Reverse mathematics and computable combinatorics

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Part One: Background

The computability-theoretic perspective

We are interested in statements of the form

$$\forall X \, [\Phi(X) \to \exists Y \, \Psi(X, Y)],$$

where Φ and Ψ are some kind of properties of X and Y.

We think of this as a problem, “given X satisfying Φ, find Y satisfying Ψ”.

We call the X such that $\Phi(X)$ holds the instances of the problem,
and the Y such that $\Psi(X, Y)$ holds the solutions to X for this problem.

Typically, we look at problems whose instances and solutions are subsets of $\mathbb{N}$,
and where the properties Φ and Ψ are arithmetical.

Question. Given an instance of a problem, how complex are its solutions?

The proof-theoretic perspective

Reverse mathematics is motivated by a foundational question:

Question. Which axioms do we really need to prove a given theorem?

The question leads to the idea of the strength of a theorem. Which theorems does it imply? Which imply it? Which is it equivalent to?

Example. Over ZF, the axiom of choice is equivalent to Zorn's lemma.

Set theory is too strong to calibrate the strength of "ordinary theorems".

We would like results of the form

Over the theory T, theorem P implies/is equivalent to theorem Q,

where T is weak enough to not prove everything, yet robust enough to accommodate a decent amount of basic coding and representation.

Subsystems of second-order arithmetic

Second-order arithmetic, Z_2, is a two-sorted theory with variables for numbers and sets of numbers, and the usual symbols of arithmetic.

The axioms of Z_2 are those of Peano arithmetic, and the comprehension scheme: if φ is a formula (with set parameters) then $\{x \in \mathbb{N} : \varphi(x)\}$ exists.

We restrict which formulas φ we allow to obtain various subsystems:

RCA_0	Δ^0_1 formulas (definitions of computable sets).
WKL_0	Formulas defining paths through infinite binary trees.
ACA_0	Arithmetical formulas.
ATR_0	Arithmetical formulas iterated along ctbl well orders.
Π^1_1CA	Π^1_1/Σ^1_1 formulas.

Subsystems of second-order arithmetic

$$\begin{array}{c} \mathrm{ZFC} \\ \Downarrow \\ \mathrm{Z}_2 \\ \Downarrow \\ \Pi^1_1\text{-}\mathrm{CA}_0 \\ \Downarrow \\ \mathrm{ATR}_0 \\ \Downarrow \\ \mathrm{ACA}_0 \\ \Downarrow \\ \mathrm{WKL}_0 \\ \Downarrow \\ \mathrm{RCA}_0 \end{array}$$

Measuring complexity

Computability theory:

- Does every instance compute a solution to itself?
- Does every instance have an arithmetically-definable solution?
- Is there a computable instance all of whose solutions compute $\emptyset'$?

Reverse mathematics/proof theory:

- We look at subsystems of second-order arithmetic, RCA_0, WKL, ACA_0, ...
- Is the theorem provable in RCA_0?
- Is the theorem provable in ACA_0?
- Does the theorem imply ACA over RCA_0?

There is well-understood interplay between these viewpoints.

Semantics

A model M of Z_2 is a pair $(\mathbb{N}, \mathcal{S})$, where $\mathbb{N}$ is a possibly nonstandard version of ω, and $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$.

When $\mathbb{N} = \omega$, M is called an ω-model, and can be identified with $\mathcal{S}$.

Models of subsystems of Z_2 correspond to closure points under natural computability-theoretic operations.

RCA_0	ω-models closed under $\leq_T$ and $\oplus$ (disjoint union).
WKL_0	ω-models closed under existence of completions of PA
ACA_0	ω-models closed under the jump operator, $A \mapsto A'$.

Classical reverse mathematics

Most (countable) classical mathematics can be developed within Z_2.

Initial focus was on classifying theorems in terms of the "big five".

Theorem. The following are provable in RCA_0.

- (Simpson). Baire category theorem, intermediate value theorem.
- (Brown; Simpson). Urysohn's lemma, Tietze extension theorem.
- (Rabin). Existence of algebraic closures of countable fields.

