Ramsey's Theorem and Products in the Weihrauch Degrees

Damir D. Dzhafarov
University of Connecticut

October 22, 2016

Joint work with Denis Hirschfeldt, Ludovic Patey, and Arno Pauly.

Traditional reverse mathematics.

Work with subsystems of second order arithmetic, Z_2:

- Base theory RCA_0;

- Stronger systems $WKL_0 < ACA_0 < ATR_0 < \Pi^1_1\text{-}CA_0$.

Initial focus was on a kind of zoological classification of theorems in terms of the "big five". More recent focus has been on exceptions.

There is fruitful interplay between reverse mathematics and effective mathematics. In essence, they are two halves of a single endeavor.

RCA_0 has limited comprehension, but classical logic still applies.

- Non-uniform decisions in proofs over RCA_0 are allowed.

- Multiple appeals to a premise/hypothesis of a theorem are allowed.

Problems.

A problem is a pair $(I, \mathbf{S})$, where $I \subseteq 2^\omega$ is a set of instances, and $\mathbf{S} : I \to \wp(2^\omega)$ assigns to each $X \in I$ a set $\mathbf{S}(X) \subseteq 2^\omega$ of solutions to X.

If you like (?), a problem is a multifunction $I \rightrightarrows 2^\omega$.

All of the principles we look at typically have the form

$$(\forall X)[\phi(X) \to \exists Y[\theta(X, Y)]],$$

where ϕ and θ are arithmetical predicates.

These can be naturally regarded as problems:

- Let $I = \{X : \phi(X)\}$.

- Let $\mathbf{S}(X) = \{Y : \theta(X, Y)\}$ for each $X \in I$.

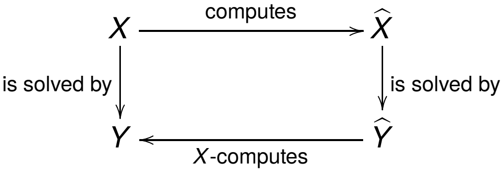
Stronger measures of strength.

Let P and Q be problems.

P is computbaly reducible to Q, written $P \leq_c Q$, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, computes a P-solution Y to X.

So the following diagram commutes:



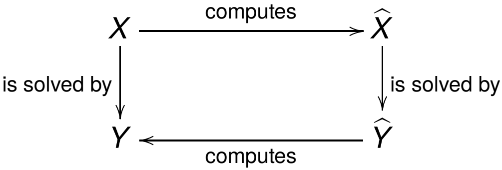
Stronger measures of strength.

Let P and Q be problems.

P is strongly computably reducible to Q, written $\mathrm{P} \leq_{sc} \mathrm{Q}$, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, ~~together with X~~, computes a P-solution Y to X.

So the following diagram commutes:



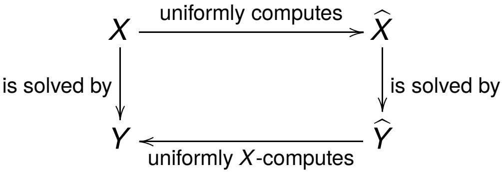
Stronger measures of strength.

Let P and Q be problems.

P is Weihrauch reducible to Q, written P $\leq_W$ Q, if

- every instance X of P uniformly computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, uniformly computes a P-solution Y to X.

So the following diagram commutes:



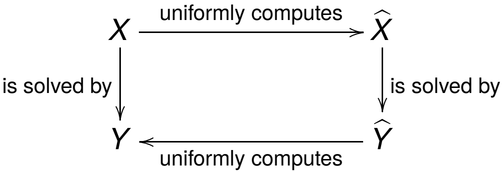
Stronger measures of strength.

Let P and Q be problems.

P is strongly Weihrauch reducible to Q, written $P \leq_{sW} Q$, if

- every instance X of P uniformly computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, ~~together with X~~, uniformly computes a P-solution Y to X.

So the following diagram commutes:



Stronger measures of strength.

Let P and Q be problems.

