$ and is homogeneous for f with color c. □

The preceding argument can be easily formalized using $\mathsf{B}\Sigma^0_2$. Since $\mathsf{S}^2\mathsf{PTT}^2_k$ implies SPT^2_k, and since SPT^2_k implies $\mathsf{B}\Sigma^0_2$ by Proposition 4.3.5 and Theorem 4.3.3, we obtain the following corollary:

Corollary 4.5.9. *For $i \in \{0, 1\}$, RCA_0 proves:*

1. *for all $k \geq 1$, $\mathsf{S}^i\mathsf{TT}^2_k \leftrightarrow \mathsf{S}^i\mathsf{PTT}^2_k \leftrightarrow \mathsf{S}^i\mathsf{IPTT}^2_k$;*
2. *$\mathsf{S}^i\mathsf{TT}^2 \leftrightarrow \mathsf{S}^i\mathsf{PTT}^2 \leftrightarrow \mathsf{S}^i\mathsf{IPTT}^2$.*

We do not know how to extend Theorem 4.5.8, or Corollary 4.5.9, to hold for other versions of stability. As we discuss next, this poses an obstacle to proving the equivalence of TT^2_2 and PTT^2_2, or at least to adapting our proof above of the equivalence of RT^2_2 with PT^2_2. This exposes another implicit yet seemingly fundamental reliance on linearity in the classical setting.

We begin by formulating versions of COH suitable for our purposes. Fix $k \geq 1$ and $i \in \{1, 2, 3, 4, 5\}$.

Tree cohesive principle ($\mathsf{C}^i\mathsf{TT}^2_k$). *For every $f : [2^{<\omega}]^2 \to k$ there exists $T \cong 2^{<\omega}$ such that $f \upharpoonright [T]^2$ is i-stable.*

From this definition, it is clear that for all k and i, TT^2_k is equivalent over RCA_0 to $\mathsf{S}^i\mathsf{TT}^2_k + \mathsf{C}^i\mathsf{TT}^2_k$. Our proof of Theorem 4.3.6 proceeded by showing that PT^2_k implies ADS, and hence also COH. While emulating this idea for trees does not apparently establish the desired result, that over RCA_0, PTT^2_2 implies TT^2_2, it produces a partial step towards this result.

Definition 4.5.10.

1. A *tree-linear* ordering on $T \subseteq 2^{<\omega}$ is a partial ordering $\leq_L$ such that for all comparable $\sigma, \tau \in T$, either $\sigma \leq_L \tau$ or $\tau \leq_L \sigma$.
2. Given a tree-linear ordering $\leq_L$ on $2^{<\omega}$, we call a subset T of $2^{<\omega}$ *ascending* for $\leq_L$ if for all $\sigma, \tau \in T$, $\sigma \preceq \tau$ if and only if $\sigma \leq_L \tau$, and we call S *descending* if instead $\sigma \preceq \tau$ if and only if $\tau \leq_L \sigma$.

The following tree version of ADS can be formalized in RCA_0:

Tree ascending or descending sequence principle (TADS). *For every tree-linear order $\leq_L$ on $2^{<\omega}$ there exists $T \cong 2^{<\omega}$ that is either an ascending sequence or a descending sequence under this order.*

The proof of the next proposition is based on Hirschfeldt and Shore's proof of Proposition 2.10 in [27].

Proposition 4.5.11. *For all $k \geq 1$, RCA_0 proves that TADS implies $\mathsf{C}^4\mathsf{TT}^2_k$.*

Proof. We argue in RCA_0. Fix $f : [2^{<\mathbb{N}}]^2 \to k$, and for each $\sigma \in 2^{<\mathbb{N}}$, let σ_f be the string

$$f(\sigma \upharpoonright 0, \sigma) f(\sigma \upharpoonright 1, \sigma) \cdots f(\sigma \upharpoonright |\sigma| - 1, \sigma) \in k^{<\omega}.$$

Define a tree-linear ordering $\leq_L$ on $2^{<\mathbb{N}}$ as follows: For $\sigma \preceq \tau$, let $\sigma \leq_L \tau$ if σ_f lexicographically precedes τ_f, and otherwise let $\tau \leq_L \sigma$. Apply TADS to obtain $T \cong 2^{<\mathbb{N}}$ which is, say, ascending for $\leq_L$ (the descending case being analogous). We claim that $f \upharpoonright [T]^2$ is 4-stable. To see this, let σ and $\sigma' \succ \sigma$ in T be given. Let ρ_σ be the lexicographically greatest string in $2^{<\mathbb{N}}$ of length $|\sigma| + 1$ such that there exists a $\tau \succ \sigma'$ in T with ρ_σ lexicographically preceding τ_f; this exists because there are only finitely many strings of length $|\sigma| + 1$ and because $0^{|\sigma|+1}$ lexicographically precedes τ_f for all $\tau \succ \sigma'$. Fix the least τ corresponding to the definition of ρ_σ. Since T is ascending, τ_f must lexicographically precede ρ_f for all $\rho \succeq \tau$ in T, and hence so must ρ_σ. In particular, ρ_σ must lexicographically precede $\rho_f \upharpoonright |\sigma| + 1$ for all such ρ, which implies that ρ_σ in fact equals ρ_f. Thus, for all $\rho \succeq \tau$, $f(\sigma, \rho) = \rho_\sigma(|\sigma|)$. Since σ and σ' were chosen arbitrarily, this proves the claim. □

Proposition 4.5.12. *Over RCA_0, PTT^2_2 implies TADS.*

Proof. Fix a tree-linear ordering $\leq_L$ on $2^{<\mathbb{N}}$. Define $f : [2^{<\mathbb{N}}]^2 \to 2$ by

$$f(\sigma, \tau) = \begin{cases} 0 & \text{if } \sigma \leq_L \tau, \\ 1 & \text{if } \tau \leq_L \sigma, \end{cases}$$

for all $\sigma \preceq \tau$. Let $S = \langle S_0, S_1 \rangle$ be a p-homogeneous set for f, as given by PTT^2_2. We define $T = \{\sigma_\alpha : \alpha \in 2^{<\mathbb{N}}\} \cong 2^{<\mathbb{N}}$ which is either ascending or descending for $\leq_L$. Let $\sigma_\emptyset$ be the root node of S_0, and suppose that for some $\alpha \succeq \emptyset$, we have defined $\sigma_\alpha \in S$. Let $\sigma_{\alpha 0}$ and $\sigma_{\alpha 1}$ be the least incomparable extensions of σ_α in S_1 if $|\alpha|$ is even, in S_0 if $|\alpha|$ is odd. Then T exists by Δ^0_1-comprehension and clearly $T \cong 2^{<\mathbb{N}}$. Furthermore, by p-homogeneity there exists $c < 2$ such that for every $\alpha \in 2^{<\mathbb{N}}$ and $i < 2$, we have $f(\sigma_\alpha, \sigma_{\alpha i}) = c$, so by definition of f, either $\sigma_\alpha \leq_L \sigma_{\alpha i}$ for all α and i, or $\sigma_{\alpha i} \leq_L \sigma_\alpha$ for all α and i. Thus, T is either ascending or descending for $\leq_L$, as desired. □

Corollary 4.5.13. $\mathsf{RCA}_0 \vdash \mathsf{PTT}^2_2 \leftrightarrow \mathsf{S}^4\mathsf{PTT}^2_2 + \mathsf{C}^4\mathsf{TT}^2_2$.

One way, then, to prove the equivalence of PTT^2_2 with TT^2_2, would be to get Corollary 4.5.9 to work for 4-stability, i.e, to show that $\mathsf{S}^4\mathsf{PTT}^2_2$ is equivalent to $\mathsf{S}^4\mathsf{TT}^2_2$ over RCA_0. Another way would be to strengthen Proposition 4.5.11 by replacing $\mathsf{C}^4\mathsf{TT}^2_2$ with $\mathsf{C}^1\mathsf{TT}^2_2$ or $\mathsf{C}^2\mathsf{TT}^2_2$. We do not know if either of these approaches is viable.

4.5.3 Δ^0_2 upper bounds

In this section, we show that the Δ^0_2 upper bound that holds for infinite homogeneous sets of computable stable colorings of pairs of numbers, transfers to the tree setting for each of the notions of stability introduced in Definition 4.5.1. In the linear case, this bound follows at once from Lemma 1.3.7, which establishes a correspondence between the infinite homogeneous sets of computable stable colorings and infinite subsets and co-subsets of Δ^0_2 sets. For colorings of pairs of binary strings, we shall only have such a correspondence for 1-stable colorings, forcing us, for i-stable colorings, $i \in \{2,3,4,5\}$, to establish the Δ^0_2 upper bound via a separate argument.

We begin with 1-stable colorings.

Definition 4.5.14. Fix $k \geq 1$ and a 1-stable $f : [2^{<\omega}]^2 \to k$.

1. The *eventual color* of $\sigma \in 2^{<\omega}$ (with respect to f) is the unique $c < k$ such that there exists $n > |\sigma|$ such that $f(\sigma, \tau) = c$ for all τ of length $\geq n$.
2. For each $c < k$, let A^f_c be the collection of all $\sigma \in 2^{<\omega}$ with eventual color c.

Note that the sets A^f_c, $c < k$, partition $2^{<\omega}$, and that they are each Δ^0_2 in f.

Lemma 4.5.15. *For all $k \geq 1$ and all 1-stable $f : [2^{<\omega}]^2 \to k$, if $S \cong 2^{<\omega}$ contains only strings with the same eventual color, then S computes a homogeneous set for f isomorphic to $2^{<\omega}$.*

Proof. Suppose every $\sigma \in S$ has eventual color $c < k$. We define a subset $H = \{\sigma_\alpha : \alpha \in 2^{<\omega}\} \cong 2^{<\omega}$ of S inductively as follows: let $\sigma_\emptyset$ be the root node of S, and having defined σ_α for some α, let $\sigma_{\alpha 0}$ and $\sigma_{\alpha 1}$ be the least pair of incomparable extensions of σ_α in S such that for every $\beta \preceq \alpha$ and $i < 2$, $f(\sigma_\beta, \sigma_{\alpha i}) = c$. Such extensions exist since for every $\sigma \in S$, $f(\sigma, \tau) = c$ for cofinitely many τ. Thus, H is isomorphic to $2^{<\omega}$, and by construction, it is homogeneouf for f. □

Corollary 4.5.16. *For all $k \geq 1$ and all computable 1-stable $f : [2^{<\omega}]^2 \to k$, there exist disjoint Δ^0_2 subsets $A^f_0, \ldots, A^f_{k-1}$ of $2^{<\omega}$ such that $2^{<\omega} = A^f_0 \cup \cdots \cup A^f_{k-1}$ and any subset of any A^f_c isomorphic to $2^{<\omega}$ computes a homogeneous set for f isomorphic to $2^{<\omega}$.*

Theorem 4.5.17. *For all $k \geq 1$, every computable 1-stable $f : [2^{<\omega}]^2 \to k$ has a Δ^0_2 homogenous set isomorphic to $2^{<\omega}$.*

Proof. By Lemma 4.4.2, there is some $c < k$ such that A^f_c computes a subset of itself isomorphic to $2^{<\omega}$, and this in turn computes an homogeneous set for f isomorphic to $2^{<\omega}$ by Lemma 4.5.15. □

Combinatorially, the situation changes when we move from 1-stable to 2-stable colorings. For if $f : [2^{<\omega}]^2 \to k$ is merely 2-stable, the sets A^f_c no longer need to partition $2^{<\omega}$, because strings no longer all need to have an eventual color. For example, if $k = 2$, then $2^{<\omega}$ is a disjoint union of A^f_0, A^f_1, and a "mixed" set

$$A^f_{0,1} = \{\sigma \in 2^{<\omega} : (\forall n)(\forall c < 2)(\exists \tau \succ \sigma)[|\tau| > n \wedge f(\sigma, \tau) = c]\}.$$

This mixed set does not, as we shall see, necessarily have the property described in Corollary 4.5.16, that any subset $S \cong 2^{<\omega}$ of it computes a homogeneous set for f isomorphic to $2^{<\omega}$.

Theorem 4.5.18. *There exists a computable 2-stable $f : [2^{<\omega}]^2 \to 2$ and a subset $T \cong 2^{<\omega}$ of $A^f_{0,1}$ which computes no homogeneous set for f isomorphic to $2^{<\omega}$.*

Proof. We build a 2-stable $f : [2^{<\omega}]^2 \to 2$ with $A^f_{0,1} = 2^{<\omega}$ such that f has no computable homogeneous set isomorphic to $2^{<\omega}$. Since $A^f_{0,1}$ is then computable, the result follows. Fix a computable stable coloring $g : [\omega]^2 \to 2$ with no computable infinite homogeneous set (e.g., let g be the coloring of Theorem 1.3.2). Define $f : [2^{<\omega}]^2 \to 2$ by

$$f(\sigma, \tau) = \begin{cases} g(|\sigma|, |\tau|) & \text{if } \tau \succeq \sigma 0, \\ 1 - g(|\sigma|, |\tau|) & \text{if } \tau \succeq \sigma 1, \end{cases}$$

for all $\sigma \prec \tau$. To see that f is 2-stable, fix σ and, by stability of g, choose $c < 2$ and n such that for all $m \geq n$, $g(|\sigma|, m) = c$. Then for all $\rho \succ \tau \succ \sigma$ with $|\tau| \geq n$, we have $f(\sigma, \rho) = f(\sigma, \tau)$, as needed. Furthermore, if $\tau \succeq \sigma 0$ then $f(\sigma, \rho) = c$, whereas if $\tau \succeq \sigma 1$ then $f(\sigma, \rho) = 1 - c$. Thus, $\sigma \in A^f_{0,1}$, and since σ was arbitrary, this means $A^f_{0,1} = 2^{<\omega}$. To complete the proof, we show that every homogeneous set for f isomorphic to $2^{<\omega}$ computes an infinite homogeneous set for g. Let $H \cong 2^{<\omega}$ be homogeneous for f, and for each $i \in \omega$, let σ_i be the lexicographically least string in H of length i. Consider the sets

$$H_0 = \{|\sigma_i| : i \in \omega \wedge \sigma_{i+1} \succeq \sigma_i 0\} \text{ and } H_1 = \{|\sigma_i| : i \in \omega \wedge \sigma_{i+1} \succeq \sigma_i 1\}.$$

Since $\sigma_0 \sigma_1 \cdots \in 2^\omega$ is computable from H, so are each of H_0 and H_1. Moreover, both are homogeneous for g, and at least one must be infinite, which proves the claim. □

Thus, we can not extend Theorem 4.5.17 to even 2-stable colorings by appealing to a result similar to Corollary 4.5.16. Even so, we can modify the argument from the theorem to directly obtain it for weaker forms of stability.

Recall that we can think of a 5-stable coloring $f : [2^{<\omega}]^2 \to k$ as one for which the induced maps $f_\sigma : 2^{<\omega} \to k$ are eventually constant on the subtree above some node. Neither the map sending $\sigma \in 2^{<\omega}$ to the least root of such a subtree, nor the map sending σ to the value of the induced map on this subtree, need to be computable from f, but both can clearly be computed from the jump of f.

Theorem 4.5.19. *For all $k \geq 1$ and all $i \in \{0, 1, 2, 3, 4, 5\}$, every computable i-stable $f : [2^{<\omega}]^2 \to k$ has a Δ^0_2 homogenous set isomorphic to $2^{<\omega}$.*

Proof. It suffices to prove the result assuming f is 5-stable. First, build a Δ^0_2 set $S = \{\sigma_\alpha : \alpha \in 2^{<\omega}\} \cong 2^{<\omega}$ as follows: let $\sigma_\emptyset = \emptyset$, and having defined σ_α for some α, use $\emptyset'$ to find the least $\tau \succ \sigma_\alpha$ such that for every $\rho \succeq \tau$, $f(\sigma, \rho) = f(\sigma, \tau)$, and let $\sigma_{\alpha 0} = \tau 0$ and $\sigma_{\alpha 1} = \tau 1$. It is then not difficult to see that f is 1-stable on S, and that the eventual color of a string σ_α within S is given by $f(\sigma_\alpha, \sigma_{\alpha 0})$ (or by $f(\sigma_\alpha, \sigma_{\alpha 0})$). Thus, we can define disjoint S-computable sets $A_0, \ldots, A_{k-1}$ that partition S by letting A_c for $c < k$ be the set of $\sigma \in S$ whose eventual color in S is c. By Lemma 4.4.2, some A_c computes a subset of itself isomorphic to $2^{<\omega}$, and since every string in this subset has the same eventual color, by Lemma 4.5.15 it in turn computes a homogeneous set for f isomorphic to $2^{<\omega}$. □

4.6 Questions

The main questions pertaining to Section 4.3 are about which of the implications established there can be reversed. Of particular note are the following, stated here for simplicity for two colors:

Question 4.6.1. Does IPT^2_2 imply RT^2_2 over RCA_0? Does IPT^2 imply RT^2?

Of course, the equivalence of IPT^2_2 with RT^2_2 would follow from an affirmative answer to the question of whether SRT^2_2 implies RT^2_2. Separating RT^2_2 and $\mathsf{IPT}^2_2 + \mathsf{B}\Sigma^0_2$, therefore, is likely to be at least as hard as obtaining a negative answer to that question.

On the computability-theoretic side of our investigation, we do not have a precise characterization of the relationship between the Turing degrees of infinite homogeneous, p-homogeneous, and increasing p-homogeneous sets. A strong connection could be inferred from an affirmative answer to the following question:

Question 4.6.2. Given a computable coloring $f : [\omega]^2 \to 2$, does there exists a computable coloring $g : [\omega]^2 \to 2$ such that every p-homogeneous (respectively, increasing p-homogeneous) set for g computes an infinite homogeneous (respectively, a p-homogeneous) set for f?

For an alternative approach, we can formulate polarized and increasing polarized analogues of Ramsey degrees, as discussed in Chapter 3.

Definition 4.6.3. A degree $\mathbf{d}$ is *p-Ramsey*, respectively *ip-Ramsey*, if every computable coloring of pairs has a p-homogeneous, respectively increasing p-homogeneous, set of degree at most $\mathbf{d}$

That every Ramsey degree is p-Ramsey is immediate by Remark 4.1.2, and every p-Ramsey degree is obviously ip-Ramsey.

Question 4.6.4. Is every ip-Ramsey degree a p-Ramsey degree? Is every p-Ramsey degree a Ramsey degree?

(Notice that while we could also formulate the polarized and increasing polarized analogues of s-Ramsey degrees, these would all coincide by Proposition 4.2.2.)

Regarding Section 4.5, the obvious questions concern how the various tree principles relate to their linear analogues. The most central of these, such as whether the tree theorem for pairs implies ACA_0, are listed in Section 3 of [7]. We can additionally ask:

Question 4.6.5. Is PTT^2_2 equivalent to TT^2_2?

Question 4.6.6. Can any of the one-way arrows in Figure 4.1 be reversed for tree principles?

In view of the the profusion of versions of stability in the tree setting, one can additionally ask:

Question 4.6.7. Do any forms of stability yield provably distinct results in reverse mathematics or computability theory?

Question 4.6.8. Does Theorem 4.5.8 fail for 3-stable colorings? Does Proposition 4.5.11 fail for 2-stable colorings?

CHAPTER 5
EQUIVALENTS OF THE AXIOM OF CHOICE

5.1 Introduction

A large number of statements in set theory are equivalent to the axiom of choice over Zermelo–Fraenkel set theory (ZF). In this chapter, which is joint work with Carl Mummert, we examine what happens when some of these statements are interpreted in the setting of second-order arithmetic, where the only "sets" available are sets of natural numbers. This interpretation allows us to study computability-theoretic and proof-theoretic aspects of choice principles in the spirit of reverse mathematics. Our results show that the re-interpreted statements need not be trivial, as may be suspected. Instead, these principles demonstrate a wide range of reverse-mathematical strengths.

The history of the axiom of choice is presented in detail by Moore [49]. The main facet of interest for our purposes is that, after Zermelo introduced the axiom of choice in 1904, set theorists began to obtain results proving other set-theoretic principles equivalent to it (relative to choice-free axiomatizations of set theory). These equivalence results, and their further development, now constitute a program in set theory, which has been documented in detail by Jech [33] and by Rubin and Rubin [52, 53].

This program provides us with a large collection of statements from which to choose. We begin in Section 5.2 with Zorn's lemma, which is perhaps the most well-known equivalent of the axiom of choice but which, we shall see, will be of only limited interest in second-order arithmetic. In Sections 5.3 and 5.4, we turn to other maximality principles with more complex and interesting behavior. Our focus is on statements closely related to the following two equivalents of the axiom of choice:

- every family of sets has a $\subseteq$-maximal subfamily with the finite intersection property;
- if φ is a property of finite character and A is any set, there is a $\subseteq$-maximal subset B of A such that B has property φ.

We avoid studying principles that concern countable well-orderings. Such principles have been thoroughly explored in the context of reverse mathematics by Friedman and Hirst [23] and by Hirst [31]. We also do not study direct formalizations of choice principles in arithmetic. These have been studied by Simpson [59, Section VII.6].

Two countable forms of equivalents of the axiom of choice are already provable in RCA_0. These are the principle that every set of natural numbers can be well ordered and the principle that every sequence of non-empty sets of natural numbers has a choice function. We shall show that several other equivalents of the axioms of choice require stronger subsystems to prove.

5.2 Zorn's lemma

Zorn's lemma is one of the best known equivalents of the axiom of choice, so we begin by studying the strength of countable versions of this principle. The reverse mathematics results in this section are relatively elementary, providing a warm-up for the more technical results of the following sections.

Working in RCA_0, we define a *countable poset* to be a set $P \subseteq \mathbb{N}$ with a reflexive, antisymmetric, transitive relation $\leq_P$. As usual, we may freely convert $\leq_P$ into an irreflexive, transitive relation $<_P$.