Theorem. The following are equivalent to WKL_0 over RCA_0.

- (Brown; Friedman). Heine-Borel theorem for $[0, 1]$.
- (Orevkov; Shoji and Tanaka). Brouwer fixed-point theorem.
- (Friedman, Simpson, and Smith). Prime ideal theorem.

Classical reverse mathematics

Theorem. The following are equivalent to ACA_0 over RCA_0.

- (Friedman). Bolzano-Weierstrass theorem.
- (Dekker). Existence of bases in vector spaces.
- (Friedman, Simpson, and Smith). Maximal ideal theorem.

Theorem. The following are equivalent to ATR_0 over RCA_0.

- (Steel; Friedman and Hirst). Comparability of well-orderings.
- (Simpson) Lusin's separation theorem.
- (Steel; Simpson) Open/clopen determinacy for ω^ω.

Theorem. The following are equivalent to Π^1_n-CA_0 over RCA_0.

- (Dzhafarov and Mummert.) Teichmüller-Tukey lemma for Σ^1_n formulas.

The big five phenomenon

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, *etc.*	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

From *The Gödel Hierarchy and Reverse Mathematics*, by Stephen Simpson.

Irregular theorems

Natural question: What are the exceptions to this classification?

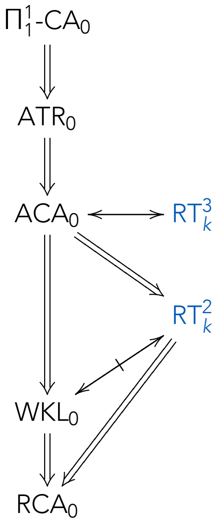
$[X]^n$ = set of all $\langle x_0, \ldots, x_{n-1}\rangle \in X^n$ with $x_0 < \cdots < x_{n-1}$.

RT^n_k. For every coloring $c : [\omega]^2 \to 2$, there exists an infinite homogeneous set for c.

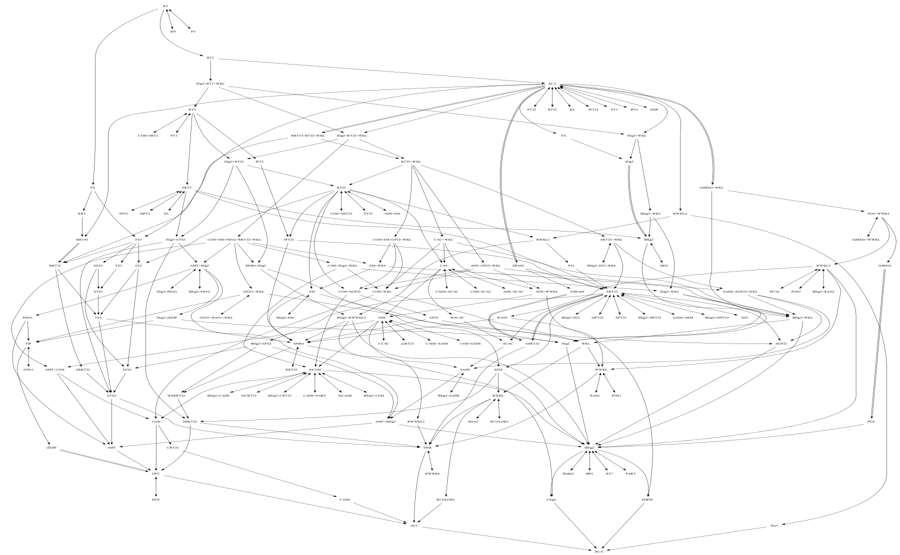
Theorem. RT^2_2 is "irregular", but RT^3_2 is not.

- (Specker). RCA_0 proves RT^n_2 if and only if $n = 1$.
- (Jockusch). For $n \geq 3$, $\mathsf{RT}^n_k \leftrightarrow \mathsf{ACA}_0$ over RCA_0.
  For WKL_0 does not prove RT^2_2.
- (Seetapun). $\mathsf{RT}^2_2 \not\to \mathsf{ACA}_0$ over RCA_0.
- (Liu). $\mathsf{RT}^2_2 \not\to \mathsf{WKL}_0$ over RCA_0.