We have the following implications:

$$
\begin{array}{ccccc}
 & & \mathsf{P} \leq_{sW} \mathsf{Q} & & \\
 & \swarrow & & \searrow & \\
\mathsf{P} \leq_{sc} \mathsf{Q} & & & & \mathsf{P} \leq_{W} \mathsf{Q} \\
 & \searrow & & \swarrow & \\
 & & \mathsf{P} \leq_{c} \mathsf{Q} & & \\
 & & \Downarrow & & \\
 & & \mathsf{Q} \models_{c} \mathsf{P} & &
\end{array}
$$

(Q *computably entails* P, i.e., every ω-model of Q is a model of P)

Because of induction issues, it does not follow that $\mathsf{RCA}_0 \vdash \mathsf{Q} \rightarrow \mathsf{P}$.

A brief history.

Weihrauch reducibility:

- Weihrauch (1992)
- Marcone and Gherardi (2008)
- Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2015)

Computable reducibility:

- Dzhafarov (2015)
- Hirschfeldt and Jockusch (2015)

Growing body of work applying both reducibilities.

A brief history.

Weihrauch reducibility:

- Jockusch (1989)
- Weihrauch (1992)
- Marcone and Gherardi (2008)
- Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2015)

Computable reducibility:

- Dzhafarov (2015)
- Hirschfeldt and Jockusch (2015)

Growing body of work applying both reducibilities.

Jockusch's result

Recall that a function $f : \omega \to \omega$ is diagonally non-recursive relative to a set X if $f(e) \neq \Phi_e^X(e)$ for all e.

For each $k \geq 2$, we define:

(DNR$_k$) For every set X, there is an $f : \omega \to k$ that is diagonally non-recursive relative to X.

Obviously, if $k > j$ then $\mathrm{DNR}_k \leq_{sW} \mathrm{DNR}_j$.

Theorem (Jockusch). Fix $k > j \geq 2$.

- $\mathrm{DNR}_j \leq_c \mathrm{DNR}_k$;
- $\mathrm{DNR}_j \not\leq_W \mathrm{DNR}_k$.

Ramsey's theorem.

For a set X, let $[X]^2$ denote the set of all $\langle x, y\rangle \in X^2$ with $x < y$. A k-coloring of $[X]^2$ is a function $c : [X]^2 \to k$. A set $Y \subseteq X$ is homogeneous for c if $c \upharpoonright [Y]^2$ is constant.

(RT^2_k) For very coloring $c : [\omega]^2 \to k$, there exists an infinite homogeneous set for c.

Classical results.

- (Jockusch). Every computable $c : [\omega]^2 \to k$ has a Π^0_2 infinfinite hom. set, but not always a Σ^0_2 one; $\mathrm{ACA}_0 \vdash \mathrm{RT}^2_k$; $\mathrm{RCA}_0 \nvdash \mathrm{RT}^2_k$.
- (Seetapun). Every computable coloring $c : [\omega]^2 \to k$ has an infinite homogeneous set that does not compute $0'$; $\mathrm{RCA}_0 + \mathrm{RT}^2_k \nvdash \mathrm{ACA}$.
- (Liu). Every computable coloring $c : [\omega]^2 \to k$ has an infinite homogeneous set whose degree is not PA; $\mathrm{RCA}_0 + \mathrm{RT}^2_k \nvdash \mathrm{WKL}$.

Some examples.

Fix $k > j \geq 1$. Over RCA_0, RT^2_k and RT^2_j are equivalent.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). $\mathsf{RT}^2_k \not\leq_{sW} \mathsf{RT}^2_j$.

Theorem (Hirschfeldt and Jockusch; Brattka and Rakotoniaina).

$\mathsf{RT}^2_k \not\leq_{W} \mathsf{RT}^2_j$.

Theorem (Patey). $\mathsf{RT}^2_k \not\leq_{c} \mathsf{RT}^2_j$.

Another example: two definitions of stable posets—SCAC due to Hirschfeldt and Jockusch; WSCAC due to Kastermans, Lempp, Lerman, and Solomon. Over RCA_0, SCAC $\leftrightarrow$ WSCAC.

Theorem (Astor, Dzhafarov, Solomon, and Suggs).

WSCAC $\not\leq_c$ SCAC. (So SCAC $<_c$ WSCAC.)

Products.

Let P_0 and P_1 be problems.

Parallel product.

$\mathsf{P}_0 \times \mathsf{P}_1$ is the problem whose instances are all pairs $\langle X_0, X_1 \rangle$ such that X_i is an instance of P_i for each i, and the solutions of $\langle X_0, X_1 \rangle$ are all pairs $\langle Y_0, Y_1 \rangle$ such that Y_i is a P_i-solution to X_i for each i.

