The Lovász local lemma and restrictions of Hindman's theorem

Damir D. Dzhafarov
University of Connecticut

June 6, 2018

Joint work with Csima, Hirschfeldt, Jockusch, Solomon, and Westrick.

Hindman's finite sums theorem

Given $A \subseteq \mathbb{N}$, let $\mathrm{FS}(A)$ denote the set of all finite non-empty sums of elements of A.

Hindman's theorem (HT). For every $k \geq 1$ and every $c : \mathbb{N} \to k$, there is an infinite set $H \subseteq \mathbb{N}$ such that c is constant on $\mathrm{FS}(H)$.

When we restrict HT to k-colorings for a specific k, we denote it by HT_k.

- Original proof by Hindman (1972), simplified by Baumgartner (1974).
- Ultrafilter proof by Galvin and Glazer (1977).
- Dynamics proof by Furstenburg and Weiss (1978).
- Reverse mathematics: Blass, Hirst, and Simpson (1987).
- A much simpler combinatorial proof by Towsner (2012).

Comparison with Ramsey's theorem

Given $A \subseteq \mathbb{N}$ and $n \geq 1$, let $[A]^n = \{(x_1, \dots, x_n) \in A^n : x_1 < \cdots < x_n\}$.

A set $H \subseteq \mathbb{N}$ is homogeneous for $c : [\mathbb{N}]^n \to k$ if c is constant on $[H]^n$.

Ramsey's theorem (RT). For all $n, k \geq 1$, every $c : [\mathbb{N}]^n \to k$ has an infinite homogeneous set.

RT^n_k denotes the restriction to a specific n and k.

There are also many proofs of RT, but many are quite elementary.

Example. How do you build 3-element solution to RT?

- Trivial for $n = 1$ and $n = 3$, not meaningful for $n > 3$.
- Given $c : [\omega]^2 \to 2$, how do you build a 3-element homogeneous set?

A 3-element solution to HT

Claim. Every $c : \mathbb{N} \to \{R, B\}$ is constant on $\mathrm{FS}(F)$ for some 3-element set F.

Proof. WLOG, say $c(0) = B$. We may assume $\exists^\infty x\, [c(x) = B]$.

If there exist positive $x < y$ with $c(x) = c(y) = c(x + y) = B$, take $F = \{0, x, y\}$. So assume not.

Fix $x_1 < x_2 < \cdots < x_6$ such that $c(x_i) = B$ for each i and the difference between any two consecutive x_i's is different.

Let $d_i = x_{i+1} - x_i$.

x_1 x_2 x_3 x_4 x_5 x_6

d_1 d_2 d_3 d_4 d_5

A 3-element solution to HT

$$\begin{array}{ccccccccccc} x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 \\ & d_1 & & d_2 & & d_3 & & d_4 & & d_5 & \end{array}$$

By assumption, it must be that $c(d_i) = R$ for each i.

A 3-element solution to HT

$$\begin{array}{ccccccccccc} x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 \\ & d_1 & & d_2 & & d_3 & & d_4 & & d_5 & \end{array}$$

By assumption, it must be that $c(d_i) = R$ for each i.

A 3-element solution to HT

$$\begin{matrix} x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 \\ & d_1 & & d_2 & & d_3 & & d_4 & & d_5 & \end{matrix}$$

By assumption, it must be that $c(d_i) = R$ for each i.

Similarly, the sum of any consecutive d_i's must also be colored R by c.

Finally, it cannot be that $c(d_1 + d_4) = c(d_2 + d_5) = c(d_1 + d_2 + d_4 + d_5) = B$.

So if $c(d_1 + d_4) = R$, we can take $F = \{d_1,\ d_2 + d_3,\ d_4\}$.

If $c(d_2 + d_5) = R$, we can take $F = \{d_2,\ d_3 + d_4,\ d_5\}$.

And if $c(d_1 + d_2 + d_4 + d_5) = R$, we can take $F = \{d_1 + d_2,\ d_3,\ d_4 + d_5\}$.

HT and reverse mathematics

Blass, Hirst, and Simpson (1987) proved that every computable instance of HT has a solution computable from $0^{(\omega+2)}$, but not necessarily $0'$.

Adapting Jockusch's results on RT^3_2, they showed that there is a computable instance all of whose solutions compute $0'$.

Theorem (Blass, Hirst, and Simpson, 1987)**.**

- HT is provable in ACA_0^+.
- Over RCA_0, HT_2 implies ACA_0.