Zorn's lemma, variant 1 (ZL-1). *If a non-empty countable poset has the property that every linearly ordered subset is bounded above, then every element of the poset is below some maximal element.*

Zorn's lemma, variant 2 (ZL-2). *If a non-empty countable poset has the property that every linearly ordered subset is bounded above, then there is a non-empty set consisting of the maximal elements of the poset.*

Zorn's lemma, variant 3 (ZL-3). *If a non-empty countable poset has the property that every linearly ordered subset is bounded above, then there is a function that assigns to each element of the poset a maximal element above it.*

Of these three principles, ZL-1 is the most natural countable analogue of Zorn's lemma, but we shall see that it is already provable in RCA_0. We shall show that ZL-2 is equivalent to ACA_0 over RCA_0, as can be expected. Principle ZL-3 is of greater interest; it can be viewed as a uniform version of ZL-1. We shall show it is also equivalent to ACA_0 over RCA_0.

Theorem 5.2.1. ZL-1 *is provable in* RCA_0.

Proof. Working in RCA_0, let $\langle P, \leq_P\rangle$ be a countable poset in which every linearly ordered subset of P is bounded above. Write $P = \langle p_i : i \in \mathbb{N}\rangle$. We shall build a sequence $\langle q_i : i \in \mathbb{N}\rangle$ by induction. Let q_0 be an arbitrary element of P. At stage $i+1$, if $q_i <_P p_i$ then put $q_{i+1} = p_i$, and otherwise put $q_{i+1} = q_i$. This inductive construction can be carried out in RCA_0. Moreover, a Π^0_1 induction in RCA_0 shows that if $i < j$ then $q_i \leq_P q_j$.

Let $L = \{q_i : i \in \mathbb{N}\}$. To decide if a fixed $p_i \in P$ is in L, it is only necessary to simulate the construction up to stage $i+1$. Therefore L is a Δ^0_1 set, and so RCA_0 proves that L exists. Moreover, L is linearly ordered; if two elements of L are incomparable, then two elements of the original sequence $\langle q_i : i \in \mathbb{N}\rangle$ are incomparable, which is impossible.

By assumption, there is some $i \in \mathbb{N}$ such that p_i is an upper bound for L. In particular, it must be that $q_i <_P p_i$, which means by construction that $p_i = q_{i+1} \in L$. Moreover, because p_i is an upper bound for L, it must be that $q_{i+j} = p_i$ for all $j \geq 1$.

Now suppose there is some $p_j \in P$ with $p_i <_P p_j$. It can not be that $j < i$, because this would imply $p_j \leq_P q_i \leq_P p_i$. However, if $i < j$ then, at stage j, the construction would select $q_{j+1} = p_j$, contradicting our result that $q_{j+1} = p_i$. Thus p_i is a maximal element above q_0. □

Theorem 5.2.2. *Each of* ZL-2 *and* ZL-3 *is equivalent to* ACA_0 *over* RCA_0.

Proof. For any countable poset P satisfying the hypothesis of ZL-2, the set of maximal elements of P is definable by an arithmetical formula and is non-empty by Theorem 5.2.1. Thus, ACA_0 implies ZL-2.

Next, we show that ZL-2 implies ZL-3 over RCA_0. Let $\langle P, \leq_P \rangle$ be any countable poset such that every element of P is below at least one maximal element, and by ZL-2 let M be the set of maximal elements of P. Define a function $m\colon P \to P$ by the rule

$$m(p) = q \iff (q \in M) \wedge (p \leq_P q) \wedge (\forall r <_{\mathbb{N}} q)[p \leq_P r \to r \notin M].$$

Then m is a function with domain P such that for each p, $m(p)$ is a maximal element with $p \leq_P m(p)$. Moreover, the definition of m is Δ^0_0 relative to M and $\leq_P$, so we can form m in RCA_0.

Finally, we show that ZL-3 implies ACA_0 over RCA_0. Fix any one-to-one function f. We shall construct a poset $\langle P, \leq_P \rangle$. Let $P = \{p_{i,s} : i, s \in \mathbb{N}\}$. The order $\leq_P$ on P is defined by cases. If $i \neq j$ then $p_{i,s}$ and $p_{j,t}$ are incomparable for all $s, t \in \mathbb{N}$. Given $i, s, t \in \mathbb{N}$, with $s \neq t$, define $p_{i,t} <_P p_{i,s}$ to hold if either $f(s) = i$, or $f(t) \neq i$ and $t > s$. Thus, for a fixed i, if there is no s with $f(s) = i$ then we have a maximal chain

$$\cdots <_P p_{i,2} <_P p_{i,1} <_P p_{i,0},$$

while if $f(s) = i$ then, because f is one-to-one, we have a maximal chain

$$\cdots <_P p_{i,2} <_P p_{i,1} <_P p_{i,0} <_P p_{i,s}.$$

In particular, for each i, either $p_{i,0}$ is a maximal element of P or there is an s with $f(s) = i$ and $p_{i,s}$ is a maximal element of P. (This gives, as a corollary, a direct reversal of ZL-2 to ACA_0 over RCA_0.)

Now, working in RCA_0, assume there is a function $m\colon P \to P$ taking each $p \in P$ to a $\leq_P$-maximal q with $p \leq_P q$. Fix $i \in \mathbb{N}$. Either $m(p_{i,0}) = p_{i,0}$, in which case i is in the range of f if and only if $f(0) = i$, or else $m(p_{i,0}) = p_{i,s}$ for some $s > 0$, in which case $f(s) = i$. Thus we have

$$i \in \operatorname{range}(f) \iff (\exists s)[f(s) = i] \iff (\forall s)[m(p_{i,0}) = p_{i,s} \Rightarrow f(s) = i].$$

Therefore the range of f exists by Δ^0_1 comprehension. This completes the reversal. □

5.3 Intersection properties

We next study several principles asserting that every countable family of sets has a $\subseteq$-maximal subfamily with certain intersection properties (see Definition 5.3.2). We shall show that, although these principles are all equivalent to the axiom of choice in set theory, they can have vastly different strengths when formalized in second-order arithmetic. In particular, we find new examples of principles that are weaker than ACA_0 and incomparable with WKL_0.

Definition 5.3.1.

1. We define a *family of sets* to be a sequence $A = \langle A_i : i \in \omega \rangle$ of sets. A family A is *non-trivial* if $A_i \neq \emptyset$ for some $i \in \omega$.

2. Given a family of sets A and a set X, we say A *contains* X, and write $X \in A$, if $X = A_i$ for some $i \in \omega$. A family of sets B is a *subfamily* of A if every set in B is in A, that is, $(\forall i)(\exists j)[B_i = A_j]$.

3. Two sets $A_i, A_j \in A$ are *distinct* if they differ extensionally as sets.

Our definition of a subfamily is intentionally weak; see Proposition 5.3.7 below and the remarks preceding it.

Definition 5.3.2. Let $A = \langle A_i : i \in \omega \rangle$ be a family of sets and fix $n \geq 2$. Then A has the

- D_n *intersection property* if the intersection of any n distinct sets in A is empty;

- $\overline{D}_n$ *intersection property* if the intersection of any n distinct sets in A is non-empty;

- F *intersection property* if for every $m \geq 2$, the intersection of any m distinct sets in A is non-empty.

Definition 5.3.3. Let $A = \langle A_i : i \in \omega \rangle$ and $B = \langle B_i : i \in \omega \rangle$ be families of sets, and let P be any of the properties in Definition 5.3.2. Then B is a *maximal* subfamily of A with the P intersection property if B has the P intersection property, and for every subfamily C of A that does also, if B is a subfamily of C then C is a subfamily of B.

It is straightforward to formalize Definitions 5.3.1–5.3.3 in RCA_0.

Given a family $A = \langle A_i : i \in \omega \rangle$ and some $J \in \omega^\omega$, we use the notation $\langle A_{J(i)} : i \in \omega \rangle$ for the subfamily $\langle B_i : i \in \omega \rangle$ where $B_i = A_{J(i)}$. We call this the subfamily *defined* by J. Given a finite set $\{j_0, \ldots, j_n\} \subset \omega$, we let $\langle A_{j_0}, \ldots, A_{j_n} \rangle$ denote the subfamily $\langle B_i : i \in \mathbb{N} \rangle$ where $B_i = A_{j_i}$ for $i \leq n$ and $B_i = A_{j_n}$ for $i > n$. Note that such a subfamily can still contain A_i for infinitely many i, because there could be a j such that $A_j = A_i$ for infinitely many i. We call a subfamily of A *finite* if it contains only finitely many distinct A_i.

Let P be any of the properties in Definition 5.3.2. We are interested in the following maximality principles:

P **intersection principle** ($P\mathsf{IP}$). *Every non-trivial family of sets has a maximal subfamily with the P intersection property.*

For $P = D_n$ and $P = \overline{D}_n$, the set-theoretic principle corresponding to $P\mathsf{IP}$ is, in the notation of Rubin and Rubin [53], $\mathsf{M}\,8\,(P)$. For $P = F$, it is $\mathsf{M}\,14$. For additional references concerning the set-theoretic forms, and for proofs of their equivalences with the axiom of choice, see Rubin and Rubin [53, pp. 54–56, 60].

Remark 5.3.4. Although we do not make it an explicit part of the definition, all of the families $\langle A_i : i \in \omega\rangle$ we construct in our results will have the property that for each i, A_i contains $2i$ and otherwise contains only odd numbers. This will have the advantage that if we are given an arbitrary subfamily $B = \langle B_i : i \in \omega\rangle$ of some such family, we can, for each i, uniformly B-computably find a j such that $B_i = A_j$. If A is computable, each subfamily B will then be of the form $\langle A_{J(i)} : i \in \omega\rangle$ for some $J \in \omega^\omega$ with $J \equiv_T B$.

5.3.1 Implications over RCA_0 and equivalences to ACA_0

Our next sequence of propositions establishes the basic relations that hold among the principles we have defined. We begin with the following upper bound on their strength:

Proposition 5.3.5. *For any property P in Definition 5.3.2, $P\mathsf{IP}$ is provable in ACA_0.*

Proof. Suppose $A = \langle A_i : i \in \mathbb{N}\rangle$ is a non-trivial family of sets. If A has a finite maximal subfamily with the P intersection property, then we are done. Otherwise, we define a function $p\colon \mathbb{N} \to \mathbb{N}$ as follows: let $p(0)$ be the least j such that $\langle A_j\rangle$ has the P intersection property, and given $i \in \mathbb{N}$, let $p(i+1)$ be the least $j > p(i)$ such that $\langle A_{p(0)}, \ldots, A_{p(i)}, A_j\rangle$ has the P intersection property. Then p exists by arithmetical comprehension, and by assumption it is total. It is not difficult to see that $B = \langle A_{p(i)} : i \in \mathbb{N}\rangle$ is a maximal subfamily of A with the P intersection property. □

Proposition 5.3.6. *For each standard $n \geq 2$, the following are provable in RCA_0:*

1. *$F\mathsf{IP}$ implies $\overline{D}_n\mathsf{IP}$;*
2. *$\overline{D}_{n+1}\mathsf{IP}$ implies $\overline{D}_n\mathsf{IP}$.*

Proof. To prove (1), let $A = \langle A_i : i \in \mathbb{N}\rangle$ be a non-trivial family of sets. We may assume that A has no finite maximal subfamily with the $\overline{D}_n$ intersection property. Define a new family $\widetilde{A} = \langle \widetilde{A}_i : i \in \mathbb{N}\rangle$ by recursion. For all $i \neq j$, let $2i \in \widetilde{A}_i$ and $2j \notin \widetilde{A}_i$. Now suppose s is such that the $\widetilde{A}_i$ have been defined precisely on the odd numbers less than $2s+1$. Consider all finite sets $F \subseteq \{0, \ldots, s\}$ such that $|F| \geq n+1$ and for every $F' \subseteq F$ of size n there is an $x \leq s$ belonging to $\bigcap_{i \in F'} A_i$. If no such F exists, then enumerate $2s+1$ into the complement of $\widetilde{A}_i$ for all $i \in \mathbb{N}$. Otherwise, list these sets as $F_0, \ldots, F_k$. For each $j \leq k$, enumerate $2(s+j)+1$ into $\widetilde{A}_i$ if $i \in F_j$, and into the complement of $\widetilde{A}_i$ if $i \notin F_j$.

The family $\widetilde{A}$ exists by Δ^0_1 comprehension, and is non-trivial by construction. Let $\widetilde{B} = \langle \widetilde{B}_i : i \in \mathbb{N}\rangle$ be a maximal subfamily of $\widetilde{A}$ with the F intersection property. Now each $\widetilde{B}_i$ contains exactly one even number, and if $2j \in \widetilde{B}_i$ then $\widetilde{B}_i = \widetilde{A}_j$. We define a family

$B = \langle B_i : i \in \mathbb{N} \rangle$, where $B_i = A_j$ for the unique j such that $2j \in \widetilde{B}_i$. We claim that this is a maximal subfamily of A with the $\overline{D}_n$ intersection property.

It is not difficult to see that B has the $\overline{D}_n$ intersection property. Indeed, let $A_{i_0}, \ldots, A_{j_{n-1}}$ be any n distinct members of B, and assume the indices have been chosen so that $\widetilde{A}_{i_j} \in \widetilde{B}$ for all $j < n$. Then $\bigcap_{j<n} \widetilde{A}_{i_j} \neq \emptyset$, so by construction we can find a finite set F of size $\geq n+1$ such that $i_j \in F$ for all j and $\bigcap_{i \in F'} A_i \neq \emptyset$ for every n-element $F' \subset F$. In particular, $\bigcap_{j<n} A_{i_j} \neq \emptyset$.

To show that B is maximal, we first argue that it is not a finite subfamily. Assume otherwise. Say the distinct members of B are $A_{i_0}, \ldots, A_{i_m}$, where the indices have been chosen so that $\widetilde{A}_{i_j} \in \widetilde{B}$ for all $j \leq m$. Now we can find a finite set F of size $\geq n+1$ such that $i_j \in F$ for all j and $\bigcap_{i \in F'} A_i \neq \emptyset$ for every n-element $F' \subset F$. If $m = 0$, this is because of our assumption on A, and if $m > 0$, this is because $\bigcap_{j \leq m} \widetilde{A}_{i_j} \neq \emptyset$. Our assumption on A also implies that $\langle A_i : i \in F \rangle$ can not be a maximal subfamily with the $\overline{D}_n$ intersection property. We can therefore fix a k so that $A_k \neq A_i$ for all $i \in F$ and $\bigcap_{i \in F'} A_i \neq \emptyset$ for every n-element $F' \subset F \cup \{k\}$. Then by construction, $\widetilde{A}_k \cap \bigcap_{j \leq m} \widetilde{A}_{i_j} \neq \emptyset$. Of course, the same is true if we replace any i_j in the intersection by any i such that $A_i = A_{i_j}$. And since for every i such that $\widetilde{A}_i \in B$ we have $A_i = A_{i_j}$ for some $j \leq m$, it follows that the intersection of any finite number of members of $\widetilde{B}$ with $\widetilde{A}_k$ is non-empty. By maximality of $\widetilde{B}$, $\widetilde{A}_k \in \widetilde{B}$ and hence $A_k \in B$. This is the desired contradiction.

Now suppose $A_k \notin B$ for some k, so that necessarily $\widetilde{A}_k \notin \widetilde{B}$. Since $\widetilde{B}$ is maximal, and since B is not finite, we can consequently find a finite set F of size $\geq n+1$ satisfying the following:

- for all $i \neq j$ in F, $A_i \neq A_j$;
- for all $i \in F$, $\widetilde{A}_i \in \widetilde{B}$;
- $\widetilde{A}_k \cap \bigcap_{i \in F} \widetilde{A}_i = \emptyset$.

By construction, this means there is an n-element subset F' of $F \cup \{k\}$ with $\bigcap_{i \in F'} A_i = \emptyset$, and clearly k must belong to F'. Since $A_i \in B$ for all $i \in F$, and in particular for all $i \in F' - \{k\}$, we conclude that B is maximal with respect to property $\overline{D}_n$. This completes the proof that $F\mathsf{IP}$ implies $\overline{D}_n\mathsf{IP}$.

A similar argument can be used to show (2). The construction of $\widetilde{A}$ is simply modified so that, instead of looking at finite sets $F \subseteq \{0, \ldots, s\}$ with $|F| \geq n+1$, it only considers only those with $|F| = n+1$. The details are left to the reader. □

An apparent weakness of our definition of subfamily is that we can not, in general, effectively decide which members of a family are contained in a given subfamily. The next proposition demonstrates that if we strengthen the definition of subfamily to make this problem decidable, all the intersection principles we study become equivalent to arithmetical comprehension.

Proposition 5.3.7. *Let P be any of the properties in Definition 5.3.2. The following are equivalent over* RCA_0*:*

1. ACA_0;

2. *every non-trivial family of sets $\langle A_i : i \in \mathbb{N}\rangle$ has a maximal subfamily B with the P intersection property, and the set $I = \{i \in \mathbb{N} : A_i \in B\}$ exists.*

Proof. The argument that (1) implies (2) is a refinement of the proof of Proposition 5.3.5. In the case where A does not have a finite maximal subfamily with the P intersection property, we can take for I the range of the function p defined in that proof.

To show that (2) implies (1), we work in RCA_0 and let $f \colon \mathbb{N} \to \mathbb{N}$ be a one-to-one function. For each i, let

$$A_i = \{2i\} \cup \{2x + 1 : (\exists y \leq x)[f(y) = i]\}.$$

noting that $i \in \operatorname{range}(f)$ if and only if A_i is not a singleton, in which case A_i contains cofinitely many odd numbers. Consequently, for every finite $F \subset \mathbb{N}$ of size ≥ 2, $\bigcap_{i\in F} A_i \neq \emptyset$ if and only if each $i \in F$ is in the range of f.

Apply (2) with $P = D_n$ to the family $A = \langle A_i : i \in \mathbb{N}\rangle$ to find the corresponding subfamily B and set I. Because B is a maximal subfamily with the D_n intersection property, there are at most $n - 1$ distinct j such that $j \in \operatorname{range}(f)$ and $A_j \in B$. And for each i not equal to any such j, we have

$$i \in \operatorname{range}(f) \iff A_i \notin B \iff i \notin I.$$

Thus the range of f exists. We reach the same conclusion if we instead apply (2) with $P = F$ or $P = \overline{D}_n$ to A. In this case, B_i is not a singleton for all $i \in \mathbb{N}$, and we have

$$i \in \operatorname{range}(f) \iff A_i \in B \iff i \in I. \qquad \square$$

We close this subsection by showing that the above reversal to ACA_0 goes through for $P = D_n$ even with our weak definition of subfamily.

Proposition 5.3.8. *For each standard $n \geq 2$, $D_n\mathsf{IP}$ is equivalent to ACA_0 over RCA_0.*

Proof. Fix a one-to-one function $f \colon \mathbb{N} \to \mathbb{N}$, and let A be the family defined in the preceding proposition. Let $B = \langle B_i : i \in \mathbb{N}\rangle$ be the family obtained from applying $D_n\mathsf{IP}$ to A. As above, there can be at most $n - 1$ many j such that $j \in \operatorname{range}(f)$ and $A_j \in B$. For i not equal to any such j, we have

$$i \in \operatorname{range}(f) \iff A_i \notin B \iff (\forall k)[2i \notin B_k].$$

This gives us a Π^0_1 definition of the range of f. Since the range of f is also definable by a Σ^0_1 formula, it follows by Δ^0_1 comprehension that the range of f exists. $\square$

We do not know whether the implications from $F\mathsf{IP}$ to $\overline{D}_n\mathsf{IP}$ or from $\overline{D}_{n+1}\mathsf{IP}$ to $\overline{D}_n\mathsf{IP}$ are strict. However, all of our results in the sequel hold equally well for $F\mathsf{IP}$ as they do for $\overline{D}_2\mathsf{IP}$. Thus, we shall formulate all implications over RCA_0 involving these principles as being to $F\mathsf{IP}$ and from $\overline{D}_2\mathsf{IP}$.

5.3.2 Non-implications and conservation results

In contrast to Proposition 5.3.8 , $F\mathsf{IP}$ and the principles $\overline{D}_n\mathsf{IP}$ for $n \geq 2$ are all strictly weaker than ACA_0. This section is dedicated to a proof of this non-implication, as well as to results showing that $F\mathsf{IP}$ does not imply WKL_0 and $D_2\mathsf{IP}$ is not provable in WKL_0. These results will be further sharpened by Proposition 5.3.24 below.

Proposition 5.3.9. *There is an ω-model of $\mathsf{RCA}_0 + F\mathsf{IP}$ consisting entirely of low sets. Therefore $F\mathsf{IP}$ does not imply ACA_0 over RCA_0.*

Proof. Given a computable non-trivial family $A = \langle A_i : i \in \omega\rangle$ of sets, consider the notion of forcing whose conditions are strings $\sigma \in \omega^{<\omega}$ such that some $x \leq \sigma(|\sigma| - 1)$ belongs to $A_{\sigma(i)}$ for all $i < |\sigma| - 1$, and $\sigma' \leq \sigma$ if $\sigma' \upharpoonright |\sigma'| - 1 \succeq \sigma \upharpoonright |\sigma| - 1$. Now fix any $A_i \neq \emptyset$, say with $x \in A_i$, and let $\sigma_0 = ix$. Given σ_{2e} for some $e \in \omega$, ask if there is a condition $\sigma \leq \sigma_{2e}$ such that $\Phi_e^{\sigma \upharpoonright |\sigma|-1}(e)\downarrow$. If so, let σ_{2e+1} be the least such σ of length greater than $|\sigma_{2e}|$, and if not, let $\sigma_{2e+1} = \sigma_{2e}$. Given σ_{2e+1}, ask if there is a condition $\sigma \leq \sigma_{2e+1}$ such that $\sigma(i) = e$ for some $i < |\sigma| - 1$. If so, let σ_{2e+2} be the least such σ, and if not, let $\sigma_{2e+2} = \sigma_{2e+1}$. A standard argument establishes that $J = \bigcup_{e\in\omega} (\sigma_e \upharpoonright |\sigma_e| - 1)$ is low, and hence so is $B = \langle A_{J(i)} : i \in \omega\rangle$. It is clear that B is a maximal subfamily of A with the F intersection property. Iterating and dovetailing this argument produces the desired ω-model.

