Reverse mathematics and the game-theoretic framework

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Reverse math, in one slide

Reverse mathematics is a foundational program for calibrating the computable and proof-theoretic content of mathematical principles.

Various subsystems of Z_2 are used as benchmarks against which to test the strength of theorems we are interested in: RCA_0, WKL, ACA_0, ...

RCA_0 consists of the algebraic axioms about the natural numbers, plus Δ^0_1-comprehension and Σ^0_1-induction.

A model of RCA_0 is a pair $(N, \mathcal{S})$, where N is a (possibly nonstandard) first-order structure, and $\mathcal{S} \subseteq \mathcal{P}(N)$ is closed under Δ^0_1-definability.

An ω-model is a model $(N, \mathcal{S})$ with $N = \omega$, which can thus be identified just with $\mathcal{S}$. If $\mathcal{S} \models RCA_0$ then $\mathcal{S}$ is a Turing ideal.

The computability-theoretic perspective

We are interested in statements of the form

$$\forall X\,[\Phi(X) \to \exists Y\,\Psi(X, Y)],$$

where Φ and Ψ are some kind of properties of X and Y.

We think of this as a problem, "given X satisfying Φ, find Y satisfying Ψ".

We call the X such that $\Phi(X)$ holds the instances of the problem,
and the Y such that $\Psi(X, Y)$ holds the solutions to X for this problem.

Typically, we look at problems whose instances and solutions are subsets of $\mathbb{N}$,
and where the properties Φ and Ψ are arithmetical.

Basic question. Given an instance of a problem, how complex are its solutions?

Measuring complexity

Computability theory:

- Does every instance compute a solution to itself?
- Does every instance have an arithmetically-definable solution?
- Is there a computable instance all of whose solutions compute $\emptyset'$?

Reverse mathematics/proof theory:

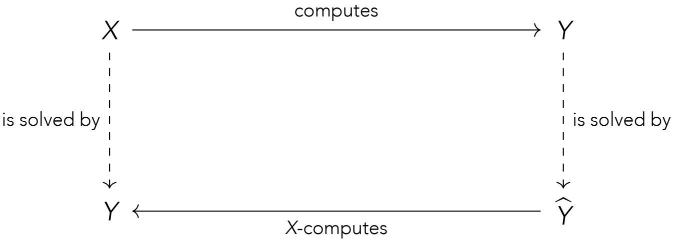
- Is the theorem provable in RCA_0?
- Is the theorem provable in ACA_0?
- Does the theorem imply ACA over RCA_0?

There is well-understood relationship between these viewpoints, somewhat complicated by induction issues.

Other ways to compare problems

Defn. Let P, Q be problems. P is computably reducible to Q, written $\mathsf{P} \leq_c \mathsf{Q}$, if:

- for every P-instance X, there is a Q-instance $Y \leq_T X$, such that
- for every solution $\widehat{Y}$ to Y, there is a solution $\widehat{X}$ to X with $\widehat{X} \leq_T X \oplus \widehat{Y}$.

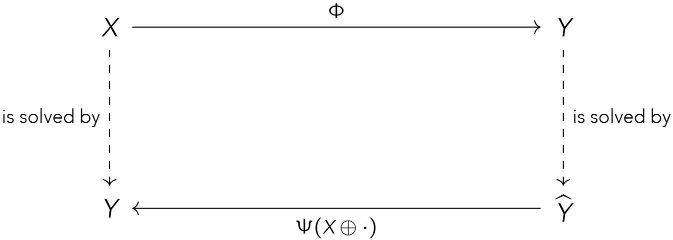


First defined by **Dzhafarov (2015)**, but informally used widely in RM literature.

Other ways to compare problems

Defn. Let P, Q be problems. P is Weihrauch reducible to Q, written $\mathsf{P} \leq_{\mathsf{W}} \mathsf{Q}$, if there are Turing functionals Φ, Ψ such that:

- for every P-instance X, we have that $\Phi(X)$ is a Q-instance, and
- for every Q-solution $\widehat{Y}$ to $\Phi(X)$, we have that $\Psi(X \oplus \widehat{Y})$ is a P-solution to X.