Coproduct.

$\mathsf{P}_0 \sqcup \mathsf{P}_1$ is the problem whose instances are all pairs $\langle i, X \rangle$ for $i < 2$ such that X is an instance of P_i, and the solutions of $\langle i, X \rangle$ are all pairs $\langle i, Y \rangle$ such that Y is a P_i-solution of X.

Products.

Let P_0 and P_1 be problems.

(Regular) product.

$\mathsf{P}_0 \sqcap \mathsf{P}_1$ is the problem whose instances are all pairs $\langle X_0, X_1 \rangle$ such that X_i is an instance of P_i for each i, and the solutions of $\langle X_0, X_1 \rangle$ are all pairs $\langle i, Y \rangle$ for $i < 2$ such that Y is a P_i-solution to X_i.

Theorem (Pauly; Brattka and Gherardi). The Weihrauch degrees under $\leq_W$ form a lattice with $\sqcup$ as sup and $\sqcap$ as inf.

Same is true for the computable degrees and the strongly computable degrees. The strong Weihrauch degrees only form a lower-semilattice.

Open question.

Do the strong Weihrauch degrees under $\leq_W$ form a lattice?

Stable/cohesive split.

A coloring $c : [\omega]^2 \to k$ is stable if $(\forall x)[\lim_y c(x, y)$ exists].

(SRT^2_k) For very stable coloring $c : [\omega]^2 \to k$, there exists an infinite homogeneous set for c.

A set X is cohesive for $\langle Y_0, Y_1, \ldots \rangle$ if $(\forall i)[|X \cap Y_i| < \infty \vee |X \cap \overline{Y}_i| < \infty]$.

(COH) For every family $\langle Y_0, Y_1, \ldots \rangle$ there exists an infinite set X cohesive for this family.

Each of these can in a certain way be regarded as a version of RT^1_k.

Theorem (Cholak, Jockusch, and Slaman).

Over RCA_0, $\mathrm{RT}^2_k \leftrightarrow \mathrm{SRT}^2_k + \mathrm{COH}$.

Compositional product.

Let P_0 and P_1 be problems.

Composition.

$\mathsf{P}_0 \circ \mathsf{P}_1$ is the problem whose instances are the P_1-instances X such that every P_1-solution Y of X is a P_0-instance, and the solutions to X are all the Z such that Z is a P_0-solution to some P_1-solution Y to X.

Compositional product.

$\mathsf{P}_0 * \mathsf{P}_1 = \sup_{\leq_W}\{\mathsf{Q}_0 \circ \mathsf{Q}_1 : (\forall i < 2)[\mathsf{Q}_i \leq_W \mathsf{P}_i]\}$.

This is the problem of applying P_1 once and then applying P_0 once.

Theorem (Brattka and Pauly).

$\mathsf{P}_0 * \mathsf{P}_1$ exists for all P_0 and P_1.

Stable/cohesive split, revisited.

Theorem (Brattka and Rakotoniaina).

$\mathrm{SRT}^2_k \sqcup \mathrm{COH} \leq_W \mathrm{SRT}^2_k \times \mathrm{COH} \leq_W \mathrm{SRT}^2_k * \mathrm{COH}$

$\mathrm{SRT}^2_k \sqcup \mathrm{COH} \leq_W \mathrm{RT}^2_k \leq_W \mathrm{SRT}^2_k * \mathrm{COH}$.

That $\mathrm{RT}^2_k \leq_W \mathrm{SRT}^2_k * \mathrm{COH}$ is the CJS proof that $\mathrm{SRT}^2_k + \mathrm{COH} \to \mathrm{RT}^2_k$.

Question (Brattka).

1) Do any of the reductions above reverse?

2) In particular, is $\mathrm{SRT}^2_k * \mathrm{COH} \leq_W \mathrm{RT}^2_2$?

One might expect a positive answer to (2) since $\mathrm{RT}^2_k \to \mathrm{SRT}^2_k + \mathrm{COH}$.

Main theorem.

Theorem (Dzhafarov, Hirschfeldt, Patey, and Pauly).