The second part of the proposition follows from the fact that every ω-model of ACA_0 must contain a set of degree $\mathbf{0}'$, which is not low. $\square$

We shall establish the result that $F\mathsf{IP}$ does not even imply WKL_0 by showing $F\mathsf{IP}$ is conservative for the following class of sentences:

Definition 5.3.10 (Hirschfeldt, Shore and Slaman [27, p. 5819])**.** A sentence in L_2 is *restricted* Π^1_2 if it is of the form

$$(\forall X)[\varphi(X) \rightarrow (\exists Y)\psi(X, Y)],$$

where φ is arithmetical and ψ is Σ^0_3.

Many familiar principles are equivalent to restricted Π^1_2 sentences over RCA_0, including the defining axiom of WKL_0. We discuss several others in the next subsection.

The study of restricted Π^1_2 conservativity was initiated by Hirschfeldt and Shore [27, Corollary 2.21] in the context of the principle COH. Subsequently, it was extended by Hirschfeldt, Shore, and Slaman [28, Corollary 3.15 and the penultimate paragraph of Section 4] to the principles AMT and $\Pi^0_1\mathsf{G}$ (see Section 5.3.3 below for definitions). The conservation proofs for the latter two principles differ from the original only in the choice of forcing notion (Mathias forcing for COH, Cohen forcing for AMT and $\Pi^0_1\mathsf{G}$). A similar proof goes through, *mutatis mutandis*, for the notion of forcing from the proof of Proposition 5.3.9, giving the following conservation result (we refer the reader to either of the above-cited articles for details):

Theorem 5.3.11. *The principle $F\mathsf{IP}$ is conservative over RCA_0 for restricted Π^1_2 sentences. Therefore $F\mathsf{IP}$ does not imply WKL_0 over RCA_0.*

The preceding results lead to the question of whether $F\mathsf{IP}$, or any one of the principles $\overline{D}_n\mathsf{IP}$, is provable in RCA_0, or at least in WKL_0. We show in the next theorem that $F\mathsf{IP}$ fails in any ω-model of WKL_0 consisting entirely of sets of hyperimmune-free Turing degree. Recall that a Turing degree is *hyperimmune* if it bounds the degree of a function not dominated by any computable function, and a degree which is not hyperimmune is *hyperimmune-free*. A model of the kind we are interested in can be obtained by iterating and dovetailing the hyperimmune-free basis theorem of Jockusch and Soare [39, Theorem 2.4], which asserts that every computable infinite subtree of $2^{<\omega}$ has an infinite path of hyperimmune-free degree.

Theorem 5.3.12. *There exists a computable non-trivial family of sets for which any maximal subfamily with the $\overline{D}_2$ intersection property must have hyperimmune degree.*

To motivate the proof, which will occupy the rest of this subsection, we first discuss the simpler construction of a computable non-trivial family for which any maximal subfamily with the $\overline{D}_2$ intersection property must be non-computable. This, in turn, is perhaps best motivated by thinking about how a proof of the contrary could fail.

Suppose we are given a computable non-trivial family $A = \langle A_i : i \in \mathbb{N}\rangle$. The most direct method of building a maximal subfamily $B = \langle B_i : i \in \mathbb{N}\rangle$ with the $\overline{D}_2$ intersection property, assuming A has no finite such subfamily, is to let $B_0 = A_i$ for the least i such that $A_i \neq \emptyset$, then let $B_1 = A_j$ for the least $j > i$ such that $A_i \cap A_j \neq \emptyset$, and so on. Of course, this subfamily need not be computable, but we could try to temper our strategy to make it computable. An obvious such attempt is the following: we first search through the members of A in some effective fashion until we find the first one that is non-empty, and we let this be B_0; then, having defined $B_0, \ldots, B_n$ for some n, we search through A again until we find the first member not among the B_i but intersecting each of them, and let this be B_{n+1}. Now while this strategy yields a subfamily B which is indeed computable and has the $\overline{D}_2$ intersection property, B need not be maximal. For example, suppose the first non-empty set we discover is A_1, so that we set $B_0 = A_1$. It may be that A_0 intersects A_1, but that we discover this only after discovering that A_2 intersects A_1, so that we set $B_1 = A_2$. It may then be that A_0 also intersects A_2, but that we discover this only after discovering that A_3 intersects A_1 and A_2, so that we set $B_2 = A_3$. In this fashion, it is possible for us to never put A_0 into B, even though it ends up intersecting each B_i.

We can exploit precisely this difficulty to build a family $A = \langle A_i : i \in \omega\rangle$ for which neither the strategy above, nor any other computable strategy, succeeds. We proceed by stages, at each one enumerating at most finitely many numbers into at most finitely many

A_i. By Remark 5.3.4, it suffices to ensure that for each e, either Φ_e is not total, or else $\langle A_{\Phi_e(i)} : i \in \omega \rangle$ is not a maximal subfamily with the $\overline{D}_2$ intersection property. We next discuss how to satisfy a single such requirement. Of course, in the full construction there will be other requirements, but these will not interfere with one another.

At stage s, we look for the longest non-empty string $\sigma \in \omega^{<\omega}$ such that for all $i < |\sigma|$, $\Phi_{e,s}(i) \downarrow = \sigma(i)$, and for all $i, j < |\sigma|$, $A_{\sigma(i)}$ and $A_{\sigma(j)}$ have been intersected by stage s. At the first stage that we find such a σ, we define t_e to be some number large enough that A_{t_e} does not yet intersect $A_{\Phi_e(i)}$ for any i. We then start defining numbers $p_{e,0}, p_{e,1}, \ldots$ as follows: at each stage, if we do not find a longer such σ, or if t_e is in the range of this σ, we do nothing; otherwise, we choose the least n such that $p_{e,n}$ has not yet been defined, and define it be some number not yet in the range of Φ_e and large enough that $A_{p_{e,n}}$ does not intersect A_{t_e}. We call $p_{e,n}$ a *follower* for σ. Then for any $p_{e,m}$ that is already defined and is a follower for some $\tau \preceq \sigma$, we intersect $A_{p_{e,m}}$ with $A_{\sigma(i)}$ for all i. Also, if $\sigma(i) = p_{e,m}$ for some i and m, then for the largest such m and for all j with $\sigma(j) \neq p_{e,m}$, we intersect $A_{\sigma(j)}$ with A_{t_e}.

Now suppose that Φ_e is total and that the subfamily it defines is a maximal one with the $\overline{D}_2$ intersection property. The idea is that A_{t_e} should behave as A_0 did in the motivating example above, by never entering the subfamily but intersecting all of its members, thereby giving us a contradiction. For the first part, note that if $\Phi_e(i) = t_e$ for some i then $\sigma(i) = t_e$ for some string σ as above, and that necessarily $A_{\sigma(j)} \cap A_{t_e} = \emptyset$ for some j. But any string we find at a subsequent stage will extend σ and hence have t_e in its range, so we shall never make $A_{\sigma(j)} = A_{\Phi_e(j)}$ intersect $A_{t_e} = A_{\Phi_e(i)}$. Thus, t_e can not be the range of Φ_e. We conclude that $p_{e,n}$ is defined for every n. For the second part, note that since each $p_{e,n}$ is a follower for some initial segment of Φ_e, each $A_{\Phi_e(i)}$ is eventually intersected with $A_{p_{e,n}}$. By maximality, then, $p_{e,n}$ belongs to the range of Φ_e for all n, which means that each $A_{\Phi_e(i)}$ is eventually also intersected with A_{t_e}.

This basic idea is the same one that we now use in our proof of Theorem 5.3.12. But since here we are concerned with more than just computable subfamilies, it no longer suffices to just play against those of the form $\langle A_{\Phi_e(i)} : i \in \omega \rangle$. Instead, we must consider all possible subfamilies $\langle A_{J(i)} : i \in \omega \rangle$ for $J \in \omega^\omega$, and show that if J defines a maximal subfamily with the $\overline{D}_2$ intersection property then there exists a function $f \leq_T J$ such that for all e, either Φ_e is not total or it does not dominate f. Accordingly, we must now define followers $p_{e,n}$ not only for those $\sigma \in \omega^{<\omega}$ that are initial segments of Φ_e, but for all strings that look as though they can be extended to some such $J \in \omega^\omega$. We still enumerate the followers linearly as $p_{e,0}, p_{e,1}, \ldots$, even though the strings they are defined as followers for no longer have to be compatible.

Looking ahead to the verification, fix any J that defines a maximal subfamily with the $\overline{D}_2$ intersection property. We describe the intuition behind defining $f \leq_T J$ that escapes domination by a single computable function Φ_e. (Of course, there are much easier ways to define f to achieve this, but this definition is close to the one that will be used in the full construction.) Much as in the more basic argument above, the construction will ensure that there are infinitely many n such that $p_{e,n}$ is a follower for some initial segment of J and belongs to the range of J. Then, $f(x)$ can be thought of as telling us how far to go along J in order to find one more $p_{e,n}$ in its range. More precisely, f is defined along with a sequence $\sigma_0 \prec \sigma_1 \prec \cdots$ of initial segments of J. For each x, σ_{x+1} is an extension of σ_x whose range contains a follower $p_{e,n}$ for some τ with $\sigma_x \preceq \tau \prec \sigma_{x+1}$, and $f(x+1)$ is a number large enough to bound an element of $\bigcap_{i<|\sigma_{x+1}|} A_{\sigma_{x+1}(i)}$. The idea behind this definition is that if f actually is dominated by Φ_e, then we can modify our basic strategy so that in deciding which members of A to intersect with A_{t_e} in the construction, we consider not initial segments of Φ_e as before, but strings $\sigma \in \omega^{<\omega}$ that look like initial segments of J. Then, just as before, we can show that no such string σ can have t_e in its range, and yet that $A_{\sigma(i)}$ is eventually intersected with A_{t_e} for all i. Thus we obtain the same contradiction we got above, namely that J does not have t_e in its range and hence can not be maximal after all.

The main obstacle to this approach is that we do not know which computable function will dominate f, if f is in fact computably dominated, and so we can not use its index in the definition of f. One way to remedy this is to make $f(x)$ large enough to find not only the next $p_{e,n}$ in the range of J for some fixed e, but the next $p_{e,n}$ for each $e < x$. This, in turn, demands that we define followers in such a way that $p_{e,n}$ is defined for every e and n, regardless of whether Φ_e is total. But then we must define followers $p_{e,n}$ even for strings that already contain t_e in their range, since we do not know ahead of time that this will not happen for all sufficiently long strings. In the construction, then, we distinguish between two types of followers, those defined as followers for strings that have t_e in their range, and those defined as followers for strings that do not. We shall see in the verification that this is not a serious complication.

We turn to the formal details. We adopt the convention that for all $e, x, y, s \in \omega$, if $\Phi_{e,s}(x) \downarrow = y$, then $e, x, y \leq s$, and $\Phi_{e,s}(z) \downarrow$ for all $z < x$. Let $s_{e,x}$ denote the least s such that $\Phi_{e,s}(x) \downarrow$, which may of course be undefined if Φ_e is not total. Then to show that some function is not computably dominated it suffices to show it is not dominated by the map $x \mapsto s_{e,x}$ for any e.

Proof of Theorem 5.3.12. We build a computable $A = \langle A_i : i \in \omega \rangle$ by stages. Let $A_{i,s}$ be the set of elements which have been enumerated into A_i by stage s, which will always be finite. Say a non-empty string $\sigma \in \omega^{<\omega}$ is *bounded* by s if it satisfies the following conditions:

- $|\sigma| \leq s$;
- for all $i < |\sigma|$, $\sigma(i) \leq s$;
- for all $i, j < |\sigma|$, there is a $y \leq s$ with $y \in A_{\sigma(i),s} \cap A_{\sigma(j),s}$.

Construction. For all $i \neq j$, let $2i \in A_i$ and $2j \notin A_i$. At stage $s \in \omega$, assume inductively that for each e, we have defined finitely many numbers $p_{e,n}$, $n \in \omega$, each labeled as either a *type 1 follower* or a *type 2 follower* for some string $\sigma \in \omega^{<\omega}$. Call a number x *fresh* if x is larger than s and every number that has been mentioned during the construction so far.

We consider consecutive substages, at substage $e \leq s$ proceeding as follows.

Step 1. If t_e is undefined, define it to be a fresh number. If t_e is defined but $s_{e,0} = s$, redefine t_e to be a fresh large number. In the latter case, change any type 1 follower $p_{e,n}$ already defined to be a type 2 follower (for the same string).

Step 2. Consider any $\sigma \in \omega^{<\omega}$ bounded by s. Choose the least n such that $p_{e,n}$ has not been defined, and define it to be a fresh number. Then, for each $i < |\sigma|$, enumerate a fresh odd number into $A_{p_{e,n}} \cap A_{\sigma(i)}$. If there is an $i < |\sigma|$ such that $\sigma(i) = t_e$, call $p_{e,n}$ a type 1 follower for σ, and otherwise, call $p_{e,n}$ a type 2 follower for σ.

Step 3. Consider any $p_{e,n}$ defined at a stage before s, and any $\sigma \in \omega^{<\omega}$ bounded by s that extends the string that $p_{e,n}$ was defined as a follower for. If $p_{e,n}$ is a type 1 follower then, for each $i < |\sigma|$, enumerate a fresh odd number into $A_{p_{e,n}} \cap A_{\sigma(i)}$. If $p_{e,n}$ is a type 2 follower, then do this only for the σ such that $\sigma(i) \neq t_e$ for all i.

Step 4. Suppose there is an x such that $\Phi_{e,s}(x)\downarrow$, and $s = s_{e,x}$ for the largest such x. Call a string $\sigma \in \omega^{<\omega}$ *viable* for e at stage s if there exist $\sigma_0 \prec \cdots \prec \sigma_x = \sigma$ satisfying:

- $|\sigma_0| = 1$;
- for each $i \leq x$, σ_i is bounded by $s_{e,i}$;
- for each $i < x$ and $j \leq i$, there exists a k with $|\sigma_i| \leq k < |\sigma_{i+1}|$ and an n such that $p_{j,n}$ is defined and is a follower for some τ with $\sigma_i \preceq \tau \prec \sigma_{i+1}$, and $\sigma_{i+1}(k) = p_{j,n}$.

If $x > e$, let $k^{\sigma}_{e,x}$ be the least k that satisfies the last condition above for $i = x - 1$ and $j = e$.

Call s an e-*acceptable* stage if, for every string σ viable for e at this stage, we have:

- $k^{\sigma}_{e,x}$ is defined;
- $A_{\sigma(k^{\sigma}_{e,x}),s} \cap A_{t_e,s} = \emptyset$;
- there is an $i < k^{\sigma}_{e,x}$ such that:
  - $\sigma(i) = p_{e,n}$ for some n;
  - $A_{\sigma(i),s} \cap A_{t_e,s} = \emptyset$;
  - for all $j \leq i$ and all τ viable for e at stage s, $\sigma(j) \neq \tau(k^{\tau}_{e,x})$.

If s is e-acceptable, then for each viable σ, choose the largest such $i < k^{\sigma}_{e,x}$, and enumerate a fresh odd number into $A_{\sigma(j)} \cap A_{t_e}$ for each $j \leq i$. (Note that the least stage that can be e-acceptable is $s_{e,e+1}$.)

Step 5. If $e < s$, go to the next substage. If $e = s$, then for each i and each x less than or equal to the largest number mentioned during the construction at stage s and and not enumerated into A_i, enumerate x into the complement of A_i. Then go to stage $s + 1$.

End construction.

Verification. It is clear that A is a computable non-trivial family. Suppose $B = \langle B_i : i \in \mathbb{N} \rangle$ is a maximal subfamily of A with the $\overline{D}_2$ intersection property. Choose the unique $J \in \omega^\omega$ such that $B_i = A_{J(i)}$ for all i.

Claim 5.3.13. *For each $e \in \omega$ and each $\sigma \prec J$, there is an $n \in \omega$ such that $p_{e,n}$ is a follower for some τ with $\sigma \preceq \tau \prec J$ and $A_{p_{e,n}} \in B$.*

Proof. First, notice that for each $\sigma \preceq J$, there are infinitely many s that bound σ. Hence, since at any such stage s of the construction (specifically, at step 2 of substage e), $p_{e,n}$ gets defined for a new $n \in \omega$, it follows that $p_{e,n}$ gets defined for all n. Second, note that t_e necessarily gets defined during the construction, and then gets redefined at most once. We use t_e henceforth to refer to its final value.

Fix $\sigma \prec J$ and $m \in \omega$, and let s be a stage by which $p_{e,n}$ has been defined for all $n \leq m$. Let τ be either σ if $A_{t_e} \notin B$ or $\sigma(i) = t_e$ for some $i < |\sigma|$, or an initial segment of J extending σ long enough that there exists a $i < |\tau|$ with $\tau(i) = t_e$. By our observation above, there exists a $t \geq \max\{s, e\}$ that bounds τ. Let $p_{e,n}$ be the follower for τ defined at stage t, substage e, step 2, of the construction, so that necessarily $n > m$. Note that $p_{e,n}$ is a type 2 follower if and only if $A_{t_e} \notin B$.

Choose any υ with $\tau \preceq \upsilon \prec J$, and let $u > t$ be large enough to bound υ. Then at stage u, substage e, step 3, of the construction, $A_{p_{e,n}}$ is made to intersect $A_{\upsilon(i)}$ for each $i < |\upsilon|$ (in case $p_{e,n}$ is a type 2 follower, this is because $\upsilon(i) \neq t_e$ for all i). Since υ was arbitrary, it follows that $A_{p_{e,n}} \cap A_{J(i)}$ for all $i \in \omega$. Hence, by maximality of B, it must be that $A_{p_{e,n}} \in B$. $\square$

Now we define a function $f : \mathbb{N} \to \mathbb{N}$, and a sequence $\sigma_0 \prec \sigma_1 \prec \cdots$ of initial segments of J. Let $\sigma_0 = J \upharpoonright 1$ and $f(0) = 2J(0)$, and assume that we have $f(x)$ and σ_x defined for some $x \geq 0$. Let $f(x+1)$ be the least s such that there exists a $\sigma \in \omega^{<\omega}$ satisfying:

- $\sigma_x \prec \sigma \prec J$;
- σ is bounded by s;
- for each $j \leq x$, there exists a k with $|\sigma_x| \leq k < |\sigma|$ and an n such that $p_{j,n}$ is defined by stage s of the construction and is a follower for some τ with $\sigma_x \preceq \tau \prec \sigma$, and $\sigma(k) = p_{j,n}$.

Let σ_{x+1} be the least σ satisfying the above conditions. By the preceding claim, $f(x)$ and σ_x are defined for all x.

Clearly, $f \leq_T B$. Seeking a contradiction, suppose e is such that $f(x) \leq s_{e,x}$ for all x. A simple induction then shows that σ_x is viable for e at stage $s_{e,x}$. In particular, for every x, there is a σ viable for e at stage $s_{e,x}$. We fix the present value of e for the remainder of the proof, including in the following claims:

Claim 5.3.14. *If σ is viable for e at stage $s_{e,0}$, then $A_{\sigma(0)}$ is not intersected with A_{t_e} before step 4 of substage e of the first e-acceptable stage.*

Proof. Note that necessarily $|\sigma| = 1$, and that $s_{e,0}$ is not e-acceptable. At step 1 of substage e of stage $s_{e,0}$, t_e gets redefined to be a fresh number. Viability at stage $s_{e,0}$ just means that σ is bounded by $s_{e,0}$, and hence $A_{\sigma(0)}$ can not intersect A_{t_e} at the end of this step. Hence, if we let s be the stage at which $A_{\sigma(0)}$ is first intersected with A_{t_e}, then $s \geq s_{e,0}$. Suppose the intersection takes place at step k of substage i of stage s. Then in particular, this point in the construction comes strictly after step 4 of substage e of stage $s_{e,0}$.

It suffices to prove the claim under the following assumption: there is no σ' viable for e at stage $s_{e,0}$ such that $A_{\sigma(0)}$ is first intersected with A_{t_e} before step k of substage i of stage s. Note also that k must be 3 or 4, since the only other step at which different members of A are intersected is step 2, but one of the two sets intersected there is always indexed by a fresh number.

First suppose $k = 3$. Then it must be that for some n, and for some τ extending the string ρ for which $p_{i,n}$ is a follower, we are intersecting $A_{p_{i,n}}$ with $A_{\tau(i)}$ for all $i < |\tau|$. Since t_e can not equal $p_{i,m}$ for any m, it must be that $\sigma(0) = p_{i,n}$, and hence that there is a a $j < |\tau|$ such that $\tau(j) = t_e$. Now $\sigma(0)$ is bounded by $s_{e,0}$ and hence is not fresh after step 4 of substage e of stage $s_{e,0}$, whereas $p_{i,n}$, when defined, is defined to be a fresh number. Thus, since $\sigma(0) = p_{i,n}$, $p_{i,n}$ must be defined as a follower for ρ before step 4 of substage e of stage $s_{e,0}$. At that point in the construction, by definition, ρ has to be bounded, so ρ must also be bounded by $s_{e,0}$. In particular, $\rho(j)$ must be viable for e at stage $s_{e,0}$, for every j. This means $\rho(j) \neq t_e$, since t_e is certainly not viable at stage $s_{e,0}$. But since τ has to be bounded by $s_{e,0}$ in order for us to be considering it, it must be that $A_{\rho(j)}$ and A_{t_e} are intersected at some earlier point in the construction. This contradicts our assumption above.

Now suppose $k = 4$ but $i \neq e$. Then it must be that s is i-acceptable. Since t_e can not equal t_i, and since members of A are only intersected at step 4 with A_{t_i}, it must be that $\sigma(0) = t_i$. There must also be a $\tau \in \omega^{<\omega}$ such that τ is viable for i at stage s and $\tau(j) = t_e$ for some $j < |\tau|$. Since s is i-acceptable, $s_{i,0}$ is defined. Now $\sigma(0)$ is bounded by $s_{e,0}$ and hence is not fresh after step 4 of substage e of stage $s_{e,0}$, whereas at step 1 of substage i of stage $s_{i,0}$, t_i is redefined to be a fresh number. Thus, since $\sigma(0) = t_i$, step 1 of substage i of stage $s_{i,0}$ can not happen after step 4 of substage e of stage $s_{e,0}$. So, since the one bit string $\tau(0)$ has to be viable for i at step $s_{i,0}$ by definition of viability, it follows that $\tau(0)$ is also viable at stage $s_{e,0}$. Hence, $\tau(0) \neq t_e$ since t_e is not viable at stage $s_{e,0}$. But since τ has to be bounded by $s_{e,0}$, it must be that $A_{\tau(0)}$ and A_{t_e} are intersected at some earlier point in the construction. This again gives us a contradiction.

