The first-order parts of Weihrauch degrees

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Classical reverse mathematics
Reverse mathematics

Measures the strengths of (countable versions, or countable representations of) theorems of ordinary mathematics.

Subsystems of second-order arithmetic ($\mathbb{Z}_2$) serve as benchmarks.

Base subsystem. $\text{RCA}_0$ consists of:

- $\text{PA}^-$;
- recursive comprehension axiom ($\Delta^0_1$ comprehension);
- $\Sigma^0_1$ induction.

Stronger subsystems.

- $\text{WKL}_0 = \text{RCA}_0 + \text{Weak König’s lemma (WKL)}$;
- $\text{ACA}_0 = \text{RCA}_0 + \text{arithmetical comprehension (ACA)}$. 
Some principles

Second-order statements.

- **Weak König’s lemma (WKL)**: every infinite tree \( T \subseteq 2^\mathbb{N} \) has an infinite branch.

- **Weak weak König’s lemma (WWKL)**: every infinite tree \( T \subseteq 2^\mathbb{N} \) of positive measure has an infinite branch.

- **Ramsey’s theorem \((\text{RT}_k^n)\)**: every coloring \( c : [\omega]^n \to k \) has an infinite homogeneous set.

Kirby-Paris hierarchy

- \( B \Gamma \) is the following scheme: for every formula \( \phi \in \Gamma \),

\[
(\forall k)[(\forall x < k)(\exists y)\phi(x, y) \rightarrow (\exists j)(\forall x < k)(\exists y < j)\phi(x, y)].
\]

- \( I \Sigma_1^0 < B \Sigma_2^0 < I \Sigma_2^0 < B \Sigma_3^0 < I \Sigma_3^0 < \cdots \).
Reverse mathematics zoo
First-order parts

**Defn.** Let $T$ be a statement in the language of $Z_2$. The first-order part of $T$ is the set of arithmetical consequences of $\text{RCA}_0 + T$.

**Examples.**
- The first-order part of $\text{RCA}_0$ and $\text{WKL}_0$ is $\Sigma^0_1$-PA.
- The first-order part of $\text{ACA}_0$ is PA.

A combinatorial example.

Consider $(\forall k) \text{RT}^1_k$, i.e., the infinitary pigeonhole principle,

$$(\forall k)(\forall c : \omega \rightarrow k)(\exists H)[H \text{ infinite and } c \upharpoonright H \text{ constant}].$$

**Thm (Hirst 1987).** $\text{RCA}_0 \vdash \text{RT}^1 \iff B\Sigma^0_2$. 
The first-order part(s) of Ramsey’s theorem

\( \mathsf{RT}^n_k \): Every \( c : [\mathbb{N}]^n \rightarrow k \) has an infinite homogeneous set.

Thm.

- (Jockusch 1972). For all \( k \) and all \( n \geq 3 \), \( \mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{ACA}_0 \).

- (Liu 2011). \( \mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{WKL} \).

Thm (Hirst 1987). \( \mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{B} \Sigma^0_2 \) and \( \mathsf{RCA}_0 + (\forall k) \mathsf{RT}^2_k \vdash \mathsf{B} \Sigma^0_3 \).

Thm (Cholak, Jockusch, and Slaman 2001). \( \mathsf{RCA}_0 + (\forall k) \mathsf{RT}^2_k \) is \( \Pi^1_1 \)-conservative over \( \mathsf{I} \Sigma^0_3 \).

Thm (Slaman and Yokoyama 2016). \( \mathsf{RCA}_0 + \mathsf{RT}^2_2 \) is \( \Pi^1_1 \)-conservative over \( \mathsf{B} \Sigma^0_3 \).

Thm (Chong, Slaman, and Yang 2017). \( \mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{I} \Sigma^0_2 \).
Reverse math, the reboot
Instance-solution problems

Typical theorems studied in reverse mathematics have the canonical form

$$(\forall X)[\phi(X) \rightarrow (\exists Y)\psi(X, Y)],$$

where $\phi$ and $\psi$ are arithmetical predicates of reals.

We view this as a problem: given $X$ such that $\phi(X)$, find $Y$ such that $\psi(X, Y)$.

Defn. A problem is a partial multifunction $P : \subseteq \omega^\omega \Rightarrow \omega^\omega$.

The $P$-instances are the elements of $\text{dom}(P)$.