Dev'd by **Weihrauch (1990)**, then **Gherardi & Marcone (2008)** and **Brattka & Marcone (2011)**. Ind. by **Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2016)**.

A refinement of reverse math in ω-models

Clearly, if $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ then $\mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$.

If P and Q are Π^1_2 statements and $\mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$, then $\mathsf{Q} \to \mathsf{P}$ in all ω-models.

Example.

Recall that RT^n_k is Ramsey's theorem for k-colorings of n-tuples of ω.

- instances: colorings $c : [\omega]^n \to k = \{0, 1, \ldots, k-1\}$;
- solutions to such a c: all infinite $H \subseteq \omega$ such that c is constant on $[H]^n$.

For each n, it is easy to see that $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_j \leftrightarrow \mathsf{RT}^n_k$ for all j and k.

Thm (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). $\mathsf{RT}^1_k \not\leq_{\mathrm{W}} \mathsf{RT}^1_j$ for all $j < k$.

Thm (Patey). $\mathsf{RT}^n_k \not\leq_{\mathrm{c}} \mathsf{RT}^n_j$ for all $j < k$ and $n \geq 2$.

Hirschfeldt-Jockusch games

Let P and Q be problems. $G(\mathsf{Q} \to \mathsf{P})$ is the following game:

move	Player 1	Player 2
1	X_0 P-instance	$Y_1 \leq_T X_0$ either P-solution to X_0 or Q-instance
2	X_1 Q-solution to Y_1	$Y_2 \leq_T X_0 \oplus X_1$ either P-solution to X_0 or Q-instance
⋮	⋮	⋮
n	X_{n-1} Q-solution to Y_{n-1}	$Y_n \leq_T X_0 \oplus \cdots \oplus X_{n-1}$ either P-solution to X_0 or Q-instance
⋮	⋮	⋮

Player 2 wins if it ever plays a P-solution to X_0, or if Player 1 has no valid move.

Player 1 wins otherwise. When either player wins, the game stops.

A game-theoretic characterization

Thm (Hirschfeldt and Jockusch). Let P and Q be problems. If every ω-model of $\mathsf{RCA}_0 + \mathsf{Q}$ is a model of P, then Player 2 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$. Otherwise, Player 1 has a winning strategy.

Recall that $\leq_W \implies \leq_c \implies$ implication over ω-models.

Note that this also captures computable reducibility: $\mathsf{P} \leq_c \mathsf{Q}$ iff Player 2 has a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$ that wins in exactly two moves.

Defn. A strategy for Player 2 is computable if there is a Turing functional Φ such that $Y_n = \Phi(n, X_0 \oplus \cdots \oplus X_{n-1})$.

Now $\mathsf{P} \leq_W \mathsf{Q}$ iff Player 2 has a computable winning strategy for $G(\mathsf{Q} \to \mathsf{P})$ that wins in exactly two moves.

Counting moves

Example. Player 2 has a computable winning strategy for $G(\mathrm{RT}^1_2 \to \mathrm{RT}^1_{<\infty})$.

- Player 1 plays $c : \omega \to 3 = \{0, 1, 2\}$.
- Player 2 plays $d_1 : \omega \to 2$ defined by $d_0(x)$ is 0 if $c(x) = 0$ and 1 otherwise.
- Player 1 plays an infinite homogeneous set $H_1 = \{y_0 < y_1 < \cdots\}$ for d_1.
- Player 2 plays $d_2 : \omega \to 2$ defined by $d_1(x)$ is 0 if $c(y_x) = 1$ and 1 otherwise.
- Player 1 plays an infinite homogeneous set H_2 for d_2.
- Player 2 plays $\{y_x \in H_1 : x \in H_2\}$ as an infinite homogeneous set for c.

Thm (Hirschfeldt and Jockusch). There is no n s.t. Player 2 has a computable winning strategy for $G(\mathrm{RT}^1_2 \to \mathrm{RT}^1_{<\infty})$ that wins in $\leq n$ moves.

Compare with: $\mathrm{RCA}_0 \vdash \mathrm{RT}^1_2$, but by results of **Hirst**, $\mathrm{RCA}_0 \nvdash \mathrm{RT}^1_{<\infty}$.