We conclude that $k = 4$ and $i = e$, that is, that $A_{\sigma(0)}$ is first intersected with A_{t_e} at step 4 of substage e of stage s. This forces s to be e-acceptable, so the claim is proved. □

Claim 5.3.15. *Suppose $x > e$ and $\sigma \in \omega^{<\omega}$ is viable for e at stage $s_{e,x}$. Then for some $i < |\sigma|$, $\sigma(i) = p_{e,n}$ for some n and $A_{\sigma(i)}$ and A_{t_e} are disjoint through the end of stage $s_{e,x}$.*

Proof. We proceed by induction on x, beginning with $x = e + 1$. Fix σ. By construction, $s_{e,x}$ is the first stage that can be e-acceptable, so by the preceding claim, $A_{\sigma(0)}$ has empty intersection with A_{t_e} at the beginning of step 4 of substage e of this stage. Hence, $\sigma(i) \neq t_e$ for all $i < |\sigma|$ since σ must be bounded by $s_{e,x}$. Now by viability, there is an i and an

n such that $\sigma(i) = p_{e,n}$ and is a follower for some τ with $\sigma(0) \preceq \tau \prec \sigma$. It follows that $p_{e,n}$ is a type 2 follower. Furthermore, it is easy to see that for any type 2 follower $p_{e,m}$, $A_{p_{e,m}}$ can only be made to intersect A_{t_e} at step 4 of substage e of an e-acceptable stage. Thus, $A_{\sigma(i)}$ must be disjoint from A_{t_e} at the beginning of step 4 of substage e of stage $s_{e,x}$. Additionally, if $s_{e,x}$ is not e-acceptable, then nothing is done at step 4 of substage e, and hence $A_{\sigma(i)}$ is not intersected with A_{t_e} during the course of the rest of the stage. If $s_{e,x}$ is e-acceptable, then in fact there must exist a i as above, namely $i = k^{\sigma}_{e,x}$, such that $A_{\sigma(i)}$ is not intersected with A_{t_e} at step 4 of substage e, and hence not during the course of the rest of the stage either. This proves the base case of the induction.

Now let $x > e$ be given and suppose the claim holds for x. Given $\sigma \in \omega^{<\omega}$ viable for e at stage $s_{e,x+1}$, there is some $\tau \prec \sigma$ viable for e at stage $s_{e,x}$. If $s_{e,x+1}$ is not e-acceptable, then the same i witnessing that the claim holds for x and τ witnesses also that it holds for $x+1$ and σ. This is because $\tau(i)$ is necessarily a type 2 follower, and $A_{\tau(i)}$ is consequently not intersected with A_{t_e} until step 4 of substage e of some e-acceptable stage after stage $s_{e,x}$. If $s_{e,x+1}$ is e-acceptable, then just as in the base case, viability of σ implies that for $i = k^{\sigma}_{e,x+1}$, $A_{\sigma(i)}$ does not intersect A_{t_e} at the beginning of step 4 of substage e of stage $s_{e,x+1}$, and is not made to do so by its end. □

Claim 5.3.16. *There exist infinitely many e-acceptable stages.*

Proof. Fix any stage $s = s_{e,x}$ for $x > e$, and assume there is not any e-acceptable stage greater than s. For each σ viable for e at stage s, let i_σ be the largest i satisfying the statement of the preceding claim. Then $\sigma(i_\sigma)$ is a type 2 follower, so by our assumption, $A_{\sigma(i_\sigma)}$ is never intersected with A_{t_e} during the course of the rest of the construction.

Now for each $y \geq x$ and each σ viable for e at stage $s_{e,y+1}$, $k^{\sigma}_{e,y+1}$ is defined and $\sigma(k^{\sigma}_{e,y+1})$ is a follower $p_{e,n}$ for some string extending a $\tau \prec \sigma$ viable for e at stage $s_{e,y}$. Since followers are always defined to be fresh numbers, if $k^{\tau}_{e,y}$ is defined then $\sigma(k^{\tau}_{e,y}) = p_{e,m}$ for some $p_{e,m}$ defined strictly before $p_{e,n}$ in the construction.

Thus, for any sufficiently large $y > x$, it must be that for each σ viable at stage $s_{e,y}$, $\sigma(k^{\sigma}_{e,y}) \neq \tau(k)$ for all τ viable at stage s and all $k \leq j_\tau$. Moreover, since $A_{\tau(i_\tau)} \cap A_{t_e} = \emptyset$ and $\sigma(k^{\sigma}_{e,y})$ is a follower for some extension of some such τ, it must be that $\sigma(k^{\sigma}_{e,y})$ is a type 2 follower. Hence, $A_{\sigma(k^{\sigma}_{e,y})}$ can only be intersected with A_{t_e} at step 4 of substage e of an e-acceptable stage, meaning at a stage at or before s. It follows that if y is additionally chosen large enough that, for each σ viable at stage $s_{e,y}$, the follower $\sigma(k^{\sigma}_{e,y})$ is not defined before stage s, then $A_{\sigma(k^{\sigma}_{e,y})}$ will be disjoint from A_{t_e}. But then in particular, $A_{\sigma(k^{\sigma}_{e,y})}[s_{e,y}] \cap A_{t_e}[s_{e,y}] = \emptyset$, so $s_{e,y}$ is an e-acceptable stage greater than s. This is a contradiction, so the claim is proved. □

We can now complete the proof. First note that $A_{t_e} \notin B$, for otherwise there would have to be an x and an $i < |\sigma_x|$ such that $\sigma_x(i) = t_e$. But then σ_x would be viable for e at stage $s = s_{e,x}$, and so is in particular it would be bounded by s, meaning $A_{\sigma_x(j),s}$ would have to intersect $A_{\sigma_x(i),s} = A_{t_e,s}$ for all $j < |\sigma_x|$. This would contradict Claim 5.3.15. Now consider any e-acceptable stage $s = s_{e,x}$. By construction, there is an $i < |\sigma_x|$ such that $A_{\sigma_x(i)}$ is disjoint from A_{t_e} at the beginning of stage s, and each $A_{\sigma_x(j)}$ for $j \leq i$ is made to intersect A_{t_e} by the end of stage s. Since, by Claim 5.3.16, there are infinitely many e-acceptable stages, and since $J = \bigcup_x \sigma_x$, it follows that $A_{J(i)}$ intersects A_{t_e} for all i. In other words, B_i intersects A_{t_e} for all i, which contradicts the choice of B as a maximal subfamily of A with the $\overline{D}_2$ intersection property. □

Remark 5.3.17. Examination of the above proof shows that it can be formalized in RCA_0, because the construction is computable and the verification that the function f defined in it is total and not computably dominated requires only Σ^0_1 induction. (See [59, Definition VII.1.4] for the formalizations of Turing reducibility and equivalence in RCA_0.)

As discussed above, this has as a consequence the following corollary:

Corollary 5.3.18. *The principle $\overline{D}_2\mathsf{IP}$ is not provable in WKL_0.*

Proof. Let $\mathscr{M}$ be an ω-model of WKL_0 such that every set in $\mathscr{M}$ is of hyperimmune-free degree. Let A be the family constructed by the formalized version of Theorem 5.3.12, noting that A belongs to REC and hence to $\mathscr{M}$. Suppose $B \in \mathscr{M}$ is a maximal subfamily of A with the $\overline{D}_2$ intersection property. Then by the preceding remark, $\mathscr{M} \models$ "B has hyperimmune degree". Now the property of having hyperimmune degree is defined by an arithmetical formula, and is thus absolute to ω-models. Therefore, B has hyperimmune degree, contradicting the construction of $\mathscr{M}$. □

5.3.3 Relationships with other principles

By the preceding results, $F\mathsf{IP}$ and the principles $D_n\mathsf{IP}$ are of the irregular variety that do not admit reversals to any of the main subsystems of Z_2 mentioned in the introduction. In particular, they lie strictly between RCA_0 and ACA_0, and are incomparable with WKL_0. Many principles of this kind have been studied in the literature, and collectively they form a rich and complicated structure. Partial summaries are given by Hirschfeldt and Shore [27, p. 199]. Additional discussion of the principles is given by Montalbán [48, Section 1] and Shore [57]. In this subsection, we investigate where our intersection principles fit into the known collection of irregular principles.

We can already show that $F\mathsf{IP}$ does not imply Ramsey's theorem for pairs (RT^2_2) or any of of the main combinatorial principles studied by Hirschfeldt and Shore [27] (all of which follow from RT^2_2). The only one of these to not have been defined above is the following:

Chain antichain principle (CAC). *For every partial order $\leq_P$ on $\mathbb{N}$ there is an infinite set X that is either a chain, i.e., $x \leq_P y$ or $y \leq_P x$ for all $x, y \in X$, or an antichain, i.e., $x \nleq_P y$ and $y \nleq_P x$ for all distinct $x, y \in X$.*

Corollary 5.3.19. *None of the following principles are implied by $F\mathsf{IP}$ over RCA_0: RT^2_2, SRT^2_2, DNR, CAC, ADS, SADS, COH.*

Proof. All but the last of these principles are equivalent to restricted Π^1_2 sentences, and so for them the corollary follows by the conservation result of Proposition 5.3.11. For COH, it follows by Proposition 5.3.9 and the fact that any ω-model of COH must contain a set of p-cohesive degree [5, p. 27], and such degrees are never low [40, Theorem 2.1]. □

Our next results require several basic model-theoretic concepts. We assume some suitable development of model theory in RCA_0 (compare [59, Section II.8]). Let T be a countable, complete, consistent theory.

- A *partial type* of T is a T-consistent set of formulas in a fixed number of free variables. A *complete type* is a $\subseteq$-maximal partial type.
- A model $\mathscr{M}$ of T *realizes* a partial type Γ if there is a tuple $\vec{a} \in |\mathscr{M}|$ such that $\mathscr{M} \models \varphi(\vec{a})$ for every $\varphi \in \Gamma$. Otherwise, $\mathscr{M}$ *omits* Γ.
- A partial type Γ is *principal* if there is a formula φ such that $T \vdash \varphi \to \psi$ for every formula $\psi \in \Gamma$. A model $\mathscr{M}$ of T is *atomic* if every partial type realized in $\mathscr{M}$ is principal.
- An *atom* of T is a formula φ such that for every formula ψ in the same free variables, exactly one of $T \vdash \varphi \to \psi$ or $T \vdash \varphi \to \neg\psi$ holds. T is *atomic* if for every T-consistent formula ψ, $T \vdash \varphi \to \psi$ for some atom φ.

A classical result states that a theory is atomic if and only if it has an atomic model. Hirschfeldt, Slaman, and Shore [28] studied the strength of this theorem in the following forms (see [28, pp. 5808, 5831]):

Atomic model theorem (AMT). *Every complete atomic theory has an atomic model.*

Omitting partial types principle (OPT). *Let T be a complete theory and let S be a set of partial types of T. Then there is a model of T omitting all the non-principal partial types in S.*

Over RCA_0, AMT is strictly implied by SADS ([28, Corollary 3.12 and Theorem 4.1]). The latter asserts that every linear order of type $\omega + \omega^*$ has a suborder of type ω or ω^*, and is one of the weakest principles studied in [27] that does not hold in the ω-model REC. (An even weaker such principle, AST, was studied in [28], and is discussed in Appendix A). Thus, AMT is especially weak even among principles lying below RT^2_2. It does, however, imply part (2) of the following theorem, and therefore also OPT ([28, Theorem 5.6 (2) and Corollary 5.8]):

Theorem 5.3.20 (Hirschfeldt, Shore, and Slaman [28, Theorem 5.7])**.** *The following are equivalent over* RCA_0*:*

1. OPT*;*
2. *for every set X, there exists a set of degree hyperimmune relative to X.*

This characterization was used by Hirschfeldt, Shore and Slaman [28, p. 5831] to conclude that WKL_0 does not imply OPT. It is of interest in light of Theorem 5.3.12 above, which links $F\mathsf{IP}$ with hyperimmune degrees. Specifically, by Remark 5.3.17, we have the following:

Corollary 5.3.21. $\overline{D}_2\mathsf{IP}$ *implies* OPT *over* RCA_0*.*

The next proposition and theorem provide a partial step towards the converse of this corollary.

Proposition 5.3.22. *Let $A = \langle A_i : i \in \mathbb{N} \rangle$ be a computable non-trivial family of sets. Every set D of degree hyperimmune relative to $\mathbf{0}'$ computes a maximal subfamily of A with the F intersection property.*

Proof. We may assume that A has no finite maximal subfamily with the F intersection property. And by deleting some of the members of A if necessary, we may further assume that $A_0 \neq \emptyset$. Define a $\emptyset'$-computable function $g\colon \mathbb{N} \to \mathbb{N}$ by letting $g(s)$ be the least y such that for all finite sets $F \subseteq \{0, \dots, s\}$,

$$\bigcap_{j \in F} A_j \neq \emptyset \Rightarrow (\exists x \leq y)[x \in \bigcap_{j \in F} A_j].$$

Since D has hyperimmune degree relative to $\mathbf{0}'$, we may fix a function $f \leq_T D$ not dominated by any $\emptyset'$-computable function. In particular, f is not dominated by g.

Now define $J \in \omega^\omega$ inductively as follows: let $J(0) = 0$, and having defined $J(s)$ for some $s \geq 0$, search for the least $i \leq s$ not yet in the range of J for which there exists an $x \leq f(s)$ with

$$x \in A_i \cap \bigcap_{j \leq s} A_{J(j)},$$

setting $J(s+1) = i$ if it exists, and setting $J(s+1) = 0$ otherwise.

Clearly, $J \leq_T f$. Moreover, $\bigcap_{i \leq s} A_{J(i)} \neq \emptyset$ for every s, so the subfamily defined by J has the F intersection property. We claim that for all i, if $A_i \cap \bigcap_{j \leq s} A_{J(j)} \neq \emptyset$ for every s then i is in the range of J. Suppose not, and let i be the least witness to this fact. Since f is not dominated by g, there exists an $s \geq i$ such that $f(s) \geq g(s)$ and for all $t \geq s$, $J(t) \neq j$ for any $j < i$. By construction, $J(j) \leq j$ for all j, so the set $F = \{i\} \cup \{J(j) : j \leq s\}$ is contained in $\{0, \dots, s\}$. Consequently, there necessarily exists some $x \leq g(s)$ with $x \in A_i \cap \bigcap_{j \leq s} A_{J(j)}$. But then $x \leq f(s)$, so $J(s+1)$ is defined to be i, which is a contradiction. We conclude that $\langle A_{J(i)} : i \in \omega \rangle$ is maximal, as desired. □

Theorem 5.3.23. *Let $A = \langle A_i : i \in \mathbb{N} \rangle$ be a computable non-trivial family of sets. Every non-computable c.e. set W computes a maximal subfamily of A with the F intersection property.*

Proof. As above, assume that A has no finite maximal subfamily with the F intersection property, and that $A_0 \neq \emptyset$. Fix a computable enumeration of W, denoting by W_s the set of elements enumerated into W by the end of stage s. We construct a limit computable set M by permitting, denoting by M_s the approximation to it at stage s of the construction. For each i and each n, call $\langle i, n \rangle$ a *copy* of i.

Construction.

Stage 0. Enumerate $\langle 0, 0 \rangle$ into $M[0]$.

Stage $s+1$. Assume that M_s has been defined, that it is finite and contains $\langle 0,0\rangle$, and that each i has at most one copy in M_s. For each i with no copy in M_s, let $\ell(i,s)$ be the greatest k with a copy in M_s, if it exists, such that there is an $x \leq s$ that belongs to A_i and to A_j for every $j \leq k$ with a copy in M_s.

Now consider all $i \leq s$ such that the following hold:

- $\ell(i,s)$ is defined;
- there is no j with a copy in M_s such that $\ell(i,s) < j < i$;
- for each $\langle j,n\rangle \in M_s$, if $\ell(i,s) < j$ then $W_s \upharpoonright \langle j,n\rangle \neq W_{s+1} \upharpoonright \langle j,n\rangle$.

If there is no such i, let $M_{s+1} = M_s$. Otherwise, fix the least such i, and let $M[s+1]$ be the result of removing from M_s all $\langle j,n\rangle > \ell(i,s)$, and then enumerating into it the least copy of i greater than every element of M_s and $W_{s+1} - W_s$.

End construction.

Verification. For every m, if $M_s(m) \neq M_{s+1}(m)$ then $W_s \upharpoonright m \neq W[s+1] \upharpoonright m$. Therefore, $M(m) = \lim_s M_s(m)$ exists for all m and is computable from W. Furthermore, note that $\bigcap_{\langle i,n\rangle \in M_s} A_i \neq \emptyset$ for all s. Thus, if F is any finite subset of M, then $\bigcap_{\langle i,n\rangle \in F} A_i \neq \emptyset$ since F is necessarily a subset of M_s for some s. If we now let $J : \omega \to \omega$ be any W-computable function with range equal to $\{i : (\exists n)[\langle i,n\rangle \in M]\}$, it follows that $\langle A_{J(i)} : i \in \omega\rangle$ has the F intersection property.

We claim that this subfamily is also maximal. Seeking a contradiction, suppose not, and let i be the least witness to this fact. So $A_i \cap \bigcap_{\langle j,n\rangle \in F} A_j \neq \emptyset$ for every finite subset F of M, and no copy of i belongs to M. By construction, $\langle 0,0\rangle \in M_s$ for all s and hence also to M, so it must be that $i > 0$. Let $i_0,\dots,i_r$ be the numbers less than i that have copies in M, and let these copies be $\langle i_0,n_0\rangle,\dots,\langle i_r,n_r\rangle$, respectively. Let s be large enough so that:

- there is an $x \leq s$ with $x \in A_i \cap \bigcap_{j \leq n} A_{i_j}$;
- for all $t \geq s$ and all $j \leq n$, $\langle i_j,n_j\rangle \in M_t$.

Now for all $t \geq s$, $\ell(i,t)$ is defined, and its value must tend to infinity.

Note that no copy of i can be in M_t at any stage $t \geq s$. Otherwise, it would have to be removed at some later stage, which could only be done for the sake of enumerating a copy of some number $< i$. This, in turn, could not be a copy of any of $i_0,\dots,i_r$ by choice of s, and so it too would subsequently have to be removed. Continuing in this way would result in an infinite regress, which is impossible.

It follows that for each $t \geq s$ there is some $j > \ell(i,t)$ with a copy $\langle j,n\rangle$ in M_t. Let $\langle j_t,n_t\rangle$ be the least such copy at stage t. Then $\langle j_t,n_t\rangle \leq \langle j_{t+1},n_{t+1}\rangle$ for all t, since no $m < \langle j_t,n_t\rangle$ can be put into $M[t+1]$. Furthermore, for infinitely many t this inequality must be strict, since infinitely often $\ell(i,t+1) \geq j_t$.

Now fix any $t \geq s$ so that $\ell(i,u) \geq i$ for all $u \geq t$. Then for all $u \geq t$, $W_u \upharpoonright \langle j_t,n_t\rangle$ must be equal to $W[u+1] \upharpoonright \langle j_t,n_t\rangle$. If not, we would necessarily have $W_u \upharpoonright \langle j_u,n_u\rangle \neq W[u+1] \upharpoonright \langle j_u,n_u\rangle$, and hence $W_u \upharpoonright \langle j,n\rangle \neq W[u+1] \upharpoonright \langle j,n\rangle$ for every $\langle j,n\rangle \in M_u$ with $j > \ell(i,u)$. But then some copy of i would be enumerated into $M[u+1]$, which can not happen. We conclude that for all $u \geq t$, $W_u \upharpoonright \langle j_u, n_u \rangle = W \upharpoonright \langle j_u, n_u \rangle$. Thus, given any n, we can compute $W \upharpoonright n$ simply by searching for a $u \geq t$ with $\langle j_u, n_u \rangle \geq x$. This contradicts the assumption that W is non-computable. The proof is complete. □

The above is of special interest. Heuristically, one would expect to be able to adapt a permitting argument into one showing the same result but with "every non-computable c.e. set" replaced by "every hyperimmune set". For example, the proof in [28] that OPT is implied over RCA_0 by the existence of a set of hyperimmune degree is an adaptation of a permitting argument of Csima [8, Theorem 1.2]. The basic idea is to translate receiving permissions into escaping domination by computable functions. We take a given function f not dominated by any computable one, and for each i define a a computable function g_i so that receiving permission for the ith requirement in the permitting argument (such as putting a copy of i into M) corresponds to having $f(s) \geq g_i(s)$ for some s. But if we try to do this in the case of Theorem 5.3.23, we run into the problem of seemingly needing to know f in order to define g. Intuitively, we are trying to put A_i into our subfamily at stage s, and are letting $g_i(s)$ be so large that it bounds a witness to the intersection of A_i and all the members of A put in so far. Thus, the definition of $g_i(s)$ depends on which A_j have been put in at a stage $t < s$, i.e., on which j had $f(t) > g_j(t)$ for some $t < s$. In the permitting argument this information is computable, but here it is not. We do not know of a way of get past this difficulty, and thus leave open the question of whether OPT reverses to $F\mathsf{IP}$ (or $\overline{D}_2\mathsf{IP}$) over RCA_0.

We also do not know whether the weaker implication from AMT to $F\mathsf{IP}$ is provable in RCA_0. However, the next proposition shows that is provable in RCA_0 together with additional induction axioms. In particular, every ω-model of AMT is also a model of $F\mathsf{IP}$. Thus we have a firm connection between the model-theoretic principles AMT and OPT and the set-theoretic principles $F\mathsf{IP}$ and $\overline{D}_n\mathsf{IP}$. The following definition is due to Hirschfeldt, Shore, and Slaman [28, p. 5823]:

Π^0_1 **genericity principle** ($\Pi^0_1\mathsf{G}$). *For any uniformly Π^0_1 collection of sets D_i, each of which is dense in $2^{<\mathbb{N}}$, there exists a set G such that for every i, $G \upharpoonright s \in D_i$ for some s.*

It was shown in [28, Theorem 4.3, Corollary 4.5, and p. 5826] that $\Pi^0_1\mathsf{G}$ strictly implies AMT over RCA_0, and that AMT implies $\Pi^0_1\mathsf{G}$ over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. As discussed in the previous subsection, $\mathsf{RCA}_0 + \Pi^0_1\mathsf{G}$ is conservative over RCA_0 for restricted Π^1_2 sentences, and thus it does not implyWKL_0 over RCA_0.