For each $X \in \text{dom}(P)$ the $P$-solutions to $X$ are the elements of $P(X)$.

Example. In $\text{RT}_2^2$, the instances are the colorings $c : [\omega]^2 \rightarrow 2$, and the solutions to such a $c$ are all the infinite homogeneous sets.
Computable reducibility

Defn (D. 2013). Let $P$ and $Q$ be problems. $P$ is computably reducible to $Q$, $P \leq_c Q$, if:

• every $P$-instance $X$ computes a $Q$-instance $\hat{X}$,

• for every $Q$-solution $\hat{Y}$ to $\hat{X}$, we have that $X \oplus \hat{Y}$ computes a $P$-solution $Y$ to $X$. 
Weihrauch reducibility

Defn (Weihrauch 1990). Let P and Q be problems. P is Weihrauch reducible to Q, $P \leq_W Q$, if there are Turing functionals $\Phi$, $\Psi$ s.t.:

- for every P-instance $X$, we have that $\Phi(X)$ is a Q-instance, and
- for every Q-solution $\hat{Y}$ to $\Phi(X)$, we have that $\Psi(X \oplus \hat{Y})$ is a P-solution $Y$ to $X$.

Equivalence classes under $\leq_W$ form the Weihrauch degrees, denoted $\mathcal{W}$. 
The Weihrauch lattice

Thm (Pauly 2010; Brattka and Gherardi 2011). Under suitable operations of $\lor$ and $\land$, $(\mathcal{W}, \leq_{\mathcal{W}}, \lor, \land)$ is a lattice.

Let $P_0$ and $P_1$ be problems.

- $P_0 \times P_1$ is the problem with domain $\text{dom}(P_0) \times \text{dom}(P_1)$, with the solutions to $(X_0, X_1)$ being all pairs $(Y_0, Y_1)$ such that $Y_i$ is a $P_i$-solution to $X_i$.
- $P_0^2 = P_0 \times P_0$; $P_0^{n+1} = P_0^n \times P_0$; $P_0^* = \bigcup_n P_0^n$.
- $P'_0$ is the problem with domain all $f : \omega^2 \to \omega$ such that $\lim_s f(x, s) \downarrow$ for all $x$, $X = \lim_s f \in \text{dom}(P)$, and the solutions to $f$ are all the $P_0$-solutions to $X$.
- $P_0^{(2)} = P''_0$; $P_0^{(n+1)} = (P_0^{(n)})'$.
- $P_0 \star P_1$ is the composition product of $P_1$ followed by $P_0$. Intuitively: “solve $P_1$ first, then use your solution to create an instance of problem $P_0$.”
A refinement of reverse mathematics

Implications over $\text{RCA}_0$ between $\Pi^1_2$ principles tend to be formalizations computable or Weihrauch (or stronger) reductions.

Example.

- For all $n, j, k$, we have $\text{RCA}_0 \vdash \text{RT}^n_k \leftrightarrow \text{RT}^n_j$.
- (Patey 2015.) If $j < k$ then $\text{RT}^n_k \not\leq_c \text{RT}^n_j$.

Defn. A coloring $c : [\omega]^2 \to 2$ is stable if $(\forall x) \lim_y c(x, y)$ exists. A set $X$ is limit-homogeneous for $c$ if $(\exists i)(\forall x \in X) \lim_y c(x, y) = i$.

$\text{SRT}^2_2$ is the restriction of $\text{RT}^2_2$ to stable colorings.

$\text{D}^2_2$ : Every stable coloring has an infinite limit-homogeneous set.

- (Chong, Lempp, and Yang 2011.) $\text{RCA}_0 \vdash \text{SRT}^2_2 \leftrightarrow \text{D}^2_2$.
- (D. 2016.) $\text{D}^2_2 \leq_W \text{SRT}^2_2$ but $\text{SRT}^2_2 \not\leq_W \text{D}^2_2$. 
First-order Weihrauch problems
First-order problems

**Defn.** A problem $P$ is **first-order** if $P(X) \subseteq \mathbb{N}$ for all $X \in \text{dom}(P)$.

Denote the collection of first-order problems by $\mathcal{FO}$.