Generalized games

Let P, Q be problems; Γ a set of L_2 formulas. $G^\Gamma(\mathrm{Q} \to \mathrm{P})$ is the game:

move	Player 1	Player 2
1	M - L_1-structure X_0 - P-instance s.t. $M[X_0]$ is consistent with Γ	$Y_1 \in M[X_0]$ either P-solution to X_0 or Q-instance
2	X_1 Q-solution to Y_1 s.t. $M[X_0, X_1]$ is consistent with Γ	$Y_2 \in M[X_0, X_1]$ either P-solution to X_0 or Q-instance
3	X_2 Q-solution to Y_1 s.t. $M[X_0, X_1, X_2]$ is consistent with Γ	$Y_3 \in M[X_0, X_1, X_2]$ either P-solution to X_0 or Q-instance
$\vdots$	$\vdots$	$\vdots$

Generalized games

Let P, Q be problems; Γ a set of L_2 formulas. $\widehat{G}^{\Gamma}(\mathrm{Q} \to \mathrm{P})$ is the game:

move	Player 1	Player 2
1	$(M, \mathcal{S}) \models \Gamma$ $X_0 \in \mathcal{S}$ P-instance	$Y_1 \in M[X_0]$ either P-solution to X_0 or Q-instance
2	$X_1 \in \mathcal{S}$ Q-solution to Y_1	$Y_2 \in M[X_0, X_1]$ either P-solution to X_0 or Q-instance
3	$X_2 \in \mathcal{S}$ Q-solution to Y_1	$Y_3 \in M[X_0, X_1, X_2]$ either P-solution to X_0 or Q-instance
$\vdots$	$\vdots$	$\vdots$

Characterizing provability

In order of difficulty, from hardest for Player 2, to hardest for Player 1, we have:

- $G^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ $\quad G^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$ $\quad \widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ $\quad \widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$

Prop (Dzhafarov, Hirschfeldt, and Reitzes). Let P, Q be problems. Let Γ be a consistent extension of Δ^0_1-CA by Π^1_1 formulas.

1) If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$, then Player 2 has a winning strategy for $G^{\Gamma}(\mathsf{Q} \to \mathsf{P})$.

2) Otherwise, Player 1 has a winning strategy for $\widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$.

Thm (Dzhafarov, Hirschfeldt, and Reitzes). Let P, Q be problems. Let Γ be a consistent extension of Δ^0_1-CA by Π^1_1 formulas including a universal Σ^0_1 formula.

1) If $\Gamma \vdash \mathsf{Q} \to \mathsf{P}$, there is an $n \in \omega$ such that Player 2 has a winning strategy for $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ that ensures victory in at most n moves.

2) Otherwise, Player 1 has a winning strategy for $\widehat{G}^{\Gamma+\mathsf{Q}}(\mathsf{Q} \to \mathsf{P})$.

Applications

Thm (folklore). If $\mathsf{ACA}_0 \vdash \mathsf{P}$ then there is an $n \in \omega$ such that every instance X of P has a solution computable in $X^{(n)}$.

Pf. If $\mathsf{ACA}_0 \vdash \mathsf{P}$ then $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$, where $\mathsf{Q} = (\forall X)[X' \text{ exists}]$. Fix n from theorem. Consider the game $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ where on its first move, Player 1 plays the ω-model $\{Y : Y \leq_T X^{(n)}\}$ and the P-instance X.

Note that if the Π^1_1 formulas added to Δ^0_1-CA to get Γ are true, then a winning strategy for $\widehat{G}^{\Gamma}(\mathsf{Q} \to \mathsf{P})$ yields a winning strategy for $G(\mathsf{Q} \to \mathsf{P})$.

Patey showed there is no bound on Player 2's moves in $G(\mathsf{RT}^2_2 \to \mathsf{RT}^2_{<\infty})$.

Cor. $\mathsf{RT}^2_k \nvdash \mathsf{RT}^2_{<\infty}$ even over RCA_0 + all true Π^1_1 formulas.

By an old result of **Cholak, Jockusch, and Slaman**, $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_{<\infty}$. The proof exploits a failure of arithmetical induction, a true Π^1_1 statement.

