Joins in the strong Weihrauch degrees.

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Mathematical problems.

Let X, Y be sets.

A multifunction $f :\subseteq X \rightrightarrows Y$ is a partial map from a subset of X to $\mathcal{P}(Y) - \{\emptyset\}$.

We think of f as a problem:

- the elements of $\mathrm{dom}(f) \subseteq X$ are the instances of f;

- for each $x \in \mathrm{dom}(f)$, the elements of $f(x)$ are the solutions to x (in f).

Example. Π^1_2 statements of second-order arithmetic.

- $(\forall X)[\varphi(X) \to (\exists Y)\psi(X, Y)]$.

- $f :\subseteq 2^\omega \to 2^\omega$ with $\mathrm{dom}(f) = \{X : \varphi(X)\}$ and $f(X) = \{Y : \psi(X, Y)\}$.

Weihrauch reducibility.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

f is Weihrauch reducible to g if there are Turing functionals Φ and Ψ such that:

- $\Phi(x) \in \mathrm{dom}(g)$ for all $x \in \mathrm{dom}(f)$;

- $\Psi(x, \widehat{y}) \in f(x)$ for all $\widehat{y} \in g(\Phi(x))$.

So the following diagram commutes:

$$\begin{tikzcd}
f \arrow[rr, phantom, "\leq_W"] & & g \\
x \arrow[rr] \arrow[d, dashed] & & \Phi(x) \arrow[d, dashed] \\
\Psi(x, \widehat{y}) & & \widehat{y} \arrow[ll]
\end{tikzcd}$$

Weihrauch reducibility.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

f is strongly Weihrauch reducible to g if there are Turing functionals Φ and Ψ such that:

- $\Phi(x) \in \mathrm{dom}(g)$ for all $x \in \mathrm{dom}(f)$;

- $\Psi(\widehat{y}) \in f(x)$ for all $\widehat{y} \in g(\Phi(x))$.

So the following diagram commutes:

$$\begin{tikzcd}
f & \leq_{sW} & g \\
x \arrow[rr] \arrow[d, dashed] & & \Phi(x) \arrow[d, dashed] \\
\Psi(\widehat{y}) & & \widehat{y} \arrow[ll]
\end{tikzcd}$$

Non-uniform versions.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

f is computably reducible to g if:

- each $x \in \mathrm{dom}(f)$ computes an element $\widehat{x} \in \mathrm{dom}(g)$, such that

- each $\widehat{y} \in g(\widehat{x})$, together with x, computes an element $y \in f(x)$.

So the following diagram commutes:

$$\begin{tikzcd}
f \arrow[rr, phantom, "\leq_c"] & & g \\
x \arrow[rr] \arrow[d, dashed] & & \widehat{x} \leq_T x \arrow[d, dashed] \\
y \leq_T \langle x, \widehat{y} \rangle & & \widehat{y} \arrow[ll]
\end{tikzcd}$$

Non-uniform versions.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

f is strongly computably reducible to g if:

- each $x \in \mathrm{dom}(f)$ computes an element $\widehat{x} \in \mathrm{dom}(g)$, such that

- each $\widehat{y} \in g(\widehat{x})$ computes an element $y \in f(x)$.

So the following diagram commutes:

$$\begin{tikzcd}
f \arrow[r, phantom, "\leq_{sc}"] & g \\
x \arrow[r] \arrow[d, dashed] & \widehat{x} \leq_T x \arrow[d, dashed] \\
y \leq_T \widehat{y} & \widehat{y} \arrow[l]
\end{tikzcd}$$

Relationships between reducibilities.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

We have the following implications:

$$
\begin{array}{ccccc}
 & & f \leq_{sW} g & & \\
 & \swarrow & & \searrow & \\
f \leq_{sc} g & & & & f \leq_W g \\
 & \searrow & & \swarrow & \\
 & & f \leq_c g & &
\end{array}
$$

If f and g come from Π^1_2 statements, then each of these also implies that every ω-model of $\mathrm{RCA}_0 + g$ is a model of f. Because of induction issues, it does not follow that $\mathrm{RCA}_0 \vdash g \rightarrow f$.

A brief history.

(Strong) Weihrauch reducibility:

- Weihrauch (1992)
- Brattka (1993)
- Marcone and Gherardi (2008)
- Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2016)

(Strong) computable reducibility:

- Dzhafarov (2015)
- Hirschfeldt and Jockusch (2016)
- Patey (2016)

Growing body of work applying both reducibilities.

Algebraic structure.

For each of the reducibilities, we can form the associated degree structure and study its algebraic properties.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

$f \sqcap g :\subseteq 2^\omega \times 2^\omega \to 2 \times 2^\omega$ is the following multifunction:

- $\mathrm{dom}(f \sqcap g) = \mathrm{dom}(f) \times \mathrm{dom}(g)$;

- $f \sqcap g(x_0, x_1) = f(x_0) \sqcup f(x_1)$.