**Examples.**

- **LPO**: instances: $0^n1^\omega \in 2^\omega$ for all $n \geq 0$; solutions: 0 if $n = 0$ and 1 otherwise.
- **lim$_\mathbb{N}$**: instances: convergent sequences $\langle x_i : i \in \mathbb{N} \rangle \subseteq \mathbb{N}$; solutions: $\lim_{i} x_i$.
- **C$_\mathbb{N}$**: instances: (co-enumerations of) non-empty sets $X \subseteq \mathbb{N}$; solutions: points in $X$.
- **K$_\mathbb{N}$**: instances: (co-enumerations of) non-empty bounded sets $X \subseteq \mathbb{N}$; solutions: points in $X$. 
Brattka’s question

\(C_N\) can be viewed as corresponding to \(I \Sigma^0_1\), and \(K_N\) as corresponding to \(B \Sigma^0_1\).

**Defn.**

- \(\max : \subseteq N^N \to N, p \mapsto \max\{p(n) : n \in N\}\).
- \(\min : N^N \to N, p \mapsto \min\{p(n) : n \in N\}\).

**Prop (Brattka).** \(\max \equiv_W C_N\) and \(\min \equiv_W K_N\).

We have the following hierarchy,

\[K_N <_W C_N <_W K'_N <_W C'_N <_W K''_N <_W C''_N <_W \cdots\]

which can thus be viewed as an analogue of the Kirby-Paris hierarchy.
First-order parts of Weihrauch degrees

**Defn.** Let $P$ be a problem. The first-order part of $P$, denoted $^1P$, is

$$\sup_{\leq_w}\{R \in \mathcal{FO} : R \leq_w P\}.$$ 

**Prop (DSY).** $^1P$ exists, for every $P$.

**Proof.** Let $Q$ to be the following problem:

- the instances are all pairs $(X, \Psi)$ such that $X \in \text{dom}(P)$ and $\Psi(X, Y)(0) \downarrow$ for all $P$-solutions $Y$ to $X$;
- the solutions to $(X, \Psi)$ are all $y \in \mathbb{N}$ such that $\Psi(X, Y)(0) \downarrow= y$ for some $P$-solution $Y$ to $X$.

Then $Q \equiv_w ^1P$. 
Basic facts

Obs. If \( P \in \mathcal{FO} \) then \(^1P \equiv_W P\).

Defn. Let \( P \) be a problem. Then \( P \) is

- computably true if \( P \leq_c \text{Id.} \)
- uniformly computably true if \( P \leq_W \text{Id.} \)

Prop (DSY). If \(^1P\) is uniformly computably true then \(^1(P \times Q) \equiv_W ^1Q\).

Prop (DSY). A problem \( P \) is computably true iff \( P \leq_W Q \) for some \( Q \in \mathcal{FO} \).

Proof. Clearly if \( P \leq_W Q \) for some \( Q \in \mathcal{FO} \) then \( P \) is computably true. Conversely, suppose \( P \) is computably true. Let \( Q \) be the problem whose instances are the same as those of \( P \), and the solutions are all (indices of) Turing functionals \( \Phi \) such that \( \Phi(X) \) is a \( P \)-solution to \( X \). Then \( Q \in \mathcal{FO} \) and \( P \leq_W Q \).
Non-diagonalizable problems

Defn (Hirschfeldt and Jockusch 2016). A problem $P$ is non-diagonalizable if there is a $\{0, 1\}$-valued Turing functional $\Delta$ such that for every $P$-instance $X$ and every $\sigma \in \omega^{<\omega},$

$$\Delta(X, \sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is extendible to a } P\text{-solution to } X, \\ 0 & \text{otherwise}. \end{cases}$$

Prop (DSY). If $P$ is non-diagonalizable then $^1P$ is uniformly computably true.

The converse fails.

$\text{TS}_3^1 :$ Every $c : \omega \rightarrow 3$ omits at least one color on some infinite set.

This is uniformly computable true, but not Weihrauch reducible to any non-diagonalizable problem (Hirschfeldt and Jockusch 2016).
Case studies
ACA

Defn. $J : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, p \mapsto p'$.

Note: the models of ACA$_0$ are the subsets of $\mathbb{N}^\mathbb{N}$ closed under $J$.

Defn.

- $\Sigma^0_n$-$\text{Tr}$: instances: indices of $\Sigma^0_n$ statement of second-order arithmetic; solutions: 1 if the statement is true, 0 otherwise.