Ramsey's theorem



The reverse mathematics zoo

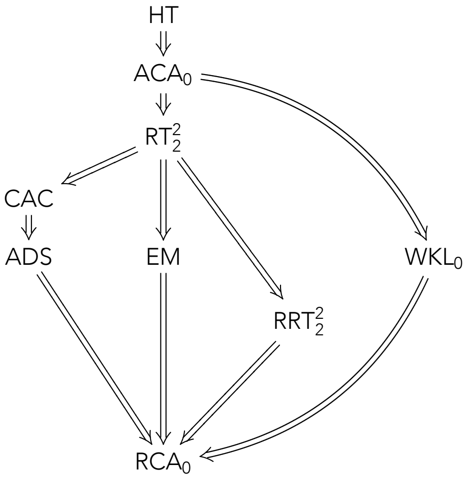


Part Two: Combinatorics and Beyond

Combinatorics below RT^2

- **Chain/antichain principle.** Every partial ordering of $\mathbb{N}$ contains an infinite chain or an infinite antichain.
- **Ascending/descending sequence principle.** Every linear ordering of $\mathbb{N}$ contains an infinite ascending or an infinite descending sequence.
- **Erdős-Moser theorem.** Every tournament on $\mathbb{N}$ has an infinite transitive subtournament.
- **Rainbow Ramsey's theorem.** For all $n, k \geq 1$ and $f : [\mathbb{N}]^n \to \mathbb{N}$ such that $|f^{-1}(n)| < k$ for all n there is an infinite $R \subseteq \mathbb{N}$ such that f is injective on $[R]^n$.
- **Hindman's theorem.** For all $k \geq 1$ and $f : \mathbb{N} \to k$ there is an infinite $I \subseteq \mathbb{N}$ and an $i < k$ such that $f(\sum F) = i$ for all non-empty finite $F \subseteq I$.

Combinatorics below RT^2



The atomic model theorem

A first-order **atomic theory** is one containing a formula that decides every other formula; an **atomic model** is one that is as small as possible.

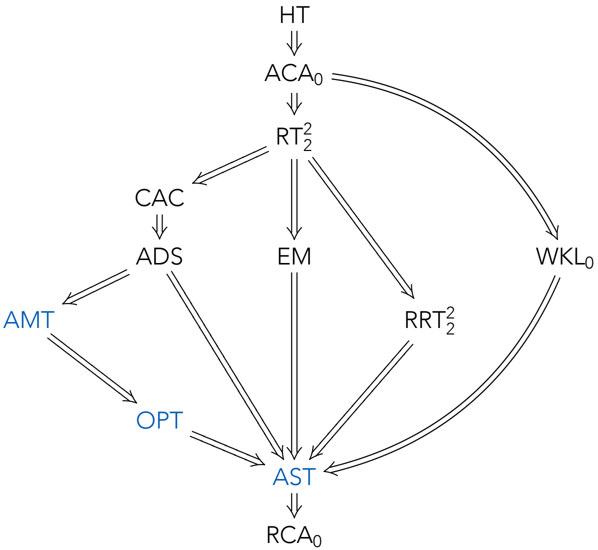
AMT. Every atomic theory has an atomic model.

There are two variants, OPT and AST, which are special cases of AMT.

Theorem (Hirschfeldt, Shore, and Slaman.)

- AMT is not provable in RCA_0, but it is extremely weak: it is implied over RCA_0 by virtually every combinatorial principle below RT^2_k.
- OPT is equivalent to the existence of hyperimmune sets, i.e., it can be characterized in terms of growth rates of computable functions.
- AST is equivalent to the existence of noncomputable sets.

The atomic model theorem



Intersection principles

A family of sets is said to have the finite intersection property (f.i.p.) if the intersection of any finitely many of its members is non-empty.

FIP. Every family of sets has a maximal subfamily with f.i.p.

NIP. Every family of sets has a maximal pairwise disjoint subfamily.