Thirty years later, this is still the state of the art.

There has been quite a bit of work on extensions of HT.

Two restrictions

Given $A \subseteq \mathbb{N}$ and $n \geq 1$, let $\mathsf{FS}^{\leq n}(A)$ denote the set of all non-empty sums of at most n elements of A.

Let $\mathsf{HT}^{\leq n}$ and $\mathsf{HT}_k^{\leq n}$ denote the obvious restrictions of HT and HT_k.

Question (Hindman, Leader and Strauss, 2003)**.** Is there a proof of $\mathsf{HT}^{\leq 2}$ that is not already a proof of the full HT?

From their paper: "It seems truly remarkable that this can be unknown."

Given $A \subseteq \mathbb{N}$ and $n \geq 1$, let $\mathsf{FS}^{=n}(A)$ denote the set of all sums of exactly n elements. Let $\mathsf{HT}^{=n}$ and $\mathsf{HT}_k^{=n}$ denote the obvious restrictions.

Obviously, $\mathsf{HT}_k \to \mathsf{HT}_k^{\leq n} \to \mathsf{HT}_k^{=n}$. Also, $\mathsf{RT}_k^n \to \mathsf{HT}_k^{=n}$.

HT for sums of length at most 2

A paradox:

- we know of no proof of $\mathsf{HT}_2^{\leq 2}$ other than the proof of the full HT,
- yet it is not at all clear how to show that $\mathsf{HT}_2^{\leq 2}$ is not computbaly true.

Recall that a coloring $c : [\mathbb{N}]^2 \to 2$ is stable if $(\forall x) \lim_y f(x, y)$ exists.

SRT_2^2 is the restriction of Ramsey's theorem to stable colorings.

Theorem (Dzhafarov, Jockusch, Solomon, and Westrick)**.**
Over RCA_0, $\mathsf{HT}_2^{\leq 2}$ implies SRT_2^2.

Thus, in particular, there is a computable instance of $\mathsf{HT}_2^{\leq 2}$ with no computable solution.

Apartness

Fix $b \geq 2$ and $x \in \mathbb{N}$. If $x = i_0 \cdot b^{e_0} + \cdots + i_t \cdot b^{e_t}$ where $i_0, \ldots, i_t \in \{1, \ldots, b-1\}$ and $e_0 < \cdots < e_t$, let $\lambda_b(x) = e_0$ and $\mu_b(x) = e_t$.

Say two natural numbers $x < y$ are <u>b-apart</u> if $\mu_b(x) < \lambda_b(y)$.

HT **with b-apartness** is the statement of HT in which all elements of the monochromatic are required to be pairwise b-apart.

Facts.

- For each k, $b \geq 2$, RCA_0 proves $HT_k \leftrightarrow HT_k$ with b-apartness.
- For each $b \geq 2$, RCA_0 proves HT $\leftrightarrow$ HT with b-apartness.

In fact, all of these are strong computable equivalences.

The proof that HT implies HT with b-apartness does not lift to also show $HT^{\leq n}$ with b-apartness implies $HT^{\leq n}$ with b-apartness.

HT with apartness

Theorem (Carlucci, Kołodziejczyk, Lepore, and Zdanowski, 2017)**.**

- For any $b \geq 2$, RCA_0 proves that $\mathsf{HT}^{\leq 2}_2$ with b-apartness implies ACA_0.
- RCA_0 proves that $\mathsf{HT}^{\leq 2}_4$ implies ACA_0.

The apartness condition is not really "cheating". It is used in most proofs of/from Hindman's theorem, and was present in the original formulation. It can also be recast as a natural principle, the Finite unions theorem.

Corollary. Our best bounds for $\mathsf{HT}^{\leq 2}$ are the same as for the full HT.

A note on strong reductions

- Our proof that $\mathsf{HT}^{\leq 2}_2 \to \mathsf{SRT}^2_2$ actually shows that $\mathsf{SRT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{\leq 2}_2$.
- Carlucci (2017) showed that $\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{\leq 2}_4$, where IPT^2_2 is the strictly stronger increasing polarized Ramsey's theorem for pairs.

HT for sums of length exactly 2

$\mathsf{HT}^{=n}_k$ is an obvious corollary of RT^n_k.

Theorem (Carlucci, Kołodziejczyk, Lepore, and Zdanowski, 2017)**.**

If $n|m$ then $\mathsf{HT}^n \leq_{\mathrm{sc}} \mathsf{HT}^m$.

