Strong computable reducibility

Damir D. Dzhafarov
University of Connecticut

September 21, 2015
A problem is a $\Pi^1_2$ statement of second-order arithmetic, thought of as

$$\text{for every } X \in \text{Inst}(P), \text{ there is a } Y \in \text{Soln}(P, X),$$

where $\text{Inst}(P)$ and $\text{Soln}(P, X)$ are arithmetically-definable sets.

Examples.

$\mathsf{RT}_k^n$. Every coloring $c : [\omega]^n \to k$ has an infinite homogeneous set.

$\mathsf{COH}$. For every family $\vec{c} = \langle c_0, c_1, \ldots \rangle$ of colorings $c_i : \omega \to 2$ there is an infinite set $H$ that is almost homogeneous for each $c_i$, i.e., if for each $i$ there is a finite set $F$ such that $H - F$ is homogeneous for $c_i$. 
Let $P$ and $Q$ be problems.

$P$ is strongly computably reducible to $Q$, written $P \leq_{sc} Q$, if every $X \in \text{Inst}(P)$ computes an $\hat{X} \in \text{Inst}(Q)$, such that every $\hat{Y} \in \text{Soln}(Q, \hat{X})$ computes a $Y \in \text{Soln}(P, X)$.

![Diagram showing the relationship between $X$, $\hat{X}$, $Y$, and $\hat{Y}$ with arrows indicating computations and solutions.]
Let $P$ and $Q$ be problems.

$P$ is **computably reducible** to $Q$, written $P \leq_c Q$, if every $X \in \text{Inst}(P)$ computes an $\hat{X} \in \text{Inst}(Q)$, such that every $\hat{Y} \in \text{Soln}(Q, \hat{X})$, together with $X$, computes a $Y \in \text{Soln}(P, X)$.
As a finer metric.

Most implications between problems are formalizations of (strong) computable or (strong) Weihrauch reductions.

**Theorem** (Cholak, Jockusch, and Slaman). $\text{RCA}_0 \vdash \text{RT}_2 \rightarrow \text{COH}$.

The proof is a formalization in $\text{RCA}_0$ that $\text{COH} \leq_s \text{RT}_2$.

We can tease apart subtle differences that $\text{RCA}_0$ alone does not see.

For all $j$ and $k$, we have $\text{RCA}_0 \vdash \text{RT}_j^n \leftrightarrow \text{RT}_k^n$.

**Theorem** (Dorais, Dzhafarov, Hirst, Mileti, Shafer). If $j > k$, then $\text{RT}_j^n \not\leq_s \text{RT}_k^n$.

**Theorem** (Hirschfeldt and Jockusch). If $j > k$, then $\text{RT}_j^n \not\leq_W \text{RT}_k^n$.

**Theorem** (Patey). If $j > k$, then $\text{RT}_j^n \not\leq_c \text{RT}_k^n$. 
Two versions of Ramsey's theorem.

A coloring \( c : [\omega]^2 \to 2 \) is stable if \( \lim_y c(x, y) \) exists for all \( x \).

\( \text{SRT}_2^2 \). Every stable coloring has an infinite homogeneous set.

**Theorem** (Cholak, Jockusch, and Slaman). \( \text{RT}_2^2 \equiv_{SW} \text{SRT}_2^2 \cdot \text{COH} \).

A set \( L \) is limit-homogeneous for a stable coloring \( c \) if there is an \( i \in \{0, 1\} \) such that \( \lim_y c(x, y) = i \) for all \( x \in L \).

\( \text{D}_2^2 \). Every stable coloring has an infinite limit-homogeneous set.

**Observation.** \( \text{SRT}_2^2 \equiv_c \text{D}_2^2 \).

Pf. Thin out a limit-homogeneous set to a homogeneous one.

**Theorem** (Chong, Lempp, and Yang). \( \text{RCA}_0 \vdash \text{SRT}_2^2 \iff \text{D}_2^2 \).
Two versions of Ramsey's theorem.

**Theorem** (Hirschfeldt and Jockusch). $\text{SRT}_2^2 \leq_W \text{D}^2_2 \bullet \text{D}^2_2$.

**Question** (Hirschfeldt and Jockusch). Does $\text{SRT}_2^2 \leq_W \text{D}^2_2$? Does $\text{SRT}_2^2 \leq_{sc} \text{D}^2_2$?

If $L$ is limit-homogeneous, but we do not know what color $i \in \{0, 1\}$ the elements in it limit to, then thinning it to a homogeneous set seems difficult.