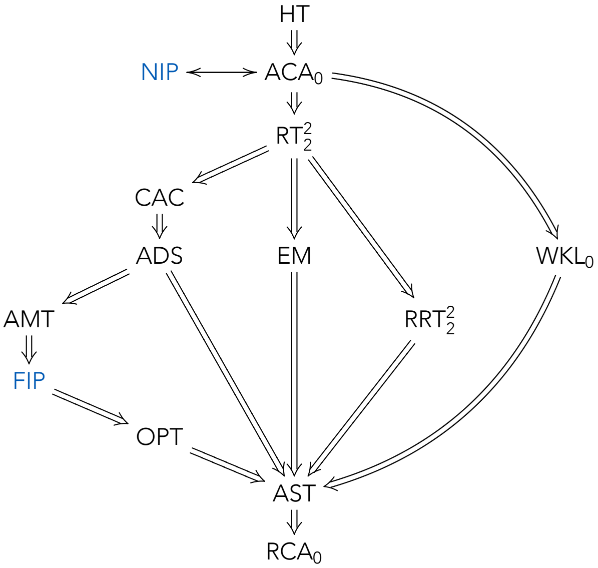
Over ZF, these principles are equivalent to choice (and so to each other).

Theorem (Dzhafarov and Mummert).

- Over RCA_0, NIP is equivalent to ACA_0.
- Over RCA_0, AMT implies FIP, which implies OPT, both strictly.

Theorem (Cholak, Downey, Igusa). FIP $\leftrightarrow$ existence of a Cohen generic.

Intersection principles



Milliken's tree theorem

For a tree T, $\mathcal{S}_\alpha(T)$ is the class of all strong subtrees of T of height $\alpha \leq \omega$.

Milliken's tree theorem. Let T be an infinite tree with no leaves. For all $n, k \geq 1$ and all $c : \mathcal{S}_n(T) \to k$ there is a $U \in \mathcal{S}_\omega(T)$ such that c is contant on $\mathcal{S}_n(U)$.

MTT^n_k. Milliken's tree theorem restricted to k-colorings of $\mathcal{S}_n(T)$.

- Generalizes many combinatorial results, including Ramsey's theorem.
- Inductive proof (on n) using the **Halpern-Laüchli theorem**.
- Every known proof actually proves a stronger, product version, PMTT^n_k.

Dobrinen (2018). What about the effectivity/reverse math of MTT?

Milliken's tree theorem

Theorem (Anglès d'Auriac, Cholak, Dzhafarov, Monin, and Patey).

- The Halpern-Laüchli theorem is computably true (and uniformly so, in an arithmetical oracle).

  Hence, $\mathsf{ACA}_0 \vdash \mathsf{PMTT}^n_k$, for all n, k.

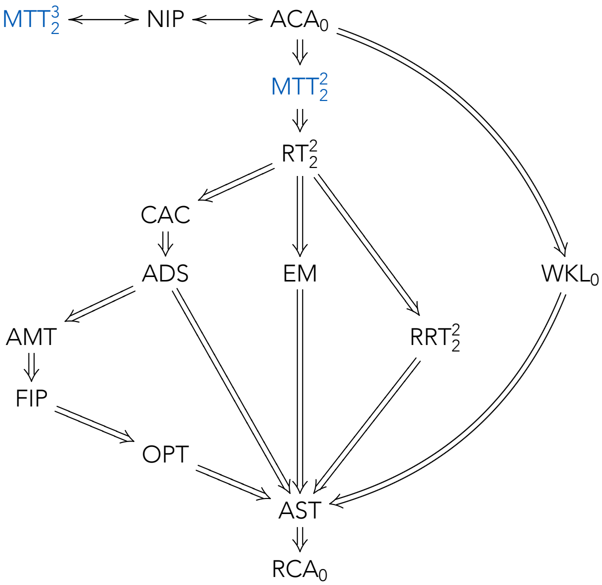
- For all $n \geq 3$ and all $k \geq 2$, $\mathsf{ACA}_0 \leftrightarrow \mathsf{MTT}^n_k \leftrightarrow \mathsf{PMTT}^n_k$.