Proposition 5.3.24. *$\Pi^0_1\mathsf{G}$ implies $F\mathsf{IP}$ over RCA_0.*

Proof. We argue in RCA_0. Let a non-trivial family $A = \langle A_i : i \in \mathbb{N} \rangle$ be given. We may assume A has no finite maximal subfamily with the F intersection property. Fix a bijection $c : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$. Given $\sigma \in 2^{<\mathbb{N}}$, we say that a number $x < |\sigma|$ is *good for σ* if:

- $\sigma(x) = 1$;
- $c(x) = \tau b$, which we call the *witness* of x, where:

– $\tau \in \mathbb{N}^{<\mathbb{N}}$,
– $b \in \mathbb{N}$,
– and there is a $y \leq b$ with $y \in \bigcap_{i<|\tau|} A_{\tau(i)}$.

We define the *good sequence* of σ to be either the empty string if there is no good number for σ, or else the longest sequence $x_0 \cdots x_n \in \mathbb{N}^{<\mathbb{N}}$, $n \geq 0$, such that:

- x_0 is the least good number for σ;
- each x_i is good, say with witness $\tau_i b_i$;
- for each $i < n$, x_{i+1} is the least good $x > x_i$ such that if τb is its witness then $\tau \succ \tau_i$.

Note that Σ^0_0 comprehension suffices to prove the existence of a function $2^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ which assigns to each $\sigma \in 2^{<\mathbb{N}}$ its good sequence.

Now for each $i \in \mathbb{N}$, let D_i be the set of all $\sigma \in 2^{<\mathbb{N}}$ that have a non-empty good sequence $x_0 \cdots x_n$, and if τb is the witness of x_n then:

- either $\tau(j) = i$ for some $j < |\tau|$,
- or $A_i \cap \bigcap_{j<|\tau|} A_{\tau(j)} = \emptyset$.

The D_i are clearly uniformly Π^0_1, and it is not difficult to see that they are dense in $2^{<\mathbb{N}}$. Indeed, let $\sigma \in 2^{<\mathbb{N}}$ be given, and define b, j, and x as follows: if the good sequence of σ is empty, choose the least $j \geq i$ such that $A_j \neq \emptyset$ and let $b \geq \min A_j$ be large enough that $x = c^{-1}(jb) \geq |\sigma|$; if the good sequence of σ is some non-empty string $x_0 \cdots x_n$ and τb_n is the witness of x_n, choose the least $j \geq i$ such that $A_j \cap \bigcap_{k<|\tau|} A_{\tau(k)} \neq \emptyset$ and let $b \geq \min A_j \cap \bigcap_{k<|\tau|} A_{\tau(k)}$ be large enough that $x = c^{-1}(\tau j b) \geq |\sigma|$. In either case, j exists because of our assumption that A is non-trivial and has no finite maximal subfamily with the F intersection property. Now define $\widetilde{\sigma} \in 2^{<\mathbb{N}}$ of length $x+1$ by

$$\widetilde{\sigma}(y) = \begin{cases} \sigma(y) & \text{if } y < |\sigma|, \\ 0 & \text{if } |\sigma| \leq y < x, \\ 1 & \text{if } y = x, \end{cases}$$

to get an extension of σ that belongs to D_i.

Apply $\Pi^0_1\mathsf{G}$ to the D_i to obtain a set G such that for all i, there is an s with $G \upharpoonright s \in D_i$. Note, that by definition, each such s must be non-zero, and $G \upharpoonright s$ must have a non-empty good sequence. Notice that if $s \leq t$ then the good sequence of $G \upharpoonright t$ extends (not necessarily properly) the good sequence of $G \upharpoonright s$. Furthermore, our assumption that A has no finite maximal subfamily with the F intersection property implies that the good sequences of the initial segments of G are arbitrarily long.

Now find the least s such that $G \upharpoonright s$ has a non-empty good sequence, and for each $t \geq s$, if $x_0 \cdots x_n$ is the good sequence of $G \upharpoonright t$, let $\tau_t b_t$ be the witness of x_n. By the preceding paragraph, we have $\tau_t \preceq \tau_{t+1}$ for all t, and $\lim_t |\tau_t| = \infty$. Let $J = \bigcup_{t \geq s} \tau_t$, which exists by Σ^0_0 comprehension. It is straightforward to check that $B = \langle A_{J(i)} : i \in \mathbb{N} \rangle$ is a maximal subfamily of A with the F intersection property. □

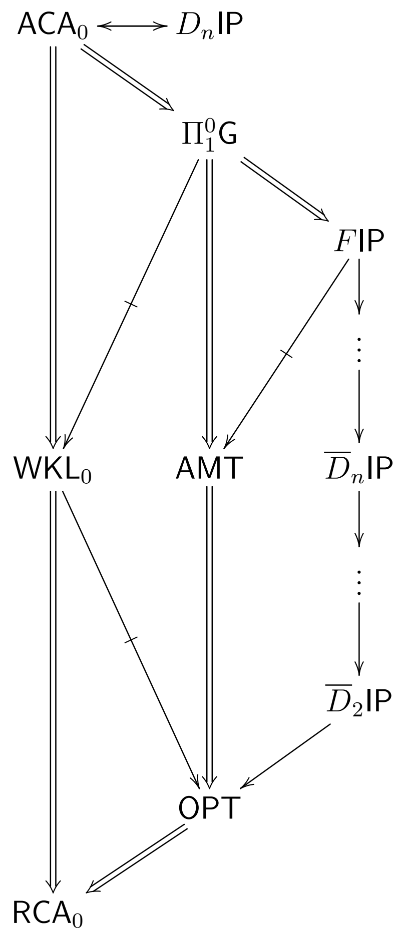

Figure 5.1: Relationship of the intersection principles to other weak principles, with $n \geq 2$ being arbitrary. Arrows denote implications provable in RCA_0, double arrows denote implications that are known to be strict, and negated arrows indicate non-implications.

We end this section with the result that $F\mathsf{IP}$ does not imply $\Pi^0_1\mathsf{G}$ or even AMT. Csima, Hirschfeldt, Knight, and Soare [9, Theorem 1.5] showed that for every set $D \leq_T \emptyset'$, if every complete atomic decidable theory has an atomic model computable from D, then D is non-low$_2$. Thus AMT can not hold in any ω-model all of whose sets have degree below a common low$_2$ Δ^0_2 degree. In conjunction with Theorem 5.3.23 (2), this fact allows us to separate $F\mathsf{IP}$ and AMT.

Corollary 5.3.25. *For every non-computable c.e. set W, there exists an ω-model $\mathscr{M}$ of $\mathsf{RCA}_0 + F\mathsf{IP}$ with $X \leq_T W$ for all $X \in \mathscr{M}$. Therefore $F\mathsf{IP}$ does not imply AMT over RCA_0.*

Proof. By Sacks's density theorem, there exist c.e. sets $\emptyset <_T W_0 <_T W_1 <_T \cdots < W$. Let $\mathscr{M}$ be the ω-model whose second-order part consists of all sets X such that $X \leq_T W_i$ for some i. For each i, Theorem 5.3.23 (2) relativized to W_i implies that every W_i-computable non-trivial family of sets has a W_{i+1}-computable maximal subfamily with the F intersection property. Thus, $\mathscr{M} \models F\mathsf{IP}$. The second part follows by building $\mathscr{M}$ with W low$_2$. $\square$

5.4 Properties of finite character

The last family of choice principles we study makes use of properties of finite character, sometimes in conjunction with finitary closure operators (see Definitions 5.4.9 and 5.4.15). We shall show that these principles are equivalent to well known subsystems of arithmetic, unlike the intersection principles of the last section.

Definition 5.4.1. A formula φ with one free set variable X is said to be of *finite character* (or have the *finite character property*) if $\varphi(\emptyset)$ holds and, for every set A, $\varphi(A)$ holds if and only if $\varphi(F)$ holds for every finite $F \subseteq A$.

The following basic facts are provable in ZF.

Proposition 5.4.2. *Let $\varphi(X)$ be a formula of finite character.*

1. *If $A \subseteq B$ and $\varphi(B)$ holds then $\varphi(A)$ holds.*
2. *If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a sequence of sets such that $\varphi(A_i)$ holds for each $i \in \omega$, then $\varphi(\bigcup_{i\in\omega} A_i)$ holds.*

We restrict our attention to formulas of second-order arithmetic, and consider countable analogues of several variants of the principle asserting that for every formula of finite character, every set has a maximal subset (under inclusion) satisfying that formula. Since the empty set satisfies any formula of finite character by definition, the validity of this principle can be seen by a simple application of Zorn's lemma.

The formalism here will be simpler than that in the previous section because we are dealing only with sets and their subsets, rather than with families of sets and their subfamilies. All the intersection properties studied in Section 5.3 can, in principle, be thought of as being defined by formulas of finite character. For example, given a family $A = \langle A_i : i \in \mathbb{N}\rangle$, the formula $(\forall i)(\forall j)[A_i \cap A_j \neq \emptyset]$ has the finite character property, and if $J = \{j_0 < j_1 < \cdots\}$ is a maximal subset of $\mathbb{N}$ satisfying it, then $\langle A_{j_i} : i \in \mathbb{N}\rangle$ is a maximal subfamily of A with the $\overline{D}_2$ intersection property. However, such an analysis of $\overline{D}_2\mathsf{IP}$ would be too crude in light of Proposition 5.3.7. Therefore, our focus in this section will instead be on formulas of finite character in general, and on the strengths of principles based on formulas of finite character from restricted syntactic classes.

5.4.1 The scheme FCP

We begin with various forms of the following principle.

Definition 5.4.3. The following scheme is defined in RCA_0.

Finite character principle (FCP). *For each formula φ of finite character, which may have arbitrary parameters, every set A has a $\subseteq$-maximal subset B such that $\varphi(B)$ holds.*

In set theory, FCP corresponds to the principle $\mathsf{M}\,7$ in the catalog of Rubin and Rubin [53], and is equivalent to the axiom of choice [53, p. 34 and Theorem 4.3].

In order to better gauge the reverse-mathematical strength of FCP, we consider restrictions of the formulas to which it applies. As with other such ramifications, we shall primarily be interested in restrictions to the classes in the arithmetical and analytical hierarchies. In particular, for each $i \in \{0, 1\}$ and $n \geq 0$, we make the following definitions:

- Σ^i_n-FCP is the restriction of FCP to Σ^i_n formulas;
- Π^i_n-FCP is the restriction of FCP to Π^i_n formulas;
- Δ^i_n-FCP is the scheme which says that for every Σ^i_n formula $\varphi(X)$ and every Π^i_n formula $\psi(X)$, if $\varphi(X)$ is of finite character and

$$(\forall X)[\varphi(X) \leftrightarrow \psi(X)],$$

  then every set A has a $\subseteq$-maximal set B such that $\varphi(B)$ holds.

We also define QF-FCP to be the restriction of FCP to the class of quantifier-free formulas without parameters.

Our first result in this section is the following theorem, which will allow us to neatly characterize most of the above restrictions of FCP (see Corollary 5.4.6). We draw attention to part (2) of the theorem, where Σ^0_1 does not appear in the list of classes of formulas. The reason behind this will be made apparent by Proposition 5.4.7.

Theorem 5.4.4. *For $i \in \{0, 1\}$ and $n \geq 1$, let Γ be any of Π^i_n, Σ^i_n, or Δ^i_n.*

1. *Γ-FCP is provable in Γ-CA_0.*
2. *If Γ is Π^0_n, Π^1_n, Σ^1_n, or Δ^1_n, then Γ-FCP implies Γ-CA_0 over RCA_0.*

We shall make use of the following technical lemma in the proof (as well as in the proof of Theorem 5.4.11 below). It is needed only because there are no term-forming operations for sets in L_2. For example, there is no term in L_2 that takes a set X and a canonical index n and returns $X \cup D_n$. (Recall that each finite (possibly empty) set of natural numbers is coded by a unique natural number known as its *canonical index*, and that D_n denotes the finite set with canonical index n.) The moral of the lemma is that such terms can be interpreted into L_2 in a natural way.

The coding of finite sets by their canonical indices can be formalized in RCA_0 in such a way that the predicate $i \in D_n$ is defined by a formula $\rho(i, n)$ with only bounded quantifiers, and such that the set of canonical indices is also definable by a bounded-quantifier formula[59, Theorem II.2.5]. Moreover, RCA_0 proves that every finite set has a canonical index. We use the notation $Y = D_n$ to abbreviate the formula $(\forall i)[i \in Y \leftrightarrow \rho(i, n)]$, along with similar notation for subsets of finite sets.

Lemma 5.4.5. *Let $\varphi(X)$ be a formula with one free set variable. There is a formula $\widehat{\varphi}(x)$ with one free number variable such that RCA_0 proves*

$$(\forall A)(\forall n)[A = D_n \to (\varphi(A) \leftrightarrow \widehat{\varphi}(n))]. \tag{5.1}$$

Moreover, we may take $\widehat{\varphi}$ to have the same complexities in the arithmetical and analytic hierarchies as φ.

Proof. Let $\rho(i, n)$ be the formula defining the relation $i \in D_n$, as discussed above. We may assume φ is written in prenex normal form. Form $\widehat{\varphi}(n)$ by replacing each occurrence $t \in X$ of φ, t a term, with the formula $\rho(t, n)$.

Let $\psi(X, \bar{Y}, \bar{m})$ be the quantifier-free matrix of φ, where $\bar{Y}$ and $\bar{m}$ are sequences of variables that are quantified in φ. Similarly, let $\widehat{\psi}(n, \bar{Y}, \bar{m})$ be the matrix of $\widehat{\varphi}$. Fix any model $\mathscr{M}$ of RCA_0 and fix $n, A \in \mathscr{M}$ such that $\mathscr{M} \models A = D_n$. A straightforward metainduction on the structure of ψ proves that

$$\mathscr{M} \models (\forall \bar{Y})(\forall \bar{m})[\psi(A, \bar{Y}, \bar{m}) \leftrightarrow \widehat{\psi}(n, \bar{Y}, \bar{m})].$$

The key point is that the atomic formulas in $\psi(A, \bar{Y}, \bar{m})$ are the same as those in $\widehat{\psi}(n, \bar{Y}, \bar{m})$, with the exception of formulas of the form $t \in A$, which have been replaced with the equivalent formulas of the form $\rho(t, n)$.

A second metainduction on the quantifier structure of φ shows that we may adjoin quantifiers to ψ and $\widehat{\psi}$ until we have obtained φ and $\widehat{\varphi}$, while maintaining logical equivalence. Thus every model of RCA_0 satisfies (5.1).

Because ρ has only bounded quantifiers, the substitution required to pass from φ to $\widehat{\varphi}$ does not change the complexity of the formula. $\square$

If F is any finite set and n is its canonical index, we sometimes write $\widehat{\varphi}(F)$ for $\widehat{\varphi}(n)$.

Proof of Theorem 5.4.4 . For (1), let $\varphi(X)$ and $A = \{a_i : i \in \mathbb{N}\}$ be an instance of Γ-FCP. Define $g\colon 2^{<\mathbb{N}} \times \mathbb{N} \to 2^{<\mathbb{N}}$ by

$$g(\tau, i) = \begin{cases} 1 & \text{if } \widehat{\varphi}(\{a_j : \tau(j) \downarrow 1\} \cup \{a_i\}) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

where $\widehat{\varphi}$ is as in the lemma, and for a finite set F, $\widehat{\varphi}(F)$ refers to $\widehat{\varphi}(n)$ where n is the canonical index of F. The function g exists by Γ comprehension. By primitive recursion, there exists a function $h\colon \mathbb{N} \to 2^{<\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $h(i) = 1$ if and only if $g(h \upharpoonright i, i) = 1$. For each $i \in \mathbb{N}$, let $B_i = \{a_j : j < i \wedge h(j) = 1\}$. An induction on φ shows that $\varphi(B_i)$ holds for every $i \in \mathbb{N}$.

Let $B = \{a_i : h(i) = 1\} = \bigcup_{i \in \mathbb{N}} B_i$. Because Proposition 5.4.2 is provable in ACA_0 and hence in Γ-CA_0, it follows that $\varphi(B)$ holds. By the same token, if $\varphi(B \cup \{a_k\})$ holds for some k then so must $\varphi(B_k \cup \{a_k\})$, and therefore $a_k \in B_{k+1}$, which means that $a_k \in B$. Therefore B is $\subseteq$-maximal, and we have shown that Γ-CA_0 proves Γ-FCP.

For (2), we assume Γ is one of Π^0_n, Π^1_n, or Σ^1_n; the proof for Δ^1_n is similar. We work in RCA_0+Γ-FCP. Let $\varphi(n)$ be a formula in Γ and let $\psi(X)$ be the formula $(\forall n)[n \in X \to \varphi(n)]$. It is easily seen that ψ is of finite character and belongs to Γ. By Γ-FCP, $\mathbb{N}$ contains a $\subseteq$-maximal subset B such that $\psi(B)$ holds. For any y, if $y \in B$ then $\varphi(y)$ holds. On the other hand, if $\varphi(y)$ holds then so does $\psi(B \cup \{y\})$, so y must belong to B by maximality. Therefore $B = \{y \in \mathbb{N} : \varphi(y)\}$, and we have shown that Γ-FCP implies Γ-CA_0. □

The corollary below summarizes the theorem as it applies to the various classes of formulas we are interested in. Of special note is part (5), which says that FCP itself (that is, FCP for arbitrary L_2-formulas) is as strong as any theorem of second-order arithmetic can be.

Corollary 5.4.6. *The following are provable in* RCA_0*:*

1. Δ^0_1-FCP, Σ^0_0-FCP, *and* QF-FCP*;*
2. *for each* $n \geq 1$, ACA_0 *is equivalent to* Π^0_n-FCP*;*
3. *for each* $n \geq 1$, Δ^1_n-CA_0 *is equivalent to* Δ^1_n-FCP*;*
4. *for each* $n \geq 1$, Π^1_n-CA_0 *is equivalent to* Π^1_n-FCP *and to* Σ^1_n-FCP*;*
5. Z_2 *is equivalent to* FCP*.*

The case of FCP for Σ^0_1 formulas is anomalous. The proof of part (2) of the theorem does not go through for Σ^0_1 because this class is not closed under universal quantification. As the proof of the next proposition shows, this limitation is quite significant. Intuitively, it means that a Σ^0_1 formula $\varphi(X)$ of finite character can only control a fixed finite piece of a set X. Hence, for the purposes of finding a maximal subset of which φ holds, we can replace φ by a formula with only bounded quantifiers.

Proposition 5.4.7. Σ^0_1-FCP *is provable in* RCA_0.

Proof. Let $\varphi(X)$ be a Σ^0_1 formula of finite character. We claim that there exists a finite subset F of $\mathbb{N}$ such that for every set A, if $F \cap A = \emptyset$ then $\varphi(A)$ holds. Let $\psi(X, x)$ be a bounded-quantifier formula such that $\varphi(X) \equiv (\exists x)\, \psi(X, x)$, and fix n such that $\psi(\emptyset, n)$ holds. Note that $\psi(X, n)$ is a bounded-quantifier formula with no free number variables. Any such formula is equivalent to a quantifier-free formula, because each quantifier will be bounded by a standard natural number. In turn, each quantifier-free formula can be written as a disjunction of conjunctions of atomic formulas and their negation. So we may assume $\psi(X, n)$ is in this form. Since $\psi(\emptyset, n)$ holds, there must be a disjunct $\theta(X)$ of $\psi(X, n)$ that holds of $\emptyset$. Clearly, $\theta(X)$ can not have a conjunct of the form $t \in X$, t a term. Therefore, if we let F be the set of all terms t such $t \notin X$ is a conjunct of $\theta(X)$, we see that $\theta(A)$ holds whenever $F \cap A = \emptyset$. This completes the proof of the claim.

Now fix any set A. By the claim, there is a finite set F such that $\varphi(A \setminus F)$ holds. By Σ^0_1 induction, there is such an F of smallest size. Then if $\varphi((A \setminus F) \cup \{a\})$ holds for some $a \in A$, it can not be that $a \in F$, as otherwise $F' = F \setminus \{a\}$ would be a strictly smaller finite set than F such that $\varphi(A \setminus F')$ holds. Thus it must be that $a \in A \setminus F$, and we conclude that $A \setminus F$ is a $\subseteq$-maximal subset of A of which φ holds. □

The above proof contains an implicit non-uniformity in the choice of F of smallest size. The following proposition shows that this non-uniformity is essential, by showing that a sequential form of Σ^0_1-FCP is a strictly stronger principle.

Proposition 5.4.8. *The following are equivalent over* RCA_0*:*

1. ACA_0*;*
2. *for every family* $A = \langle A_i : i \in \mathbb{N} \rangle$ *of sets, and every* Σ^0_1 *formula* $\varphi(X, x)$ *with one free set variable and one free number variable such that for all* $i \in \mathbb{N}$*, the formula* $\varphi(X, i)$ *is of finite character, there exists a family* $B = \langle B_i : i \in \mathbb{N} \rangle$ *of sets such that for all* i*,* B_i *is a* $\subseteq$*-maximal subset of* A_i *satisfying* $\varphi(X, i)$*.*

Proof. The forward implication follows by a straightforward modification of the proof of Theorem 5.4.4 . For the reversal, let a one-to-one function $f \colon \mathbb{N} \to \mathbb{N}$ be given. For each $i \in \mathbb{N}$, let $A_i = \{i\}$, and let $\varphi(X, x)$ be the formula

$$(\exists y)[x \in X \to f(y) = x].$$

Then, for each i, $\varphi(X, i)$ has the finite character property, and for every set S that contains i, $\varphi(S, i)$ holds if and only if $i \in \text{range}(f)$. Thus, if $B = \langle B_i : i \in \mathbb{N} \rangle$ is the subfamily obtained by applying part (2) to the family $A = \langle A_i : i \in \mathbb{N} \rangle$ and the formula $\varphi(X, x)$, then

$$i \in \text{range}(f) \iff B_i = \{i\} \iff i \in B_i.$$

It follows that the range of f exists. □

Note that the proposition fails for the class of bounded-quantifier formulas of finite character in place of the class of Σ^0_1 such formulas, since part (2) is then clearly provable in RCA_0. Thus, in spite of the similarity between the two classes suggested by the proof of Proposition 5.4.7, the two do not coincide.