Proof of the main theorem

Lem. For $n \in \omega$, let $\Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n)$ be the formula

if X_0 is a P-instance then $(Y_0 = \Phi_{e_0}^{X_0}$ and Y_0 is a P-solution to X_0, or

Y_0 is a Q-instance, and

if X_1 is a Q-solution to Y_0 then $(Y_1 = \Phi_{e_1}^{X_0 \oplus X_1}$ and Y_1 is a P-solution to X_0, or

Y_1 is a Q-instance, and

if X_2 is a Q-solution to Y_1 then $(Y_2 = \Phi_{e_2}^{X_0 \oplus X_1 \oplus X_2}$ and Y_2 is a P-solution to X_0, or

$\vdots$

$\ldots (Y_n = \Phi_{e_n}^{X_0 \oplus \cdots \oplus X_n}$ and Y_n is a solution to $X_0)) \cdots)$.

Let Δ_n be $\forall X_0 \exists e_0, Y_0 \cdots \forall X_n \exists e_n, Y_n \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n)$.

If $\Gamma \vdash Q \to P$ then $\Gamma \vdash \Delta_n$ for some $n \in \omega$.

Proof of the main theorem

Suppose $\Gamma \vdash Q \rightarrow P$ but $\Gamma \nvdash \Delta_n$ for all n.

Expand L_2 to L_2' by adding a function symbol f from (strings of) first-order objects to second-order objects.

For each n, there is a model $\mathcal{M} = (M, S) \models \Gamma + \neg\Delta_n$. We can expand this to an L_2'-structure using the failure of Δ_n to define $f^{\mathcal{M}}$ as a Skolem function.

$\mathcal{M}$ and $f^{\mathcal{M}}$ then satisfies Γ together with $\Psi_n =$
$\forall e_0, Y_0 \cdots \forall e_n, Y_n \neg\Theta_n(e_0, \ldots, e_k, f(\langle\rangle), f(\langle e_0\rangle), \ldots, f(\langle e_0 \cdots e_n\rangle), Y_0, \ldots, Y_n)$.

By compactness, there is a model $\mathcal{N}$ of $\Gamma \cup \{\Psi_1, \Psi_2, \ldots\}$.

But now $f^{\mathcal{N}}$ gives a winning strategy for Player 1 in $G^{\Gamma}(Q \rightarrow P)$.

Extensions

We don't know if in our theorem we can replace $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ by $G^\Gamma(\mathsf{Q} \to \mathsf{P})$.

But we can do so for natural cases, where the strategy is definable.

Prop (Dzhafarov, Hirschfeldt, and Reitzes. Let P, Q be problems. Let Γ be a consistent extension of Δ^0_1-CA by Π^1_1 formulas. TFAE:

1) Player 2 has a computable winning strategy for $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$.

2) Player 2 has a computable winning strategy for $G^\Gamma(\mathsf{Q} \to \mathsf{P})$.

3) There is an $n \in \omega$ s.t. Player 2 has a computable winning strategy that wins every run of $\widehat{G}^\Gamma(\mathsf{Q} \to \mathsf{P})$ in at most n moves.

4) There is an $n \in \omega$ s.t. Player 2 has a computable winning strategy that wins every run of $G^\Gamma(\mathsf{Q} \to \mathsf{P})$ in at most n moves.

The n's in (3) and (4) are the same.

Variations of $\mathsf{B}\Sigma^0_2$

Fix a class Γ of L_2-formulas. Recall the following scheme:

$\mathsf{B}\Gamma$: $(\forall n)[(\forall x < n)(\exists y)\phi(x,y) \to (\exists b)(\forall x < n)(\exists y < b)\phi(x,y)]$,

where $\phi \in \Gamma$.

A frequently encountered principle in reverse math is $\mathsf{B}\Sigma^0_2$. It's not provable in RCA_0, but it shows up naturally in many RM arguments. It has many important equivalent formulations.

Thm (Hirst). Over RCA_0, $\mathsf{B}\Sigma^0_2$ is equivalent to $\mathsf{RT}^1_{<\infty}$.

By our theorem, such equivalences correspond with the existence of winning strategies for Player 2 in the game $\widehat{G}^{\mathsf{RCA}_0}$.