- PMTT^2_k does not imply ACA_0 over RCA_0.

The proof is a forcing construction, utilizing a kind of analogue of (finite) Ramsey numbers for Milliken's tree theorem.

Some applications to the study of big Ramsey degrees of various structures.

Milliken's tree theorem



Part Three: Current Trends and Questions

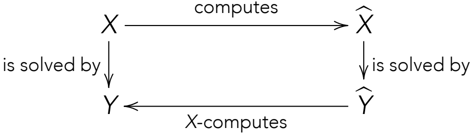
Stronger measures of strength

Let P and Q be problems.

P is computably reducible to Q, written P $\leq_c$ Q, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, computes a P-solution Y to X.

So the following diagram commutes:



(Dzhafarov '15; Hirschfeldt and Jockusch '16).

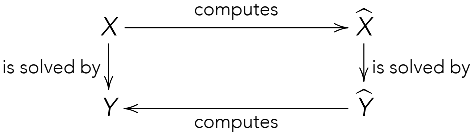
Stronger measures of strength

Let P and Q be problems.

P is strongly computably reducible to Q, written P $\leq_{sc}$ Q, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, ~~together with X~~, computes a P-solution Y to X.

So the following diagram commutes:



(Dzhafarov '15; Hirschfeldt and Jockusch '16).

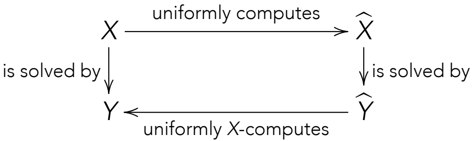
Stronger measures of strength

Let P and Q be problems.

P is Weihrauch reducible to Q, written P $\leq_W$ Q, if

- every instance X of P *uniformly* computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, *uniformly* computes a P-solution Y to X.

So the following diagram commutes:



(Weihrauch '92; Brattka; Gherardi and Marcone '08; DDHMS '16).

Stronger measures of strength

Let P and Q be problems.

We have the following implications:

$$
\begin{array}{ccccc}
 & & \mathsf{P} \leq_{sW} \mathsf{Q} & & \\
 & \swarrow & & \searrow & \\
\mathsf{P} \leq_{sc} \mathsf{Q} & & & & \mathsf{P} \leq_{W} \mathsf{Q} \\
 & \searrow & & \swarrow & \\
 & & \mathsf{P} \leq_{c} \mathsf{Q} & & \\
 & & \Downarrow & & \\
 & & \mathsf{Q} \leq_{\omega} \mathsf{P} & &
\end{array}
$$

(Q *computably entails* P, i.e., every ω-model of Q is a model of P)

Usually, if $\mathsf{P} \leq_{\omega} \mathsf{Q}$ then $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$, but not always (induction issues).

Logical/algebraic properties of reductions

Extensive work has been done on the algebraic structure of $\leq_W$ and $\leq_{sW}$.

Brattka and Gherardi '11; Higuchi and Pauly '13; Hölzl and Shafer '15; Dzhafarov '19; Brattka and Pauly '20, others.

Theorem (Brattka and Gherardi).

- There exist ops. turning the Weihrauch degrees into a distributive lattice.
- The join does not work for $\leq_{sW}$.

Theorem (Dzhafarov). There exists a join operation for $\leq_{sW}$. The resulting lattice is non-distributive.

Theorem (Higuchi and Pauly; Dzhafarov). Every countable distributive lattice embeds into the (strong) Weihrauch degrees.

Example: Ramsey's theorem for different colors

Over RCA_0, $\mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^n_2$ for all $k \geq 2$.

But to prove, say, $\mathsf{RT}^2_2 \rightarrow \mathsf{RT}^2_3$, we seem to need to use RT^2_3 twice.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer).

For all $k \geq 2$, $\mathsf{RT}^n_{2^k} \not\leq_W \mathsf{RT}^n_k$.

Theorem (Hirschfeld and Jockusch; Brattka and Rakotoniania).