5.4.2 Finitary closure operators

We can strengthen FCP by imposing additional requirements on the maximal set being constructed. In particular, we now consider requiring the maximal set to satisfy a finitary closure property as well as to satisfy a property of finite character.

Definition 5.4.9. A *finitary closure operator* is a set of pairs $\langle F, n \rangle$ in which F is (the canonical index for) a finite (possibly empty) subset of $\mathbb{N}$ and $n \in \mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *closed* under a finitary closure operator D, or D*-closed*, if for every $\langle F, n \rangle \in D$, if $F \subseteq A$ then $n \in A$.

Our definition of a closure operator is not the standard set-theoretic definition presented by Rubin and Rubin [53, Definition 6.3]. However, it is easy to see that for each operator of the one kind there is an operator of the other such that the same sets are closed under both. The above definition has the advantage of being readily formalizable in RCA_0.

The following fact expresses the monotonicity of finitary closure operators.

Proposition 5.4.10. *If D is a finitary closure operator and $A_0 \subseteq A_1 \subseteq A_2 \cdots$ is a sequence of sets such that each A_i is D-closed, then $\bigcup_{i \in \mathbb{N}} A_i$ is D-closed.*

The principle in the next definition is analogous to principle $\mathsf{AL}'3$ of Rubin and Rubin [53], which is equivalent to the axiom of choice by [53, p. 96, and Theorems 6.4 and 6.5].

Closure extension scheme (CE)**.** *If D is a finitary closure operator, φ is a formula of finite character, and A is any set, then every D-closed subset of A satisfying φ is contained in a maximal such subset.*

In the terminology of Rubin and Rubin [53], this is a "primed" statement, meaning that it asserts the existence not merely of a maximal subset of a given set, but the existence of a maximal *extension* of any given subset. Primed versions of all of the principles considered above can be formed, and can easily be seen to be equivalent to the unprimed ones. By contrast, CE has only a primed form. This is because if A is a set, φ is a formula of finite character, and D is a finitary closure operator, A need not have any D-closed subset of which φ holds. For example, suppose φ holds only of $\emptyset$, and D contains a pair of the form $\langle \emptyset, a \rangle$ for some $a \in A$.

This leads to the observation that the requirements in the CE scheme that the maximal set must both be D-closed and satisfy a property of finite character are, intuitively, in opposition to each other. Satisfying a finitary closure property is a positive requirement, in the sense that forming the closure of a set usually requires adding elements to the set. Satisfying a property of finite character can be seen as a negative requirement in light of part (1) of Proposition 5.4.2.

We consider restrictions of CE as we did restrictions of FCP above. By analogy, if Γ is a class of formulas, we use the notation Γ-CE to denote the restriction of CE to the formulas in Γ. We begin with the following analogue of Theorem 5.4.4 (1) from the previous subsection:

Theorem 5.4.11. *For $i \in \{0, 1\}$ and $n \geq 1$, let Γ be Π^i_n, Σ^i_n, or Δ^1_n. Then Γ-CE is provable in Γ-CA_0.*

Proof. We work in Γ-CA_0. Let φ be a formula of finite character in Γ, which may have parameters, and let D be a finitary closure operator. Let A be any set and let C be a D-closed subset of A such that $\varphi(C)$ holds.

For any $X \subseteq A$, let $\mathrm{cl}_D(X)$ denote the *D-closure* of X. That is, $\mathrm{cl}_D(X) = \bigcup_{i \in \mathbb{N}} X_i$, where $X_0 = X$ and for each $i \in \mathbb{N}$, X_{i+1} is the set of all $n \in \mathbb{N}$ such that either $n \in X_i$ or there is a finite set $F \subseteq X_i$ such that $\langle F, n \rangle \in D$. Because we take D to be a set, $\mathrm{cl}_D(X)$ can be defined using a Σ^0_1 formula with parameter D. Define a formula $\psi(\sigma, X)$ as

$$(\forall n)[(D_n \subseteq \mathrm{cl}_D(X \cup \{i : \sigma(i) = 1\})) \rightarrow \widehat{\varphi}(n)] \wedge \mathrm{cl}_D(X \cup \{i : \sigma(i) = 1\}) \subseteq A,$$

where $\widehat{\varphi}$ is as in Lemma 5.4.5. Note that ψ is arithmetical if Γ is Π^0_n or Σ^0_n, and is in Γ otherwise.

Define the function $f\colon \mathbb{N} \to \{0,1\}$ inductively such that $f(i) = 1$ if and only if $\psi(\{j < i : f(j) = 1\} \cup \{i\}, C)$ holds. The characterization of the complexity of ψ ensures that f can be constructed using Γ comprehension. Now let

$$B_i = \mathrm{cl}_D(C \cup \{j < i : f(j) = 1\})$$

for each $i \in \mathbb{N}$, and let $B = \bigcup_{i\in\mathbb{N}} B_i$. The construction of f ensures that $\varphi(B_i)$ implies $\varphi(B_{i+1})$ for all i, and we have assumed that φ holds of $B_0 = \mathrm{cl}_D(C) = C$. Therefore, an instance of induction shows that φ holds of B_i for all $i \in \mathbb{N}$, and thus also of B by Proposition 5.4.2. This also shows that $B \subseteq A$. Similarly, because each B_i is D-closed, the formalized version of Proposition 5.4.10 implies B is D-closed.

Finally, we check that B is a maximal D-closed extension of C in A of which φ holds. Suppose that for some $i \in A$, $B \cup \{i\}$ is D-closed and $\varphi(B \cup \{i\})$ holds. Then since $B_i \subseteq B$, we have $\mathrm{cl}_D(B_i \cup \{i\}) \subseteq B \cup \{i\}$. Thus $\varphi(F)$ holds for every finite subset F of $\mathrm{cl}_D(B_i \cup \{i\})$, so by definition $f(i) = 1$ and $B_{i+1} = \mathrm{cl}_D(B_i \cup \{i\})$. Here we are using the fact that for all sets X and all $a, b \in \mathbb{N}$, $\mathrm{cl}_D(X \cup \{a,b\}) = \mathrm{cl}_D(\mathrm{cl}_D(X \cup \{a\}) \cup \{b\})$. Since $B_{i+1} \subseteq B$, we conclude that $i \in B$, as desired. □

It follows that for most standard classes Γ, Γ-CE is equivalent to Γ-FCP. Indeed, for any class Γ we have that Γ-CE implies Γ-FCP, because any instance of the latter can be regarded as an instance of the former by adding an empty finitary closure operator. And if Γ is Π^0_n, Π^1_n, Σ^1_n, or Δ^1_n, then Γ-FCP is equivalent to Γ-CA_0 by Theorem 5.4.4 (2), and hence reverses to Γ-CE. Thus, in particular, parts (2)–(5) of Corollary 5.4.6 hold for CE in place of FCP, and the full scheme CE itself is equivalent to Z_2.

The proof of the preceding theorem does not work for $\Gamma = \Delta^0_1$, because then Γ-CA_0 is just RCA_0, and we need at least ACA_0 to prove the existence of the function f defined there (the formula $\psi(\sigma, X)$ being arithmetical at best). The next proposition shows that this can not be avoided, even for a class of considerably weaker formulas.

Proposition 5.4.12. QF-CE *implies* ACA_0 *over* RCA_0.

Proof. Assume a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ is given. Let $\varphi(X)$ be the quantifier-free formula $0 \notin X$, which trivially has finite character, and let $\langle p_i : i \in \mathbb{N}\rangle$ be an enumeration of all primes. Let D be the finitary closure operator consisting, for all $i, n \in \mathbb{N}$, of all pairs of the following form:

- $\langle \{p_i^{n+1}\}, p_i^{n+2}\rangle$;
- $\langle \{p_i^{n+2}\}, p_i^{n+1}\rangle$;
- $\langle \{p_i^{n+1}\}, 0\rangle$, if $f(n) = i$.

Notice that D exists by Δ^0_1 comprehension relative to f and our enumeration of primes.

Note that $\emptyset$ is a D-closed subset of $\mathbb{N}$ and $\varphi(\emptyset)$ holds. Thus, we may apply CE for quantifier-free formulas to obtain a maximal D-closed subset B of $\mathbb{N}$ such that $\varphi(B)$ holds. Then by definition of D, for every $i \in \mathbb{N}$, B either contains every positive power of p_i or no positive power. Now if $f(n) = i$ for some n, then no positive power of p can be in B, since otherwise p^{n+1} would necessarily be in B and hence so would 0. On the other hand, if $f(n) \neq i$ for all n then $B \cup \{p_i^{n+1} : n \in \mathbb{N}\}$ is D-closed and satisfies φ, so by maximality p_i^{n+1} must belong to B for every n. It follows that $i \in \text{range}(f)$ if and only if $p_i \in B$, so the range of f exists. □

Thus we are able to separate CE from FCP at least in terms of some of their strictest restrictions. In contrast to Corollary 5.4.6 (1) and Proposition 5.4.7, we consequently have:

Corollary 5.4.13. *The following are equivalent over* RCA_0*:*

1. ACA_0;
2. Σ^0_1-CE;
3. Σ^0_0-CE;
4. QF-CE.

We conclude this subsection with one additional illustration of how formulas of finite character can be used in conjunction with finitary closure operators. Recall the following concepts from order theory:

- a *countable join-semilattice* is a countable poset $\langle L, \leq_L \rangle$ with a maximal element 1_L and an operation $\vee_L \colon L \times L \to L$ such that for all $a, b \in L$, $a \vee_L b$, called the *join* of a and b, is the least upper bound of a and b;
- an *ideal* on a countable join-semilattice L is a subset I of L that is downward closed under $\leq_L$ and closed under $\vee_L$.

The principle in the next proposition is the countable analogue of a variant of $\mathsf{AL}'\,1$ in Rubin and Rubin [53]; compare with Proposition 5.4.17 below. For more on the computability theory of ideals on lattices, see Turlington [67].

Proposition 5.4.14. *Over* RCA_0, QF-CE *implies that every proper ideal on a countable join-semilattice extends to a maximal proper ideal.*

Proof. Let L be a countable join-semilattice. Let φ be the formula $1 \notin X$, and let D be the finitary closure operator consisting of all pairs of the following form:

- $\langle \{a, b\}, c \rangle$ where $a, b \in L$ and $c = a \vee b$;
- $\langle \{a\}, b \rangle$, where $b \leq_L a$.

Because we define a join-semilattice to come with both the order relation and the join operation, the set D is Δ^0_0 with parameters, so RCA_0 proves D exists. It is immediate that a set X is closed under D if and only if X is an ideal in L. □

5.4.3 Non-deterministic finitary closure operators

It appears that the underlying reason that the restriction of CE to arithmetical formulas is provable in ACA_0 (and more generally, why Γ-CE is provable in Γ-CA_0 if Γ is as in Theorem 5.4.11) is that our definition of finitary closure operator is very constraining. Intuitively, if D is such an operator and φ is an arithmetical formula, and we seek to extend some D-closed subset B satisfying φ to a maximal such subset, we can focus largely on ensuring that φ holds. Achieving closure under D is relatively straightforward, because at each stage we only need to search through all finite subsets F of our current extension, and then adjoin all n such that $\langle F, n\rangle \in D$. This closure process becomes far less trivial if we are given a choice of which elements to add. We now consider the case when each finite subset F can be associated with a possibly infinite set of numbers from which we must choose at least one to adjoin. We shall show that this weaker notion of closure operator leads to a stronger analogue of CE.

Definition 5.4.15. A *non-deterministic finitary closure operator* is a sequence of sets of the form $\langle F, S\rangle$ where F is (the canonical index for) a finite (possibly empty) subset of $\mathbb{N}$ and S is a non-empty subset of $\mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *closed* under a non-determnistic finitary closure operator N, or N-closed, if for each $\langle F, S\rangle$ in N, if $F \subseteq A$ then $A \cap S \neq \emptyset$.

Note that if D is a *deterministic* finitary closure operator, that is, a finitary closure operator in the stronger sense of the previous subsection, then for any set A there is a unique $\subseteq$-minimal D-closed set extending A. This is not true for non-determnistic finitary closure operators. Let N be the operator such that $\langle \emptyset, \mathbb{N}\rangle \in N$ and, for each $i \in \mathbb{N}$ and each $j > i$, $\langle \{i\}, \{j\}\rangle \in N$. Then any N-closed set extending $\emptyset$ will be of the form $\{i \in \mathbb{N} : i \geq k\}$ for some k, and any set of this form is N-closed. Thus there is no $\subseteq$-minimal N-closed set.

In this subsection we study the following non-determnistic version of CE:

Non-deterministic closure extension scheme (NCE)**.** *If N is a non-determnistic closure operator, φ is a formula of finite character, and A is any set, then every N-closed subset of A satisfying φ is contained in a maximal such subset.*

Restrictions of NCE to various syntactical classes of formulas are defined as for CE and FCP. Note that, because the union of a chain of N-closed sets is again N-closed, NCE can be proved in set theory using Zorn's lemma.

Remark 5.4.16. We might expect to be able to prove NCE from CE by suitably transforming a given non-determnistic finitary closure operator N into a deterministic one. For instance, we could go through the members of N one by one, and for each such member $\langle F, S\rangle$ add $\langle F, n\rangle$ to D for some $n \in S$ (e.g., the least n). All D-closed sets would then indeed be N-closed. The converse, however, would not necessarily be true, because a set could have F as a subset for some $\langle F, S\rangle \in N$, yet it could contain a different $n \in S$ than the one chosen in defining D. In particular, a maximal D-closed subset (of some given set) would not need to be maximal among N-closed subsets.

The next result provides a simple but concrete example of this point. Recall that an *ideal* on a countable poset $\langle P, \leq_P \rangle$ is a subset I of P downward closed under $\leq_P$ and such that for all $p, q \in I$ there is an $r \in I$ with $p \leq_P r$ and $q \leq_P r$. The next proposition is similar to Proposition 5.4.14 above, which dealt with ideals on countable join-semilattices. In the proof of that proposition, we defined a deterministic finitary closure operator D in such a way that D-closed sets were closed under the join operation. For this we relied on the fact that for every two elements in the semilattice there is a unique element that is their join. The reason we need non-determnistic finitary closure operators below is that, for ideals on countable posets, there are no longer unique elements witnessing closure under the relevant operations.

Proposition 5.4.17. *Over* RCA_0, Π^0_2-NCE *implies that every ideal on a countable poset can be extended to a maximal ideal.*

Proof. We work in RCA_0. Let $\langle P, \leq_P \rangle$ be a countable poset. Without loss of generality we may assume $P = \{p_i : i \in \mathbb{N}\}$ is infinite. We form a non-determnistic closure operator $N = \langle N_i : i \in \mathbb{N} \rangle$ by considering the following two cases: for each $i \in \mathbb{N}$,

- if $i = 2\langle j, k \rangle$ and $p_j \leq_P p_k$, let $N_i = \langle \{p_k\}, \{p_j\} \rangle$;
- if $i = 2\langle j, k, l \rangle + 1$ and $p_j \leq_P p_l$ and $p_k \leq_P p_l$, let

$$N_i = \langle \{p_j, p_k\}, \{p_n : (p_j \leq_P p_n) \wedge (p_k \leq_P p_n)\} \rangle;$$

- otherwise, let $N_i = \langle \{p_i\}, \{p_i\} \rangle$.

This construction gives a quantifier-free definition of each N_i uniformly in i, so the sequence N exists.

Let $\varphi(X)$ be the Π^0_2 formula which says that every pair of elements in X has a common upper bound in P. A straightforward proof shows that φ is of finite character and that a set $I \subseteq P$ is an ideal on P if and only if I is N-closed and $\varphi(I)$ holds. □

Mummert [50, Theorem 2.4] showed that the proposition that every ideal on a countable poset extends to a maximal ideal is equivalent to Π^1_1-CA_0 over RCA_0. Hence, Π^0_2-NCE implies Π^1_1-CA_0. By Theorem 5.4.11, Π^0_2-CE is provable in ACA_0, so we see that the idea of Remark 5.4.16 fundamentally can not work.

We shall obtain the reversal of Π^0_2-NCE to Π^1_1-CA_0 in a sharper form in Theorem 5.4.19 below. First, we establish an upper bound. The proof uses a technique involving countable coded β-models, parallel to Lemma 2.4 of Mummert [50]. In RCA_0, a *countable coded β-model* is defined as a sequence $\mathscr{M} = \langle M_i : i \in \mathbb{N} \rangle$ of subsets of $\mathbb{N}$ such that for every Σ^1_1 formula φ with parameters from $\mathscr{M}$, φ holds if and only if $\mathscr{M} \models \varphi$ [59, Definitions VII.2.1 and VII.2.3]. A general treatment of countable coded β-models is given by Simpson [59, Section VII.2].

Proposition 5.4.18. Σ^1_1-NCE *is provable in* Π^1_1-CA_0.

Proof. We work in Π^1_1-CA_0. Let φ be a Σ^1_1 formula of finite character (possibly with parameters) and let N be a non-determnistic closure operator. Let A be any set and let C be an N-closed subset of A such that $\varphi(C)$ holds.

Let $\mathscr{M} = \langle M_i : i \in \mathbb{N}\rangle$ be a countable coded β-model containing A, B, N, and any parameters of φ, which exists by [59, Theorem VII.2.10]. Using Π^1_1 comprehension, we may form the set $\{i : \mathscr{M} \models \varphi(M_i)\}$.

Working outside $\mathscr{M}$, we build an increasing sequence $\langle B_i : i \in \mathbb{N}\rangle$ of N-closed extensions of C. Let $B_0 = C$. Given i, ask whether there is a j such that:

- M_j is an N-closed subset of A;
- $B_i \subseteq M_j$;
- $i \in M_j$;
- and $\varphi(M_j)$ holds.

If there is, choose the least such j and let $B_{i+1} = M_j$. Otherwise, let $B_{i+1} = B_i$. Finally, let $B = \bigcup_{i\in\mathbb{N}} B_i$.

Because the inductive construction only asks arithmetical questions about $\mathscr{M}$, it can be carried out in Π^1_1-CA_0, and so Π^1_1-CA_0 proves that B exists. Clearly $C \subseteq B \subseteq A$. An arithmetical induction shows that for all $i \in \mathbb{N}$, $\varphi(B_i)$ holds and B_i is N-closed. Therefore, the formalized version of Proposition 5.4.2 shows that $\varphi(B)$ holds, and the analogue of Proposition 5.4.10 to non-determnistic finitary closure operators shows that B is N-closed.

Now suppose that for some $i \in A$, $B \cup \{i\}$ is an N-closed subset of A extending C and satisfying φ. Because φ is Σ^1_1, and because N is a sequence, the property

$$(\exists X)[X \text{ is } N\text{-closed} \wedge B_i \subseteq X \subseteq A \wedge i \in X \wedge \varphi(X)]$$

is expressible by a Σ^1_1 sentence, and $B \cup \{i\}$ witnesses that it is true. Thus, because $\mathscr{M}$ is a β-model, this sentence must be satisfied by $\mathscr{M}$, which means that some M_j must also witness it. The inductive construction must therefore have selected such an M_j to be B_{i+1}, which means $i \in B_{i+1}$ and hence $i \in B$. It follows that B is maximal. □

The next theorem shows that NCE for quantifier-free formulas without parameters is already as strong as Σ^1_1-FCP and Σ^1_1-CE. In particular, in view of Corollary 5.4.13, it is considerably stronger than CE for quantifier-free formulas.

Theorem 5.4.19. *For each $n \geq 1$, the following are equivalent over RCA_0:*

1. Π^1_1-CA_0;
2. Σ^1_1-NCE;
3. Σ^0_n-NCE;
4. QF-NCE.

Proof. We have already proved (1) implies (2), and it is obvious that (2) implies (3) and (3) implies (4). The reversal of (4) to (1) splits into two steps.

For the first step, note that RCA_0 can convert any finitary closure operator D into a corresponding non-determnistic closure operator N such that the notions of D-closed and N-closed coincide (note that this is the opposite of what was discussed in Remark 5.4.16). Therefore NCE for quantifier-free formulas implies ACA_0 over RCA_0 by Proposition 5.4.12.

Next, for the second step, we work in ACA_0. Let $\langle T_i : i \in \mathbb{N}\rangle$ be a sequence of subtrees of $\mathbb{N}^{<\mathbb{N}}$. To prove Π^1_1-CA_0, it is sufficient to form the set of $i \in \mathbb{N}$ such that T_i has an infinite path [59, Lemma VI.1.1]. Let A be the set of all pairs $\langle i, \sigma\rangle$ such that $\sigma \in T_i$, along with one distinguished element z that is not a pair. Let $\varphi(X)$ be the formula $z \notin X$, which has no parameters provided that z is coded by a standard natural number. Clearly, φ has the finite character property.

Write $A - \{z\} = \{a_i : i \in \mathbb{N}\}$, and define a non-determnistic finitary closure operator $N = \langle N_i : i \in \mathbb{N}\rangle$ as follows: for each $j \in \mathbb{N}$, if $a_j = \langle i, \sigma\rangle$, then

- if σ is a dead end in T_i, let $N_j = \langle \{\langle i, \sigma\rangle\}, \{z\}\rangle$;
- if σ is not a dead end in T_i, let

$$N_j = \langle \{\langle i, \sigma\rangle\}, \{\langle i, \tau\rangle : \tau \in T_i \wedge \tau \succ \sigma \wedge |\tau| = |\sigma| + 1\}\rangle.$$

Notice that N can be formed by arithmetical comprehension.