Question. When can these strategies be chosen to be computable?

Example: the limit homogeneous problem

Consider the following principle:

LH: For every $c : [\omega]^2 \to 2$ such that $\lim_y c(x, y) = 1$ for all x, there is an infinite set H such that $c(x, y) = 1$ for all $x, y \in H$.

Observation. Over RCA_0, LH is equivalent to the bounding principle $\mathsf{B}\Sigma^0_2$.

Pf. ($\leftarrow$) Assume $\mathsf{B}\Sigma^0_2$. Fix an instance $c : [\omega]^2 \to 2$ of LH. There is a Δ^0_1-definable way to thin $\mathbb{N}$ to a homogeneous set for c. $\mathsf{B}\Sigma^0_2$ is used to prove that the thinning process is total.

($\rightarrow$) We prove LH $\rightarrow$ $\mathsf{RT}^1_{<\infty}$. Fix $d : \omega \to k$ and assume no homogeneous set for d is infinite. Define $c(x, y) = 0$ if $d(x) = d(y)$ and $c(x, y) = 1$ otherwise. Then $\lim_y c(x, y) = 1$ for all x. A homogeneous set for c contradicts d only using k many colors.

Example: the limit homogeneous problem

Prop (Dzhafarov, Hirschfeldt, and Reitzes). Player 2 has no computable winning strategy for $G^{\mathsf{RCA}_0}(\mathsf{RT}^1_{<\infty} \to \mathsf{LH})$. This is true even if the number of colors is not provided along with the instance of $\mathsf{RT}^1_{<\infty}$.

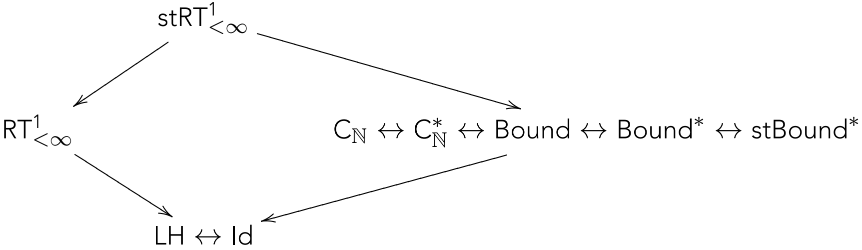
Pf idea. Player 1 begins by playing the standard model of $\mathbb{N}$ along with some initial segment of an instance of LH. Player 2, playing according to a purported computable strategy, responds by starting to define an instance of $\mathsf{RT}^1_{<\infty}$.

Player 1 guesses at which color in the coloring will be used infinitely often. Player 2's strategy must defeat all these guesses, which it can only do by increasing the number of colors. This can only happen finitely often.

Once Player 2 settles on a number of colors, Player 1 changes to playing in a 1-elementary extension of the standard model in which the portion of LH can be extended so as to have no definable solution. By 1-elementarity, Player 2's instance of $\mathsf{RT}^1_{<\infty}$ must use the same number of colors in both models.

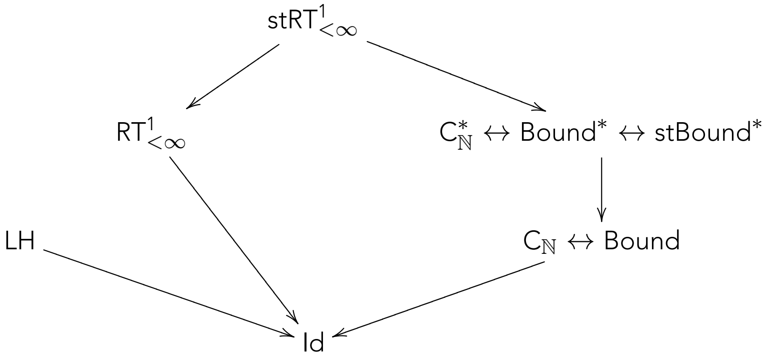
A finer measure of strength

$Q \to P$ means $P \leq_W Q$.



A finer measure of strength

Q $\to$ P means Player 2 has a computable winning strategy for $G^{\mathsf{RCA}_0}(\mathsf{Q} \to \mathsf{P})$.



Thanks for your attention!