If $k > j$, then $\mathsf{RT}^n_k \not\leq_W \mathsf{RT}^n_j$.

Theorem (Patey). If $k > j$, then $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_j$.

Each of these results is proved by a somewhat different kind of forcing construction.

The CJS decomposition

A coloring $c : [\mathbb{N}]^2 \to k$ is **stable** if there is an $i < k$ such that for every $x \in \mathbb{N}$, $c(x, y) = i$ for all sufficiently large y (i.e., for every $x \in \mathbb{N}$, $\lim_y c(x, y) = i$).

SRT^2_k. For every stable coloring $c : [\omega]^2 \to k$, there exists an infinite homogeneous set for c.

A set L is limit-homogeneous for c if $\lim_y c(x, y)$ is the same for all $x \in L$.

D^2_k. For every stable coloring $c : [\omega]^2 \to k$, there exists an infinite limit-homogeneous set for c.

Theorem (Chong, Lempp, and Yang). $\mathsf{SRT}^2_2 \leftrightarrow \mathsf{D}^2_2$ over RCA_0.

Theorem (Dzhafarov). $\mathsf{SRT}^2_2 \not\leq_W \forall k\, \mathsf{D}^2_k$ and $\mathsf{SRT}^2_2 \not\leq_{sc} \forall k\, \mathsf{D}^2_k$.

The CJS decomposition

Combinatorially:

- D^2_2 = solving an instance of RT^1_2.
- SRT^2_2 = solving an instance of RT^1_2, plus thinning.

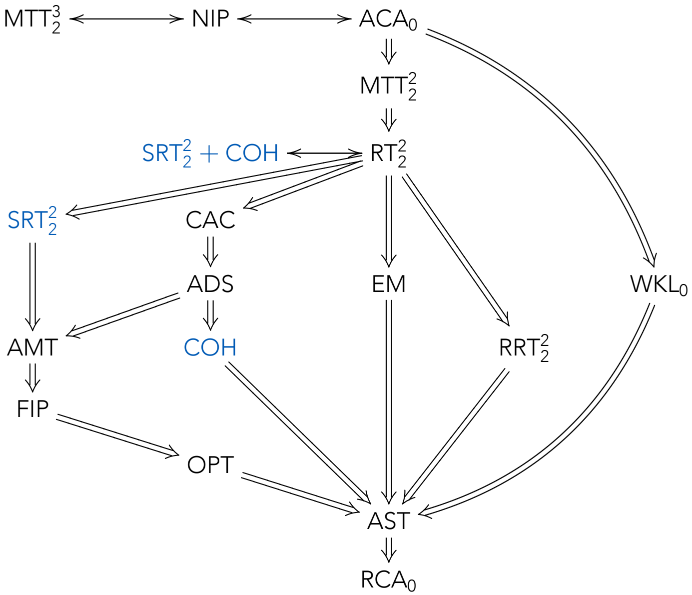
COH. For every family $\vec{X} = \langle X_0, X_1, \ldots \rangle$ there exists an infinite set Y which is **$\vec{X}$-cohesive**, i.e., for all i either $Y \cap X_i$ or $Y \cap (\omega - X_i)$ is finite.

- COH = solving ω many instances of RT^1_2 in parallel, allowing finite errors.

Theorem (Cholak, Jockusch, Slaman). $RT^2_2 \leftrightarrow SRT^2_2 + COH$ over RCA_0.

Longstanding problem: Understand the relationship between COH and SRT^2_2.

The CJS decomposition



The SRT^2_2 versus COH problem

Theorem (Chong, Slaman, Yang '13). $\mathsf{SRT}^2_2 \not\to \mathsf{COH}$ over RCA_0.

Interestingly, the proof uses non-standard methods in an essential way. The model produces a model of $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 + \neg\mathsf{COH}$ in which Σ^0_2 **induction fails**.

This set off much work to produce an ω-model separation.

Theorem (Dzhafarov '15). $\mathsf{COH} \not\leq_{sc} \forall k\, \mathsf{D}^2_k$.