Suppose B is an N-closed subset of A that satisfies φ (i.e., does not contain z). Then, for any i, whenever $\langle i, \sigma\rangle$ is in B there is some immediate extension τ of σ in T_i such that $\langle i, \tau\rangle$ is in B. Thus if $\langle i, \sigma\rangle$ is in B then there is an infinite path through T_i extending σ. In particular, if $\langle i, \emptyset\rangle$ is in B then T_i has an infinite path. Conversely, if f is an infinite path through T_i, then $B \cup \{\langle i, f \upharpoonright n\rangle : n \in \mathbb{N}\}$ is N-closed and satisfies φ.

Because $\emptyset$ is N-closed and satisfies φ, we may apply NCE for quantifier-free formulas to get a maximal extension of it within A. By the previous paragraph and the maximality of B, T_i has an infinite path if and only if $\langle i, \emptyset\rangle \in B$. Thus, the set of i such that T_i has an infinite path exists, as desired. □

Our final results characterize the strength of NCE for formulas higher in the analytical hierarchy.

Proposition 5.4.20. *For each $n \geq 1$,*

1. *Σ^1_n-NCE and Π^1_n-NCE are provable in Π^1_n-CA_0;*
2. *Δ^1_n-NCE is provable in Δ^1_n-CA_0.*

Proof. We prove part (1), the proof of part (2) being similar. Let $\varphi(X)$ be a Σ^1_n formula of finite character, respectively a Π^1_n such formula. Let N be a non-determnistic closure operator, let A be any set, and let C be an N-closed subset of A such that $\varphi(C)$ holds.

By Lemma 4.5, let $\widehat{\varphi}$ be a Σ^1_n formula, respectively a Π^1_n formula, such that

$$(\forall X)(\forall n)[X = D_n \to (\varphi(X) \leftrightarrow \widehat{\varphi}(n))].$$

We may use Π^1_n comprehension to form the set $W = \{n : \widehat{\varphi}(n)\}$. Define $\psi(X)$ to be the arithmetical formula $(\forall n)[D_n \subseteq X \to n \in W]$.

We claim that for every set X, $\psi(X)$ holds if and only if $\varphi(X)$ holds. The definitions of W and ψ ensure that $\psi(X)$ holds if and only if $\varphi(D_n)$ holds for every finite $D_n \subseteq X$, which is true if and only if $\varphi(X)$ holds because φ has finite character. This establishes the claim.

By the claim, ψ is a property of finite character and $\psi(C)$ holds. Using Σ^1_1-NCE, which is provable in Π^1_1-CA_0 by Proposition 5.4.18 and thus in Π^1_n-CA_0, there is a maximal N-closed subset B of A extending C with property ψ. Again by the claim, B is a maximal N-closed subset of A extending B with property φ. □

Corollary 5.4.21. *The following are provable in* RCA_0*:*

1. *for each* $n \geq 1$*,* Δ^1_n-CA_0 *is equivalent to* Δ^1_n-NCE*;*
2. *for each* $n \geq 1$*,* Π^1_n-CA_0 *is equivalent to* Π^1_n-NCE *and to* Σ^1_n-NCE*;*
3. Z_2 *is equivelent to* NCE*.*

Proof. The implications from Δ^1_n-CA_0, Π^1_n-CA_0, and Z_2 follow by Proposition 5.4.20. On the other hand, each restriction of NCE trivially implies the corresponding restriction of FCP, so the reversals follow by Corollary 5.4.6. □

Remark 5.4.22. The characterizations in this section shed light on the role of the closure operator in the principles CE and NCE. For $n \geq 1$, we have shown that Σ^1_n-FCP, Σ^1_n-CE, and Σ^1_n-NCE are all equivalent over RCA_0. However, QF-FCP is provable in RCA_0, QF-CE is equvalent to ACA_0 over RCA_0, and QF-NCE is equivalent to Π^1_1-CA_0 over RCA_0. Thus the closure operators in the stronger principles serve as a sort of replacement for arithmetical quantification in the case of CE, and for Σ^1_1 quantification in the case of NCE. This allows these principles to have greater strength than may be suggested by the property of finite character alone. At higher levels of the analytical hierarchy, the principles become equivalent because the complexity of the property of finite character overtakes the complexity of the closure notions.

5.5 Questions

In this section we summarize the principal questions left over from our investigation. These concern the precise strength of $F\mathsf{IP}$ and the principles $\overline{D}_n\mathsf{IP}$. While we have closely located these principles' position in the structure of statements lying between RCA_0 and ACA_0, we do not know the answers to the following questions:

Question 5.5.1. Does $\overline{D}_2\mathsf{IP}$ imply $F\mathsf{IP}$ over RCA_0? Does $\overline{D}_n\mathsf{IP}$ imply $\overline{D}_{n+1}\mathsf{IP}$?

Question 5.5.2. Does AMT imply $\overline{D}_2\mathsf{IP}$ over RCA_0? Does OPT imply $\overline{D}_2\mathsf{IP}$?

By Proposition 5.3.24, the first part of the Question 5.5.2 has an affirmative answer over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. For the second part, it may be easier to ask whether the implication can at least be shown to hold in ω-models. An affirmative answer would likely follow from an affirmative answer to the following question:

Question 5.5.3. Given a computable non-trivial family A, does every set of hyperimmune degree compute a maximal subfamily of A with the F intersection property (or at least with the $\overline{D}_2$ intersection property)?

We conjecture the answer to be no.

Our final question is less directly related to our investigation. We mention it in view of Proposition 5.4.14 above.

Question 5.5.4. What is the strength of the principle asserting that every proper ideal on a countable join-semilattice extends to a maximal proper ideal?

This question is further motivated by work of Turlington [67, Theorem 2.4.11] on the similar problem of constructing prime ideals on computble lattices. However, because a maximal ideal on a countable lattice need not be a prime ideal, Turlington's results do not directly resolve our question.

CHAPTER 6
INFINITE SATURATED ORDERS

6.1 Introduction

Saturated orders (see Definition 6.1.2) were introduced by Suck in [63] as a generalization of interval orders (see Definition 6.1.1). The latter, developed by Fishburn (see [22]), have been used extensively in the theory of measurement, utility theory, and various areas of psychophysics and mathematical psychology (see [22], Chapter 2, for examples). Suck applied the concept of saturated orders to the theory of knowledge spaces, as introduced by Doignon and Falmagne (see [11]), but he formulated it for finite orders only. Since the study of knowledge spaces in general need not be restricted to finite structures, it is natural to ask whether the concept of saturation can be formulated for partial orders of arbitrary cardinality.

In this chapter, we give such a formulation and show it to be equivalent to a certain algebraic characterization of partial orders. We then look at the proof theoretic strength of this equivalence within the framework of reverse mathematics. This answers questions of Suck raised at the *Reverse Mathematics: Foundations and Applications* workshop at the University of Chicago in November 2009. Beyond an interest in the underlying combinatorial principles, the motivation for this kind of analysis comes from seeking a possible new basis by which to judge and compare competing quantitative approaches to problems in cognitive science. The exploration of this interaction was one of the goals of the Chicago workshop.

Definition 6.1.1. Let $\mathbf{P} = (P, \leq_P)$ be a partial order.

1. An *interval representation* of $\mathbf{P}$ is a map f from P into the set of open intervals of some linear order $\mathbf{L} = (L, \leq_L)$ such that for all $p, p' \in P$, $p <_P p'$ if and only if $\ell <_L \ell'$ for all $\ell \in f(p)$ and $\ell' \in f(p')$.

2. $\mathbf{P}$ is an *interval order* if it admits an interval representation.

Definition 6.1.2 ([63], Definitions 1 and 3)**.** Let $\mathbf{P} = (P, \leq_P)$ be a finite partial order.

1. A *set representation* of $\mathbf{P}$ is an injective map $\varphi : P \to \mathcal{P}(Q)$ for some set Q such that $p <_P p'$ if and only if $\varphi(p) \subsetneq \varphi(p')$ for all $p, p' \in P$.

2. A set representation φ of $\mathbf{P}$ is *parsimonious* if $|\varphi(p)| = \left|\bigcup_{p' <_P p} \varphi(p')\right| + 1$ for all $p \in P$.

3. $\mathbf{P}$ is *saturated* if $|P| = \left|\bigcup_{p \in P} \varphi(p)\right|$ for all parsimonious set representations φ of $\mathbf{P}$.

Every finite partial order $\mathbf{P} = (P, \leq_P)$ has at least one parsimonious set representation, namely $\pi : P \to \mathcal{P}(P)$ where $\pi(p) = \{p' \in P : p' \leq_P p\}$ for all $p \in P$. Suck [63, Definition 2] calls this the *principal ideal representation* of $\mathbf{P}$. The notion of saturation arose as a means of characterizing finite orders for which this is essentially the only parsimonious set representation ([63], p. 375). Indeed, suppose $\varphi : P \to \mathcal{P}(Q)$ is parsimonious, and let $\alpha_\varphi : P \to Q$ be defined by setting $\alpha_\varphi(p)$ for each $p \in P$ to be the single element of $\varphi(p) - \bigcup_{p' <_P p} \varphi(p')$. If $\mathbf{P}$ is saturated then α_φ must be a bijection between P and $\bigcup_{p \in P} \varphi(p)$. Let $\leq_Q$ be an ordering of the latter set defined by setting $q \leq_Q q'$ for each $q, q' \in \bigcup_{p \in P} \varphi(p)$ if and only if $q = \alpha_\varphi(p)$ and $q' = \alpha_\varphi(p')$ for some $p, p' \in P$ with $p \leq_P p'$. Then α_φ is an isomorphism of $\mathbf{P}$ with $(\bigcup_{p \in P} \varphi(p), \leq_Q)$, and $\varphi(p) = \alpha_\varphi(\pi(p))$ for all $p \in P$. Thus, up to a renaming of elements, φ and π are the same representation.

A simple example of an order that is not saturated is one of type $\mathbf{2} \oplus \mathbf{2}$, i.e., one isomorphic to $(\{a, b, c, d\}, \leq)$ where $a \leq b$, $c \leq d$, $a \not\leq d$ and $c \not\leq b$ (consider the parsimonious set representation that maps a and c to $\{0\}$ and $\{1\}$ respectively, and b and d to $\{0, 2\}$ and $\{1, 2\}$ respectively). This order is also not an interval order:

Theorem 6.1.3 (Fishburn [21], p. 147; Mirkin [47])**.** *A partial order is an interval order if and only if it does not contain a suborder of type* $\mathbf{2} \oplus \mathbf{2}$.

Suck [63, Theorem 2] extended this observation by showing that every finite interval order is a saturated order. The converse, however, fails, as it is easy to build a saturated order which has a suborder of type $\mathbf{2} \oplus \mathbf{2}$. (See Figure 6.1, which also illustrates that saturated orders are not, contrary to interval orders, closed under restrictions of the domain).

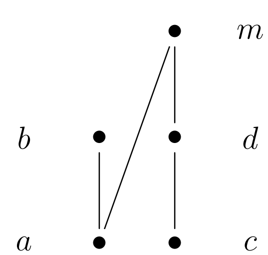

Figure 6.1: A saturated order which has a suborder of type $\mathbf{2} \oplus \mathbf{2}$ and hence is not an interval order.

If one recasts the condition of not containing a suborder of type $\mathbf{2} \oplus \mathbf{2}$ as

$$(\forall p_a, p_b, p_c, p_d \in P)[p_a \not<_P p_b \vee p_c \not<_P p_d \vee p_a \leq_P p_d \vee p_c \leq_P p_b],$$

then the next definition and theorem can be seen as providing an algebraic characterization of saturated orders similar to the characterization of interval orders provided by Theorem 6.1.3.

Definition 6.1.4 ([64], Definitions 5 and 6)**.** Let $\mathbf{P} = (P, \leq_P)$ be a finite partial order.

1. A *fan* in $\mathbf{P}$ is a subset F of P with at least two elements such that $\max F$ exists under $\leq_P$ and such that no elements of $F - \{\max F\}$ are pairwise $\leq_P$-comparable.

2. Two fans F_0 and F_1 in $\mathbf{P}$ are *parallel* if no element of F_0 is $\leq_P$-comparable with any element of F_1.

3. Two parallel fans F_0 and F_1 in $\mathbf{P}$ are *skewly topped* if there exists some $m \in P$ and some $i \in \{0, 1\}$ such that the following hold:

   (a) $m \geq_P \max F_i$;

   (b) $m \not\geq_P \max F_{1-i}$;

   (c) and $m \geq_P p$ for all $p \in F_{1-i} - \{\max F_{1-i}\}$.

Theorem 6.1.5 (Suck [64], Theorem 5)**.** *A finite partial order is saturated if and only if every two parallel fans in it are skewly topped.*

We can now state the questions of Suck mentioned above.

Question 6.1.6 (Suck)**.**

1. Does (some suitable analogue of) Theorem 6.1.5 hold for infinite partial orders?

2. If so, what are the set theoretic axioms necessary to carry out its proof?

The second part of the question is inspired by the work of Marcone [45], who investigated the reverse-mathematical content of Theorem 6.1.3. We refer the reader to Section 6.3 for a brief introduction to reverse mathematics, and Simpson [59] for a complete reference. In the next section we give an affirmative answer to part (1) of Question 6.1.6, and in Section 6.3 we consider possible answers to part (2).

6.2 The infinite case

In this section we formulate the concept of saturation for infinite partial orders and prove an analogue of Theorem 6.1.5. To begin, notice that set representations can be defined for infinite orders just as for finite ones. The other parts of Definition 6.1.2, however, need to be appropriately adjusted to the infinite setting.

Definition 6.2.1. Let $\mathbf{P} = (P, \leq_P)$ be a partial order.

1. A set representation $\varphi : P \to \mathcal{P}(Q)$ of $\mathbf{P}$ is *parsimonious* if for all $p \in P$

   (a) $\left|\varphi(p) - \bigcup_{p' <_P p} \varphi(p')\right| = 1$,

   (b) and for all $q \in \varphi(p)$, $\{q\} = \varphi(p') - \bigcup_{p'' <_P p'} \varphi(p'')$ for some $p' \leq_P p$.

2. Given a parsimonious set representation $\varphi : P \to \mathcal{P}(Q)$ of $\mathbf{P}$, define $\alpha_\varphi : P \to Q$ by $\alpha_\varphi(p) = q$ for $p \in P$ if and only if $\{q\} = \varphi(p) - \bigcup_{p' <_P p} \varphi(p')$.

3. $\mathbf{P}$ is *saturated* if and only if α_φ is injective for all parsimonious set representations φ of $\mathbf{P}$.

Part 1 (a) above is a straightforward modification of Definition 6.1.2 (2). Part 1 (b) is intended to express the idea that for a set representation $\varphi : P \to \mathcal{P}(Q)$ to be parsimonious, $\bigcup_{p\in P}\varphi(p)$ should comprise a minimal number of elements from Q.

It is not difficult to check that for finite partial orders the new definitions agree with the old:

Proposition 6.2.2. *Let* $\mathbf{P} = (P, \leq_P)$ *be a finite partial order.*

1. *A set representation of* $\mathbf{P}$ *is parsimonious according to Definition 6.1.2 if and only if it is parsimonious according to Definition 6.2.1.*

2. $\mathbf{P}$ *is saturated according to Definition 6.1.2 if and only if it is saturated according to Definition 6.2.1.*

In particular, the discussion following Definition 6.1.2 holds verbatim for infinite partial orders if parsimony and saturation are understood according to Definition 6.2.1. Thus infinite saturated orders admit essentially only one parsimonious set representation, and so the preceding definition does indeed capture the "spirit" of the concept.

We next generalize the notion of fans from Definition 6.1.4; we shall see at the end of the section why fans alone would not suffice.

Definition 6.2.3. A *bouquet* in $\mathbf{P}$ is a subset B of P with at least two elements such that $\max B$ exists under $\leq_P$. We define what it means for two bouquets to be *parallel* and *skewly topped* just as for fans.

If $\mathbf{P}$ is finite, or even just a partial order in which every element has only finitely many $\leq_P$-successors, then every two parallel bouquets B_0 and B_1 can be replaced by parallel fans F_0 and F_1 with the same respective maxima. Namely, let

$$F_i = \{b \in B_i : (\forall b' >_P b)[b' \in B_i \implies b' = \max B_i]\}$$

for each i. Then an element of P skewly tops B_0 and B_1 if and only if it skewly tops F_0 and F_1, and conversely. Thus we have:

Proposition 6.2.4. *If* $\mathbf{P} = (P, \leq_P)$ *is a finite partial order, then every two parallel fans in* $\mathbf{P}$ *are skewly topped if and only if every two parallel bouquets in* $\mathbf{P}$ *are skewly topped.*

The theorem below is the analogue of Theorem 6.1.5 for infinite partial orders. Along with the preceding two propositions, it also gives an alternative proof of Theorem 6.1.5, Suck's original one having been by induction on the size of the partial order.

Theorem 6.2.5. *A partial order is saturated if and only if every two parallel bouquets in it are skewly topped.*

Proof. ($\Longrightarrow$) Suppose B_0 and B_1 are two parallel bouquets in $\mathbf{P}$ that are not skewly topped. Let q^* be a symbol not in P, and let $Q = P \cup \{q^*\} - \{\max B_0, \max B_1\}$. Let π be the principal ideal representation of $\mathbf{P}$, and define $\varphi : P \to \mathcal{P}(Q)$ as follows: if $p \geq_P \max B_0$ or $p \geq_P \max B_1$ let $\varphi(p) = \pi(p) \cup \{q^*\} - \{\max B_0, \max B_1\}$, and otherwise let $\varphi(p) = \pi(p)$. We claim, first of all, that φ is a set representation. So fix distinct $p_0, p_1 \in P$ and note that if $p_0 \not\geq_P \max B_0, \max B_1$ and $p_1 \not\geq_P \max B_0, \max B_1$ then $\varphi(p_i) = \pi(p_i)$ for each i, meaning $\varphi(p_0) \neq \varphi(p_1)$ and $p_i <_P p_{1-i}$ if and only if $\varphi(p_i) \subsetneq \varphi(p_{1-i})$. This leaves the following cases to consider:

Case 1: for some $i, j \in \{0, 1\}$,

- $p_i \geq_P \max B_j$,
- $p_{1-i} \not\geq_P \max B_0, \max B_1$.

Clearly $\varphi(p_0) \neq \varphi(p_1)$ since $q^* \in \varphi(p_i)$ and $q^* \notin \varphi(p_{1-i})$. If p_0 and p_1 are $\leq_P$-comparable, it must be that $p_{1-i} <_P p_i$, so $\varphi(p_{1-i}) = \pi(p_{1-i}) \subsetneq \pi(p_i)$. And since $p_{1-i} \not\geq \max B_0, \max B_1$ we have $\max B_0, \max B_1 \notin \pi(p_{1-i})$, implying that $\varphi(p_{1-i}) \subseteq \pi(p_i) - \{\max B_0, \max B_1\} \subsetneq \varphi(p_i)$. Conversely, if $\varphi(p_0)$ and $\varphi(p_1)$ are comparable under inclusion, it must be that $\varphi(p_{1-i}) \subsetneq \varphi(p_i)$. Thus $\pi(p_{1-i}) \subseteq \varphi(p_i) - \{q^*\} \subseteq \pi(p_i)$. However, it can not be that $\pi(p_{1-i}) = \pi(p_i)$ since this would mean that $\max B_j \leq_P p_{1-i}$, so we must have $\pi(p_{1-i}) \subsetneq \pi(p_i)$ and hence $p_{1-i} <_P p_i$.

Case 2: for some $i, j \in \{0, 1\}$,

- $p_i \geq_P \max B_j$,
- $p_i \not\geq_P \max B_{1-j}$,
- $p_{1-i} \geq_P \max B_{1-j}$,
- $p_{1-i} \not\geq_P \max B_j$.

In this case we clearly can not have $p_{1-i} <_P p_i$ or $p_i <_P p_{1-i}$. We show that neither $\varphi(p_{1-i}) \subseteq \varphi(p_i)$ nor $\varphi(p_i) \subseteq \varphi(p_{1-i})$ can happen. Indeed, suppose it was the case that $\varphi(p_{1-i}) \subseteq \varphi(p_i)$ (the other case being symmetric). Then every $p \in B_{1-j} - \{\max B_{1-j}\}$, being an element of $\pi(p_{1-i})$, would belong to $\varphi(p_i)$ and, not being q^*, also to $\pi(p_i)$. Thus, we would have $p \leq_P p_i$, so p_i would skewly top B_0 and B_1, a contradiction.

Case 3: for some $j \in \{0, 1\}$, $p_0, p_1 \geq_P \max B_j$. Since p_0 and p_1 are distinct, we must have $p_i >_P \max B_j$ for some $i \in \{0, 1\}$. Since $\max B_0$ and $\max B_1$ are $\leq_P$-incomparable, this means that $p_i \in \varphi(p_i)$. So if $p_i \notin \varphi(p_{1-i})$ then $\varphi(p_i) \neq \varphi(p_{1-i})$. And if $p_i \in \varphi(p_{1-i})$ then $p_i <_P p_{1-i}$ and hence $p_{1-i} \in \varphi(p_{1-i}) - \varphi(p_i)$, so again $\varphi(p_i) \neq \varphi(p_{1-i})$. Now suppose $p_{1-i} <_P p_i$ for some i, so that $\pi(p_{1-i}) \subsetneq \pi(p_i)$. Then as $\varphi(p_0) = \pi(p_0) \cup \{q^*\} - \{\max B_0, \max B_1\}$ and $\varphi(p_1) = \pi(p_1) \cup \{q^*\} - \{\max B_0, \max B_1\}$, we have $\varphi(p_{1-i}) \subseteq \varphi(p_i)$ and hence $\varphi(p_{1-i}) \subsetneq \varphi(p_i)$ since $\varphi(p_{1-i}) \neq \varphi(p_i)$. Conversely, suppose $\varphi(p_{1-i}) \subsetneq \varphi(p_i)$. The only way it could fail to be the case that $\pi(p_{1-i}) \subsetneq \pi(p_i)$ is if $\max B_{1-j} \in \pi(p_{1-i}) - \pi(p_i)$. But every $p <_P \max B_{1-j}$ belongs to $\varphi(p_{1-i}) - \{q^*\}$ and hence to $\varphi(p_i) - \{q^*\} \subseteq \pi(p_i)$, meaning $p \leq_P p_i$.