Proof.

Fix $c : \mathbb{N} \to k$. Say $m = nd$. Let $H = \{x_1 < x_2 < \cdots\}$ be an infinite set such that c is constant on $\mathsf{FS}^{=m}(H)$. Now define G to be the set $\{x_1 + \cdots + x_d,\ x_{d+1} + \cdots + x_{2d+1},\ \ldots\}$. Then c is constant on $\mathsf{FS}^{=n}(G)$.

Theorem (Carlucci, Kołodziejczyk, Lepore, and Zdanowski, 2017)**.**

For any $n \geq 3$, $b \geq 2$, $\mathsf{HT}^{=n}$ with b-apartness is equivalent to ACA_0.

What about $\mathsf{HT}^{=2}$? Can we at least show it's not computably true?

Diagonalization strategy

We want to build a computable coloring $c : \mathbb{N} \to 2$.

For each e, wait for a certain-sized finite $F_e \subseteq W_e$ to be enumerated.

For sufficiently large s, ensure $F_e + s$ is not homogeneous.

Dealing with a single c.e. set W.

- Wait for some $x < y$ in W to be enumerated into W. Let $d = y - x$.
- For each $s \leq d$ let $c(s) = 0$.
- For $s > d$, having inductively defined $c \upharpoonright s$, define $c(s) = 1 - c(s - d)$.
- Now $c(y + s) = 1 - c(y + s - d) = 1 - c(x + s)$ for all large enough s.

Diagonalization strategy

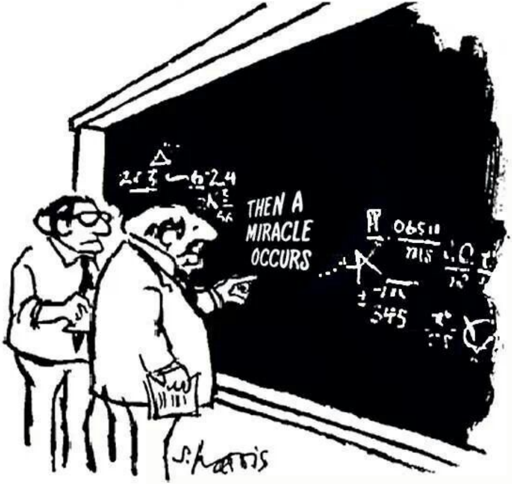
The basic strategy fails even for two c.e. sets, W_0 and W_1.

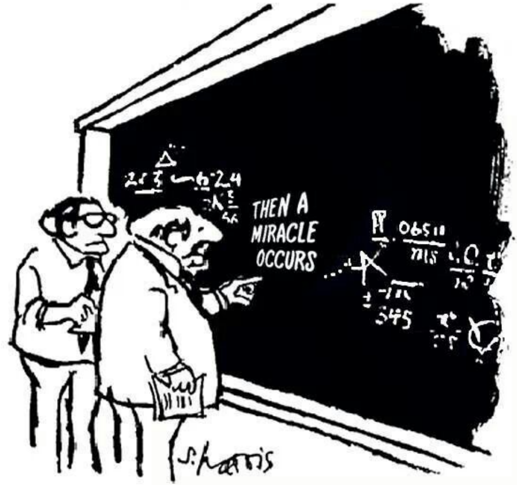
Example.

- Suppose $F_0 = \{0, 1\}$ and $F_1 = \{0, 2\}$.
- Then for all s, one of $F_0 + s$, $F_1 + s$, $F_0 + (s + 1)$ must be homogeneous.

This failure gives us some insights.

- The probability that $F_e + s$ is homogeneous is only $2^{-|F_e|+1}$.
- If $s < t$ are far enough apart, then $F_e + s$ and $F_i + t$ are disjoint.


THEN A MIRACLE OCCURS
S. Harris



* Thanks to Jason Bell and Jeff Shallit (U Waterloo).

An application of the Lovász local lemma

Consider a collection $x_0, x_1, \ldots$ of independent binary random variables.

A clause is a finite sequence $x_{n_0} = i_0 \vee \cdots \vee x_{n_k} = i_k$, where $i_0, \ldots, i_k \in \{0, 1\}$.

A CNF is an infinite conjunction of clauses.