$f \sqcup g :\subseteq 2 \times 2^\omega \to 2 \times 2^\omega$ is the following multifunction:

- $\mathrm{dom}(f \sqcup g) = \mathrm{dom}(f) \sqcup \mathrm{dom}(g)$;

- $f \sqcup g(\langle 0, x\rangle) = \{0\} \times f(x)$ and $f \sqcup g(\langle 1, x\rangle) = \{1\} \times g(x)$.

Algebraic structure.

Theorem (Pauly; Brattka and Gherardi).

- $\sqcap$ is the meet operation in the Weihrauch and strong Weihrauch degrees.

- $\sqcup$ is the join operation in the Weihrauch degrees.

Both are also true of the computable and strong computable degrees.

Corollary.

- The Weihrauch degrees, computable degrees, and strong computable degrees are lattices under $\sqcap$ and $\sqcup$.

- The strong Weihrauch degrees are a lower semi-lattice under $\sqcap$.

Joins in the strong Weihrauch degrees.

Proposition (Brattka and Pauly).

$\sqcup$ is not the join in the strong Weihrauch degrees.

Let f, g, and $h :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

Suppose $f \leq_W h$ via Φ_0 and Ψ_0, and $g \leq_W h$ via Φ_1 and Ψ_1.

How might we reduce $f \sqcup g$ to h?

- We can map $\langle 0, x_0 \rangle \in \mathrm{dom}(f \sqcup g)$ to $\Phi_0(x_0)$, and $\langle 1, x_1 \rangle$ to $\Phi_1(x_1)$.
- Given $\widehat{y} \in h(x)$, do we map it to $\Psi_0(\widehat{y})$ or to $\Psi_1(\widehat{y})$?
- The answer seems to depend on whether $x = \Phi_0(x_0)$ for some x_0, or $x = \Phi_1(x_1)$ for some x_1, or both.

Question (Brattka; Hölzl and Shafer). Is there a join operation in the strong Weihrauch degrees? Do these degrees form a lattice?

Monotone approximations.

A monotone approximation is a set $A \subseteq \omega \times \omega \times 2$ as follows:

- if $\langle n, s, i \rangle \in A$ and $\langle n, t, i \rangle \in A$ then $s = t$ and $i = j$;
- if $\langle m, s, i \rangle \in A$ and $\langle n, t, j \rangle \in A$ and $m < n$ then $s < t$.

A is total if for each n there exist s, i such that $\langle n, s, i \rangle \in A$. In this case, let $\mathbf{e}(A) = \{n : \exists s \ \langle n, s, 1 \rangle \in A\}$. So $\mathbf{e}$ defines a Turing functional.

Given a set X and a Turing functional Ψ, define $A_{\Psi(X)}$ as follows

- if $\Psi(X)(n) \downarrow = i \in \{0, 1\}$, find the least s such that $\Psi(X)(n)[s] \downarrow$ and enumerate $\langle n, s, i \rangle \in A_{\Psi(X)}$.

Usual use conventions ensure that $A_{\Psi(X)}$ is a monotone approximation. Note that if $\Psi(X)$ is total then $\mathbf{e}(A_{\Psi(X)}) = \Psi(X)$.

The join operation.

Let $f :\subseteq 2^\omega \rightrightarrows 2^\omega$ and $g :\subseteq 2^\omega \rightrightarrows 2^\omega$ be multifunctions.

Definition. Let $f \boxplus g : 2^\omega \sqcup 2^\omega \to 2^\omega \times 2^\omega$ be the following multifunction:

- $\mathrm{dom}(f \boxplus g) = \mathrm{dom}(f) \sqcup \mathrm{dom}(g)$;
- $f \boxplus g(0, x) = (A, X)$ where X is any set, and A is a total monotone approximation with $\mathbf{e}(A) \in f(x)$;
- $f \boxplus g(1, x) = (X, A)$ where X is any set, and A is a total monotone approximation with $\mathbf{e}(A) \in g(x)$.

Theorem (D). $\boxplus$ is the join operation in the strong Weihrauch degrees.

Corollary. The strong Weihrauch degrees are a lattice under $\sqcap$ and $\boxplus$.

(It is easy to see that for all f, g we have $f \boxplus g \equiv_W f \sqcup g$.)

Distributivity.

Recall that a lattice $\mathcal{L} = (L, \vee, \wedge)$ is distributive if for all $a, b, c \in L$ we have $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

Theorem (Pauly; Brattka and Gherardi). The Weihrauch lattice is distributive and embeds every countable distributive lattice.

The proof actually shows that the computable lattice and strong computable lattice are also each distributive.

And it is easy to see that each of these lattices, as well as the strong Weihrauch lattice, also embeds every countable distributive lattice.

Theorem (D). The strong Weihrauch lattice is not distributive.

Corollary. The Weihrauch and strong Weihrauch lattices are not isomorphic.

Thanks for your attention!