So, in this case, p_i would skewly top B_0 and B_1. It must thus be that $\pi(p_{1-i}) \subsetneq \pi(p_i)$ and hence that $p_{1-i} <_P p_i$, as desired.

Our next claim is that φ is parsimonious. Fixing $p \in P$, we first verify condition (1a) of Definition 6.2.1. If $p \not\geq_P \max B_0, \max B_1$, then there is nothing to show since $\varphi(p) = \pi(p)$. If $p = \max B_j$ for some $j \in \{0,1\}$, then $\varphi(p) = \bigcup_{p' <_P p} \varphi(p') \cup \{q^*\}$ and $q^* \notin \varphi(p')$ for any $p' <_P p$ since necessarily $p' \not\geq_P \max B_0, \max B_1$. If $p >_P \max B_j$ for some j, then $\varphi(p) = \bigcup_{p' <_P p} \varphi(p') \cup \{p\}$. In this case, $p \notin \varphi(p')$ for any $p' <_P p$ since as $p \neq q^*$ this would mean that $p \in \pi(p')$ and hence that $p \leq_P p' <_P p$. In any case, then, $\left|\varphi(p) - \bigcup_{p' <_P p} \varphi(p')\right| = 1$.

We now verify condition (1b) of Definition 6.2.1. Given $q \in \varphi(p)$, we either have that $q = q^*$ and $\max B_j \leq_P p$ for some $j \in \{0,1\}$, or that $q \in P$ and $q \leq_P p$. If we apply the argument just given to q instead of to p then it follows that in the former case $\{q\} = \varphi(\max B_j) - \bigcup_{p' <_P \max B_j} \varphi(p')$, and that in the latter case $\{q\} = \varphi(q) - \bigcup_{p' <_P q} \varphi(p')$.

Finally, it follows that $\mathbf{P}$ is not saturated. Indeed, as the preceding argument shows, $\alpha_\varphi(\max B_0) = q^* = \alpha_\varphi(\max B_1)$. Hence, α_φ is not injective.

($\Longleftarrow$) Fix a partial order $\mathbf{P} = (P, \leq_P)$. Fix a parsimonious set representation $\varphi : P \to \mathcal{P}(Q)$ and suppose that α_φ is not injective, so that $\alpha_\varphi(p_0) = \alpha_\varphi(p_1)$ for some distinct $p_0, p_1 \in P$. Then by definition of α_φ, it follows that p_0 and p_1 are $\leq_P$-incomparable and not minimal in P. For $i \in \{0,1\}$, let I_i be the set of all $p <_P p_i$ in P which are $\leq_P$-incomparable with p_{1-i}, and let C_i consist of all $p <_P p_i$ in P which are $\leq_P$-comparable with p_{1-i}. Note that necessarily $p <_P p_{1-i}$ for all $p \in C_i$. This implies that each I_i must be non-empty as otherwise we would have $\varphi(p) \subsetneq \varphi(p_{1-i})$ for all $p <_P p_i$ by virtue of φ being a set representation, which would mean that $\varphi(p_i) \subseteq \varphi(p_{i-1})$, and hence that $p_i \leq_P p_{1-i}$.

Thus, $I_0 \cup \{p_0\}$ and $I_1 \cup \{p_1\}$ are parallel bouquets in $\mathbf{P}$ with p_0 and p_1 as their respective maxima. Towards a contradiction, suppose $m \in P$ skewly tops these bouquets, i.e., there is an $i \in \{0,1\}$ such that $p_i <_P m$, $p_{1-i} \not\leq_P m$, and $p <_P m$ for all $p <_P p_{1-i}$. Then $\alpha_\varphi(p_{1-i}) \in \varphi(p_i) \subsetneq \varphi(m)$ and $\varphi(p) \subsetneq \varphi(m)$ for all $p <_P p_{1-i}$ and thus

$$\varphi(p_{1-i}) = \{\alpha_\varphi(p_{1-i})\} \cup \bigcup_{p <_P p_{1-i}} \varphi(p) \subseteq \varphi(m),$$

which gives $p_{1-i} \leq_P m$, a contradiction. Thus, $I_0 \cup \{p_0\}$ and $I_1 \cup \{p_1\}$ are not skewly topped. □

The theorem shows why the move from fans in the finite case to bouquets in the infinite case was necessary. For consider the partial order $\mathbf{P}$ with domain

$$P = \{l_i : i \in \mathbb{N}\} \cup \{l\} \cup \{r_i : i \in \mathbb{N}\} \cup \{r\} \cup \{t_i : i \in \mathbb{N}\},$$

and ordering $\leq_P$ defined by (the transitive closure of) the following: for all $i <_\mathbb{N} j$,

- $l_i <_P l_j <_P l$;
- $r_i <_P r_j <_P r <_P t_i <_P t_j$;

- $l_i <_P t_i$.

(See Figure 6.2.)

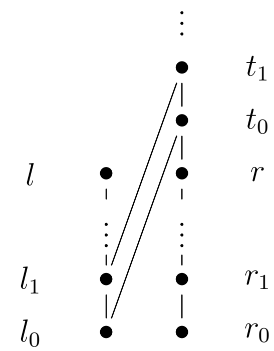

Figure 6.2: A partial order in which every two parallel fans are skewly topped, but in which not every pair of parallel bouquets is skewly topped.

Then if F_0 and F_1 are parallel fans in $\mathbf{P}$, it must be that $|F_0| = |F_1| = 2$, and that one of the two fans, say F_0, only contains elements $\leq_P$-incomparable with r, while the other only contains elements $\leq_P$-incomparable with l. Thus either $F_0 = \{l_i, l_j\}$ for some $i <_{\mathbb{N}} j$, or $F_0 = \{l_i, l\}$ for some i. In either case, F_1 must consist of some elements $<_P t_i$, and t_i must consequently skewly top F_0 and F_1. On the other hand, $B_0 = \{l_0, l_1, \ldots\} \cup \{l\}$ and $B_1 = \{r_0, r_1, \ldots\} \cup \{r\}$ are parallel bouquets in $\mathbf{P}$ which are clearly not skewly topped by any element of P. By the theorem, $\mathbf{P}$ is not saturated.

6.3 Reverse mathematics of Theorem 6.2.5

For interval orders, the equivalences between various set-theoretic and algebraic characterizations were studied in the context of reverse mathematics by Marcone [45]. For example, Theorem 6.1.3 is provable in RCA_0 ([45], Theorems 2.13 and 4.2), but other characterizations of interval orders are equivalent to stronger subsystems (recall the notion of interval representation from Definition 6.1.1):

Theorem 6.3.1 (Marcone [45], Theorem 5.6)**.** *Over* RCA_0*, the following are equivalent:*

1. WKL_0*;*

2. *a partial order is an interval order if and only if it admits an interval representation that is injective.*

For the purposes of analyzing the reverse-mathematical strength of Theorem 6.2.5, we begin by formalizing the concept of set representation in the language of second-order arithmetic.

Definition 6.3.2. The following definitions are made in RCA_0: if $\mathbf{P} = (P, \leq_P)$ is a partial order, a *set representation* of $\mathbf{P}$ is a subset φ of $P \times Q$ for some set Q such that if we abbreviate $\{q \in Q : (p, q) \in \varphi\}$ by $\varphi(p)$, then for all $p, p' \in P$, the following hold:

1. $p \neq p' \to \varphi(p) \neq \varphi(p')$,
2. and $p <_P p' \leftrightarrow \varphi(p) \subsetneq \varphi(p')$.

Parsimony is then formalized in a straightforward way, along with all the combinatorial notions from Definitions 6.1.4 and 6.2.3. Formalizing saturation, on the other hand, presents us with two options (we deliberately use the same term for both):

Definition 6.3.3. The following definitions are made in RCA_0: if $\mathbf{P} = (P, \leq_P)$ is a partial order, then

1. $\mathbf{P}$ is *saturated* if for every parsimonious set representation $\varphi \subseteq P \times Q$ of $\mathbf{P}$, it holds that for all $p_0, p_1 \in P$ and all $q_0, q_1 \in Q$, if $p_0 \neq p_1$ and $\{q_i\} = \varphi(p_i) - \bigcup_{p' <_P p_i} \varphi(p')$ for each $i \in \{0, 1\}$, then $q_0 \neq q_1$;
2. $\mathbf{P}$ is *saturated* if for every parsimonious set representation $\varphi \subseteq P \times Q$ of $\mathbf{P}$, the map $\alpha_\varphi : P \to Q$ exists and is injective.

Classically, the two definitions are, of course, equivalent. But in the present context they need not be because the existence of the map α_φ may not always be provable in RCA_0. The next pair of propositions shows that this can indeed happen. Thus, while formulating saturation according to Definition 6.3.3 (2) may be more natural, the set theoretic assumptions necessary to carry out the proof of Theorem 6.2.5 become much higher.

Proposition 6.3.4. RCA_0 *proves that a partial order is saturated according to Definition 6.3.3 (1) if and only if every two parallel bouquets in it are skewly topped.*

Proof. RCA_0 suffices to carry out the left-to-right direction of the proof of Theorem 6.2.5. For the right-to-left direction, fix a partial order $\mathbf{P} = (P, \leq_P)$ and a parsimonious set representation $\varphi \subseteq P \times Q$. Suppose there exists $p_0 \neq p_1$ in P such that $\varphi(p_0) - \bigcup_{p' <_P p_0} \varphi(p') = \varphi(p_1) - \bigcup_{p' <_P p_1} \varphi(p')$. Then we can argue as in the right-to-left direction of the proof of Theorem 6.2.5 that there exist parallel bouquets in $\mathbf{P}$ which are not skewly topped. □

Proposition 6.3.5. *Over* RCA_0*, the following are equivalent:*

1. ACA_0*;*
2. *for every parsimonious set representation φ of a partial order, the map α_φ exists;*
3. *a partial order is saturated according to Definition 6.3.3 (2) if and only if every two parallel bouquets in it are skewly topped;*
4. *a partial order is saturated according to Definition 6.3.3 (1) if and only if it is saturated according to Definition 6.3.3 (2).*

Proof. For every parsimonious set representation φ of a partial order $(P, \leq_P)$, we have that α_φ is arithmetically definable, so (1) implies (2). By Proposition 6.3.4 it follows that (2) implies (3), and obviously the equivalence of (1) and (3) implies the equivalence of (1) and (4).

It thus remains only to show that (3) implies (1). To this end, we prove from (3) that the range of every injective function $f : \mathbb{N} \to \mathbb{N}$ exists (this is equivalent to ACA_0. So fix an injective function f and define a partial order $\mathbf{P} = (P, \leq_P)$ with domain $P = \{p_{i,s} : i, s \in \mathbb{N}\}$ as follows:

- for all $i <_{\mathbb{N}} j$, let $p_{i,s} >_P p_{j,t}$ for all $s, t \in \mathbb{N}$;
- for each i and all $s <_{\mathbb{N}} t$, let $p_{i,s} <_P p_{i,t}$ if $s > 0$ and $f(t-2) = i$, and let $p_{i,s} >_P p_{i,t}$ otherwise.

In other words, fixing i, if $f(t) \neq i$ for all t, then we have $p_{i,s} >_P p_{i,t}$ for all $s <_{\mathbb{N}} t$; while if $f(t) = i$ for some t, then we have $p_{i,0} >_P p_{i,t+2} >_P p_{i,s} >_P p_{i,s'}$ for all $s <_{\mathbb{N}} s'$ in $\mathbb{N} - \{0, t+2\}$. RCA_0 suffices to show that $\mathbf{P}$ exists, that it is a linear order, and that every element has an immediate $\leq_P$-predecessor. In particular, linearity implies that there are no parallel bouquets in $\mathbf{P}$, so $\mathbf{P}$ must be saturated according to Definition 6.3.3 (2) by part (3) above.

Define

$$\varphi = \{(p, p') \in P \times P : p >_P p' \wedge (\forall i \in \mathbb{N})[\, p' \neq p_{i,0}\,]\},$$

which exists by Σ^0_0 comprehension and is clearly a set representation of $\mathbf{P}$. If we let p^- denote the immediate $\leq_P$-predecessor of each $p \in P$, then we see that $\{p^-\} = \varphi(p) - \bigcup_{p' <_P p} \varphi(p')$. Furthermore, if $q \in \varphi(p)$ for some $p = p_{i,s}$, then $p >_P q$ and $q = p_{j,t}$ for some $j \leq_{\mathbb{N}} i$ and $t >_{\mathbb{N}} 0$, so $q = p^-_{j,t'}$ for some $p_{j,t'} \leq_P p$. Thus, φ is parsimonious.

Applying Definition 6.3.3 (2) to φ, it follows that $\alpha_\varphi : P \to P$ exists and is injective, and by the preceding discussion we have $\alpha_\varphi(p) = p^-$ for all p. Let $R = \{i \in \mathbb{N} : \alpha_\varphi(p_{i,0}) \neq p_{i,1}\}$, which exists by Σ^0_0 comprehension. Then by construction of $\leq_P$, we have that $i \in R$ if and only if $p^-_{i,0} = p_{i,t+2}$ for some t such that $f(t) = i$, which in turn holds if and only if i is in the range of f. Hence, the range of f is equal to R and so consequently exists. This completes the proof. $\square$

APPENDIX A
RELATIONSHIPS BETWEEN WEAK PRINCIPLES

In this appendix, we summarize, by way of Figure A.1 below, some of the major relationships known to hold between various combinatorial principles that lie strictly between RCA_0 and ACA_0. In addition to the principles defined above, we list some others whose logical strength has been studied in conjunction with that of Ramsey's theorem for pairs. The following were defined by Hirschfeldt and Shore [27], pp. 175, 178, 180, 192, 193, and 196, respectively:

Cohesive Ramsey's theorem for pairs (CRT^2_2)**.** *For every coloring $f : [\omega]^2 \to 2$, there exists an infinite set S such that for every $x \in s$ there is a $c < 2$ satisfying $f(x, y) = c$ for all sufficiently large $y \in S$.*

Chain or antichain principle (CAC)**.** *For every partial order $\leq_P$ on ω there is an infinite set X that is either a chain under this order, i.e., $x \leq_P y$ or $y \leq_P x$ for all x, y in X, or an antichain, i.e., $x \nleq_P y$ and $y \nleq_P x$ for all $x \neq y$ in X.*

Stable ascending or descending sequence principle (SADS)**.** *For every linear order of order type $\omega + \omega^*$, there is an infinite set that is either an ascending sequence or a descending sequence under the order.*

For the next two principles, call a partial order $\leq_P$ on ω *stable* if one of the following holds:

1. for every x, either almost all y are incomprable with x, or almost all satisfy $y <_P x$;
2. for every x, either almost all y are incomprable with x, or almost all satisfy $y >_P x$.

(See also [27], Definition 3.2.)

Stable chain or antichain principle (SCAC)**.** *For every stable partial order on ω there is an infinite set that is either a chain or an antichain under this order.*

Cohesive chain or antichain principle (CCAC)**.** *For every partial order on ω there is an infinite set on which the order is stable.*

Strong cohesive principle (StCOH)**.** *For every sequence $\vec{R} = \langle R_i : i \in \omega \rangle$ of sets, there exists an infinite set S such that for every $i \in \omega$ there exists s such that for every $j \leq i$, either every $x \geq s$ in S belongs to R_j, or every such x belongs to $\overline{R_j}$.*

Clearly, CAC is equivalent over RCA_0 to $\mathsf{SCAC}+\mathsf{CCAC}$, RT^2_2 is equivalent to $\mathsf{SRT}^2_2+\mathsf{CRT}^2_2$, and ADS implies SADS. Hirschfeldt and Shore showed in Proposition 3.7 of [27] that CCAC is equivalent to ADS, and in Propositions 2.7, 2.9, and 2.10 that ADS is equivalent to $\mathsf{SADS} + \mathsf{COH}$. By Propositions 4.4 and 4.8 of that article, StCOH is equivalent over RCA_0 to $\mathsf{COH} + \mathsf{B}\Sigma^0_2$ and to $\mathsf{CRT}^2_2 + \mathsf{B}\Sigma^0_2$. We refer the reader to [27] for further results.

In [28], Hirschfeldt, Shore, and Slaman considered the weak variant of AMT:

Atomic model theorem with subenumerable types (AST). *For every complete atomic theory T, if there exists a family $\langle S_0, S_1, \ldots\rangle$ of partial types of T such that every type of T implies the same set of formulas as S_i for some i, then T has an atomic model.*

It was shown in Theorem 6.3 of [28] that AST is equivalent over RCA_0 to the statement $(\forall X)(\exists Y)[Y \not\leq_T X]$. Thus, this principle can legitimately claim the title of being the weakest to not hold in REC, at least with respect to principles that do not hold in a *topped* ω-model, i.e., one that contains some set S from which every other set in the model is computable. In particular, with the exception of induction and bounding schemes, all the principles discussed in this dissertation that are not provable in RCA_0 imply AST.

We define one final principle, whose logical strength has thus far escaped most traditional avenues of analysis.

Definition A.0.6.

1. A *tournament on a set S* is a complete directed simple graph with domain S. We identify tournaments with their edge relations and with their domains. Thus, we say a tournament is *infinite* if its domain is infinite.

2. A tournament T is *transitive* if xTy and yTz imply xTz for all $x, y, z \in T$.

3. A *subtournament* of a tournament T is any subgraph of T that is itself tournament.

Note that, by viewing the edge relation as an order relation, transitive tournaments can be identified with linear orders.

Erdős/Moser principle (EM). *Every tournament on $\mathbb{N}$ has an infinite transitive subtournament.*

It is straightforward to show that RT^2_2 implies EM, but for the reverse implication, one appears to need an application of ADS. (See, for example, Theorem 8 of [3], and the comment following it.) Whether the use of ADS can be avoided is an open question, along with the question of which other principles imply EM, and which ones are implied by it. The implication from EM to OPT noted in Figure A.1 is due to Dzhafarov, Kach, and Solomon (unpublished); it is strict, since if OPT implied EM then $\mathsf{OPT} + \mathsf{ADS}$ would imlpy RT^2_2, which is false because OPT follows from ADS and ADS is strictly weaker than RT^2_2.

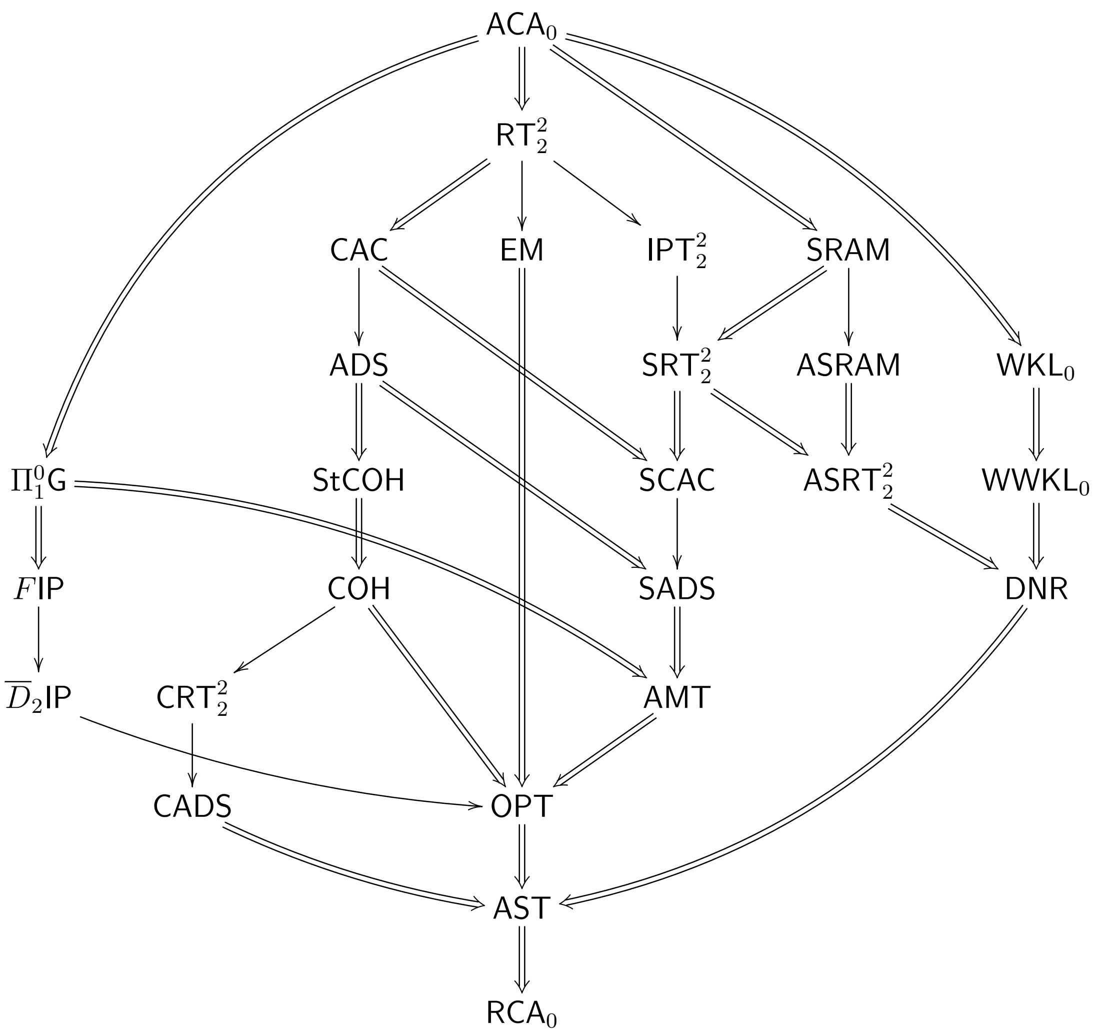

Figure A.1: Summary of relationships between various combinatorial principles lying strictly between RCA_0 and ACA_0. For clarity, equivalences of principles are omitted. Arrows denote implications provable in RCA_0, and double arrows denote implications that are known to be strict.

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