A satisfying assignment for a CNF is a map $c : \mathbb{N} \to \{0, 1\}$ such that each conjunct in the CNF has a disjunct $x_n = i$ and $c(n) = i$.

Theorem (Rumyantsev and Shen, 2014).

For every $\alpha \in (0, 1)$, there exists an $N \in \mathbb{N}$ such that every computable infinite CNF in which all clauses have size at least N, and for all $m \geq N$, every variable appears in at most $2^{\alpha m}$ clauses of size m, has a computable satisfying assignment.

An application of the Lovász local lemma

Theorem (Rumyantsev and Shen, 2014).

For every $\alpha \in (0, 1)$, there exists an $N \in \mathbb{N}$ such that every computable infinite CNF in which all clauses have size at least N, and for all $m \geq N$, every variable appears in at most $2^{\alpha m}$ clauses of size m, has a computable satisfying assignment.

Let $\alpha = 0.5$. Fix N as above. For each e, wait for $F_e \subseteq W_e$ of size $N + e$.

Take the CNF whose clauses are $\bigvee_{n \in F_e + s} x_n = 0$ and $\bigvee_{n \in F_e + s} x_n = 1$ for all sufficiently large s.

If c is a satisfying assignment and W_e is infinite, then c is not homogeneous on $F_e + s$ for all sufficiently large s.

Corollaries

Theorem (Csima, D., Hirschfeldt, Jockusch, Solomon, and Westrick)**.**

There exists a computable instance of $\mathsf{HT}_2^{=2}$ with no computable solution.

Corollary. RCA_0 does not prove $\mathsf{HT}_2^{=2}$.

A modification of the argument also yields the following:

Theorem (Csima, D., Hirschfeldt, Jockusch, Solomon, and Westrick)**.**

There exists a computable instance of $\mathsf{HT}_2^{=2}$ every solution of which computes a $\mathsf{DNC}(0')$ function.

Corollary. RCA_0 proves $\mathsf{HT}_2^{=2} \to \mathsf{RRT}_2^2$.

Here, RRT_2^2 is the Rainbow Ramsey's theorem for pairs.

Ramseyan factorization theorem

Murakami, Yamazaki, and Yokoyama introduced the following principle in connection with their work on the Ramseyan factorization theorem.

Fix $n, k \geq 1$ and $f : [\mathbb{N}]^n \to \mathbb{N}$.

RT^f_k is the statement that for every $c : \mathbb{N} \to k$ there is an infinite set $H \subseteq \mathbb{N}$ such that $c \circ f$ is constant on $[H]^n$.

If $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ for all $x_1, \ldots, x_n \in \mathbb{N}$ then $\mathsf{RT}^f_k = \mathsf{HT}^{=n}_k$.

Theorem (Murakami, Yamazaki, and Yokoyama, 2014)**.**

- RCA_0 proves $\mathsf{RT}^n_k \to (\forall f : [\mathbb{N}]^n \to \mathbb{N})\ \mathsf{RT}^f_k$.
- If $f : [\mathbb{N}]^n \to \mathbb{N}$ is a bijection then $\mathsf{RT}^f_k \leftrightarrow \mathsf{RT}^n_k$ over RCA_0.

Addition-like functions

A computable function $f : [\mathbb{N}]^2 \to \mathbb{N}$ is addition-like if

- there is a computable function g such that $y > g(x, n) \to f(x, y) > n$,
- there is a b such that $|\{y : f(x, y) = k\}| < b$ for all $x, k \in \mathbb{N}$.

Examples.

- Addition.
- Subtraction/difference.

Theorem (Csima, D., Hirschfeldt, Jockusch, Solomon, and Westrick).

For each addition-like f, there exists a computable instance of RT^f_2 all of whose solutions compute a $\mathsf{DNC}(0')$ function.

Corollary. For each addition-like f, RCA_0 proves $\mathsf{RT}^f_2 \to \mathsf{RRT}^2_2$.

Further applications

Theorem (Cholak, D., Hirschfeldt, and Patey)**.**

There exists an instance of $\mathsf{HT}_2^{=2}$ such that the class of oracles that compute a solution to c has measure 0.

OVW(2, 2) is the Ordered variable word problem for 2-element alphabets.

Miller and Solomon (2004) constructed a computable instance of OVW(2, 2) with no computable solution.

Theorem (Liu, Monin, and Patey, 2018)**.**

There exists a computable instance of OVW(2, 2) all of whose solutions compute a DNC($0'$) function.

Thanks for your attention!