Theorem (Dzhafarov '16). $\mathsf{COH} \not\leq_{W} \forall k\, \mathsf{SRT}^2_k$ and $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_2$.

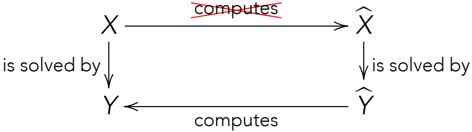
Theorem (Dzhafarov, Patey, Solomon, Westrick '17). $\mathsf{COH} \not\leq_{sc} \forall k\, \mathsf{SRT}^2_k$.

Theorem (Monin and Patey '20). $\mathsf{COH} \not\leq_{\omega} \mathsf{SRT}^2_2$.

Combinatorial reductions and separations

Often, we are able to prove stronger separations than just $\not\leq_c$, $\not\leq_{sc}$, etc.

Namely, we can often remove the effective relationship between instnces:



P is (strongly) omnisciently computably reducible to Q if

- for every instance X of P there **exists** an instance $\widehat{X}$ of Q, such that
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, with X (or not) computes a P-solution Y to X.

We write P $\leq_{oc}$ Q or P $\leq_{soc}$ Q.

Combinatorial reductions and separations

Theorem (Dzhafarov, Patey, Solomon, and Westrick).
If $k > j$, then $\mathsf{RT}^1_k \not\leq_{soc} \mathsf{RT}^1_j$.

There is a $c : \omega \to k$ such that **for every** stable $d : [\omega]^2 \to j$ there is an $i < j$ and an infinite homogeneous set H_i computing no infinite homogeneous set for c.

Main elements of proof:

- Fix M, a countable transitive model of ZFC.
- Let c bZ Cohen generic for forcing in $k^{<M}$.
- Given $d : \omega \to j$ and $i < j$, let $\mathbb{M}_i$ be Mathias forcing with conditions (F, I) such that $I \in M$ and F is monochromatic for d with color i.
- Let H_i be generic for $\mathbb{M}_i$ over a model $M' \supseteq M \cup \{c, d\}$.

Combinatorial core uses the tree labeling method (Dzhafarov '15).

Combinatorial reductions and separations

Observation. COH $\leq_{soc}$ SRT^2_2.

Proof. Fix an instance of COH, $\vec{X} = (X_0, X_1, \ldots)$. Define $c : [\mathbb{N}]^2 \to 2$ by

$$c(n, b) = \begin{cases} 0 & \text{if some intersection of } X_0, \ldots, X_n, \overline{X_0}, \ldots, \overline{X_n} \\ & \text{is finite but contains an element } x > b. \\ 1 & \text{otherwise.} \end{cases}$$

Let $H = \{n_0 < n_1 < \cdots\}$ be a homogeneous set for s, necessarily of color 1.

We can now compute from H an infinite cohesive set for $(X_0, X_1, \ldots)$.

For example, to see which of $X_0 \cap X_1$, $\overline{X_0} \cap X_1$, $X_0 \cap \overline{X_1}$, or $\overline{X_0} \cap \overline{X_1}$ is infinite, search for the least $x > n_1$ in one of these intersections. ■

Questions

What if we replace SRT^2_k by D^2_k?

Observation. For all k, $\mathsf{D}^2_k \equiv_{soc} \mathsf{RT}^1_k$.

Since $\mathsf{RT}^1_k \not\leq_{soc} \mathsf{RT}^1_j$ for all $k > j$, it is also easy to see that $\mathsf{COH} \not\leq_{soc} \mathsf{RT}^1_k$.

Open question. Is $\mathsf{COH} \leq_{oc} \mathsf{D}^2_2$? Equivalently, is $\mathsf{COH} \leq_{oc} \mathsf{RT}^1_2$?

Turing computations are **effectively continuous** transformations $2^{\mathbb{N}} \to 2^{\mathbb{N}}$. What if we weaken effectivity to continuity?

Open question. Given $\vec{X} = (X_0, X_1, \ldots)$, does there exist $c : \mathbb{N} \to 2$, every infinite hom. set for which continuously maps onto an infinite $\vec{X}$-cohesive set?

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Thank you for your attention!