- Use: instances: pairs $(X, \Gamma), X \in \mathbb{N}^\mathbb{N}, \Gamma$ a Turing functional s.t. $\Gamma(X)(0) \downarrow$; solutions: all $\ell \geq \text{use}(\Gamma(X)(0))$.

Prop (DSY). $^1J^{(n)} \equiv_W (\Sigma^0_n$-$\text{Tr}) \star \text{Use}^{(n)}$.

(Recall: $\star$ denotes the compositional product.)

In particular, $^1J^{(m)} \not\leq_W ^1J^{(n)}$ whenever $m > n$. 
**WKL**

**Obs.** $^1\text{WKL} \equiv_W ^1\text{WWKL}$. 

$C_2$ : instances: (co-enumerations of) non-empty $X \subseteq \{0, 1\}$; solutions: points in $X$. 

**Thm (DSY).**

- $^1\text{WKL} \equiv_W (C_2)^*$. 

- $^1\text{WKL}^{(n)} \equiv_W (C_2^{(n)})^* \ast \text{Use}^{(n)}$. 

Jumps are combinatorially natural:

- The principle COH is (provably in RCA$_0$, and as a Weihrauch equivalence) the jump inversion of WKL'. (More on COH below.) 

- The **Rainbow Ramsey’s theorem for bounded colorings** is the jump of DNR, a close relative of WKL (J. Miller, unpublished).
Ramsey’s theorem

Obs. $RT_2^1 \equiv_W 1RT_2^1$.

Prop. $RT_2^1 \equiv_W C'_2$.

Thm (DSY). $(\forall k) RT_k^1 \equiv_W (RT_2^1)^* \equiv_W (\forall k) RT_k^1 \equiv_W RT_2^1 \equiv_W (C'_2)^*$.

For higher exponents, we use the observation that $(RT_k^1)^{(n-1)} \leq_W RT_k^n$.

Thm (DSY). $(C_2^{(n)})^* \leq_W (\forall k) RT_k^n \leq_W (C_2^{(n)})^* \star \text{Use}^{(n)}$.

Recall $\text{SRT}_k^2$, the restriction of $RT_k^2$ to stable colorings.

Thm (DSY). $(C_2'')^* \leq_W (\forall k) \text{SRT}_k^2 \leq_W (C_2'')^* \star \text{Use}''$.

So our best bounds on the first-order parts of $(\forall k) RT_k^2$ and $(\forall k) \text{SRT}_k^2$ agree.
Bounded first-order parts
Bounding first-order parts

Defn. Let $P \in \mathcal{FO}$.

$bP$ : same instances as $P$, with the solutions to an instance $X$ being all $n \in \mathbb{N}$ such that there is a $P$-solution $y \leq n$ to $X$.

Obs.

Obviously, $^1P \leq_W bP$ for all problems $P$.

Conversely, consider $C_2 \in \mathcal{FO}$.

- $C_2 \equiv_W ^1C_2$ is not uniformly computably true.
- $bC_2$ is uniformly computably true.
**SRT\textsuperscript{2}_2 and COH**

**COH**: for every sequence $\langle c_0, c_1, \ldots \rangle$ of colorings $c_i : \omega \to 2$ there exists an infinite set $X$ s.t. for all $i$, $X$ is almost homogeneous for $c_i$.

**Thm (Cholak, Jockusch, and Slaman 2001).** $\text{RCA}_0 \vdash \text{RT}_2^2 \leftrightarrow \text{SRT}_2^2 + \text{COH}$. The implication $\text{SRT}_2^2 + \text{COH} \to \text{RT}_2^2$ is a formalization of a Weihrauch reduction: $\text{RT}_2^2 \leq_W \text{SRT}_2^2 \star \text{COH}$.

**Thm (D., Hirschfeldt, Patey, Pauly 2019).** $\text{SRT}_2^2 \star \text{COH} \not\leq_W \text{RT}_2^2$.

As mentioned, our best bounds on the first-order parts of Ramsey’s theorem for pairs and the stable Ramsey’s theorem agree. But they are not sharp.

**Thm (DSY).** $^b((\forall k) \text{SRT}_k^2 \star \text{COH}) \equiv_W ^b(\forall k) \text{RT}_k^2 \equiv_W ^b(\forall k) \text{SRT}_k^2$. 


Thanks for your attention!