**Theorem** (Dzhafarov). $\text{SRT}_2^2 \not\leq_W \text{D}^2_2$.

**Theorem** (Dzhafarov). There is a stable coloring $c$ such that every other stable coloring $d$ has an infinite limit-homogeneous set $L$ that computes no infinite homogeneous set for $c$.

**Corollary.** $\text{SRT}_2^2 \not\leq_{sc} \text{D}^2_2$. 
COH and $D_2^2$.

**Open question** (Chong, Slaman, and Yang). Does SRT$_2^2$ (or $D_2^2$) imply COH in $\omega$-models of RCA$_0$? Is COH $\leq_c$ SRT$_2^2$? Equivalently, is COH $\leq_c$ $D_2^2$?

**Theorem** (Dzhafarov, 2012). COH $\not\leq_{sc} D_2^2$.

The proof is a computable forcing argument. Any 3-generic yields a family $\langle X_0, X_1, \ldots \rangle$ witnessing the theorem, so we can find one computable in $\emptyset^{(3)}$.

**Theorem** (Hirschfeldt and Jockusch; Patey). There is a family of sets $X = \langle X_0, X_1, \ldots \rangle$ such that every stable coloring $d$ has an infinite limit-homogeneous set $L$ that computes no infinite $X$-cohesive set.

The $X$ built by Hirschfeldt and Jockusch is non-hyperarithmetical. Patey's is $\Delta^0_2$.

**Question.** Given the differences between SRT$_2^2$ and $D_2^2$ under $\leq_W$ and $\leq_{sc}$, what relationships hold between COH and SRT$_2^2$?
COH and SRT$_2^2$.

It is possible to elaborate on the proof that COH $\not\preccurlyeq_{W} D_2^2$ to obtain:

**Theorem** (Dzhafarov). COH $\not\preccurlyeq_{W} SRT_2^2$ (via a computable instance).

Homogeneous sets, unlike limit-homogeneous ones, have internal structure.

E.g., suppose we are building a family of colorings $\vec{c}$ and $\Phi\vec{c}$ is to be stable.

To build a limit-homogeneous set $L$ for $\Phi\vec{c}$, we can build a finite portion $F$ of $L$, and only later extend $\vec{c}$, say in a way to diagonalize some computation from $F$.

By Seetapun's argument, $F$ can be chosen so that its elements' limits agree.

But to build a homogeneous set $H$ for $\Phi\vec{c}$, we cannot delay building $\vec{c}$ in this way because homogeneity of any finite set directly depends on it.
Tree labeling method.

We define a certain subtree of $\omega^{<\omega}$ with labels on its nodes corresponding to diagonalization opportunities.

Paths give trivial wins (e.g., solutions that don't compute infinite sets).

If the tree is well-founded, we can use the labels to guide the construction of a homogeneous set.

**Theorem** (Dzhafarov). $\text{COH} \not\leq_{\text{sc}} \text{SRT}^2_2$.

The tree labeling method is quite powerful for separating principles under $\leq_{\text{sc}}$.

**Theorem** (Dzhafarov, Patey, Solomon, Westrick). If $j > k$ then $\text{RT}^1_j \not\leq_{\text{sc}} \text{SRT}^2_k$.

**Theorem** (Nichols). $\text{SRT}^2_2 \not\leq_{\text{sc}} \text{SPT}^2_2$. 
Hyperarithmetic instances.

The tree labeling method involves iteratively taking paths through subtrees of \( \omega^{<\omega} \) so the instances it produces are non-hyperarithmetical.

Open question. Can the tree labeling method be made more effective?

Recall that a set \( X \) has a self-modulus if there is a function \( f \equiv_T X \) such that \( X \leq_T g \) from every function \( g > f \). By a result of Solovay, \( X \) is hyperarithmetical.

Observation. If \( \text{COH} \nless_{sc} \text{SRT}^2 \) via an instance \( \vec{c} = \langle c_0, c_1, \ldots \rangle \) that has a self-modulus, then \( \text{COH} \nless_c \text{SRT}^2 \).

Theorem (Dzhafarov, Patey, Solomon, Westrick). \( \text{COH} \nless_{sc} \text{SRT}^2 \) via an instance \( \vec{c} \) computable in \( \emptyset^{(\omega)} \) (and so at least hyperarithmetic).

Open question. Can the instance \( \vec{c} \) be chosen \( \Delta^0_2 \)?
Thank you.