$\mathsf{SRT}^2_2 \times \mathsf{COH} \not\leq_W \mathsf{RT}^2_2$.

Brattka and Rokotoniaina showed:

$\mathsf{SRT}^2_k \sqcup \mathsf{COH} \leq_W \mathsf{SRT}^2_k \times \mathsf{COH} \leq_W \mathsf{SRT}^2_k * \mathsf{COH}$

$\mathsf{SRT}^2_k \sqcup \mathsf{COH} \leq_W \mathsf{RT}^2_k \leq_W \mathsf{SRT}^2_k * \mathsf{COH}$.

Every computable instance of COH and SRT^2_k has a Δ^0_2 solution.
By a result of Jockusch, not every computable instance of RT^2_k does.
Combining this with our main theorem yields:

Corollary.

None of the reductions of Brattka and Rakotoniaina reverse.

Outline of proof that $\mathsf{SRT}^2_2 \times \mathsf{COH} \not\leq_W \mathsf{RT}^2_2$.

Fix Turing reductions Φ and Ψ.

We need to construct

- a computable stable coloring $c : [\omega]^2 \to 2$,

- a computable family $Y = \langle Y_0, Y_1, \ldots \rangle$,

- and, if $d = \Phi(c \oplus Y)$ is a coloring $[\omega]^2 \to 2$, then also an infinite homogeneous set H for d, such that

- - either $\Psi^{H \oplus c \oplus Y}$ does not equal $\langle G, X \rangle$, or

- - G is not an infinite homogeneous set for c, or

- - X is not an infinite cohesive set for Y.

Outline of proof that $\mathsf{SRT}^2_2 \times \mathsf{COH} \not\leq_W \mathsf{RT}^2_2$.

Let Y be any computable instance of COH with no low solution.

We build H by a standard argument. (Seetapun or CJS.)

We make c color all pairs 0 until, if ever, $\Psi^{H \oplus c \oplus Y} = \langle G, X \rangle$ enumerates two elements into G (necessarily colored 0 by c).

If this happens, we make c color all future pairs 1.

Two possible outcomes of the Seetapun/CJS construction:

- we can extend the piece of H holding the diagonalizing computation to an infinite homogeneous set, so G will not be homogeneous for c.
- c will have a low infinite homogeneous set, so if we take this to be H then X will not be cohesive for Y.

Main theorem: strengthenings.

Our proof actually shows considerably more.

Our "backup strategy" used nothing special about COH.

The same argument would have applied for any P $\not\leq_c$ WKL.

Likewise, our "diagonalization strategy" did not need all of SRT^2_2.

(LPO) For every non-decreasing $c : \omega \to 2$, $\lim_x c(x)$ exists.

Our argument actually shows: if P $\not\leq_c$ WKL, then LPO $\times$ P $\not\leq_W$ RT^2_k.

In fact, using a more sophisticated argument, we can show:

Theorem (Dzhafarov, Hirschfeldt, Patey, and Pauly).

If P is not computably true, then LPO $\times$ P $\not\leq_W$ RT^2_k.

Related results.

(TS^1_ω) For every $c : \omega \to \omega$, there exists an i and an infinite set T such that $c(x) \neq i$ for all $x \in T$.

For a problem P, let P_* be the problem with

- the same instances as P,
- the solutions of an instance X are the finite modifications of the P-solutions of X.

Theorem (Dzhafarov, Hirschfeldt, Patey, and Pauly).

If $\mathsf{P} \not\leq_c \mathsf{WKL}$ then $(\mathsf{TS}^1_\omega)_* \times \mathsf{P} \not\leq_W \mathsf{SRT}^2_2$.

Theorem (Dzhafarov, Hirschfeldt, Patey, and Pauly).

For every $k \geq 1$, $(\mathsf{RT}^1_k)_* \times \mathsf{NON} \leq_W \mathsf{D}^2_{k+1}$.

Weihrauch quotient.

Let P_0 and P_1 be problems.

We define the following Weihrauch quotient:

$\mathsf{P}_0/\mathsf{P}_1 = \sup_{\leq_W}\{R : R \times \mathsf{P}_1 \leq_W \mathsf{P}_0\}$.

Examples.

- $\mathsf{COH}/\mathsf{COH} = \mathsf{COH}$.

- $\mathsf{WKL}/\mathsf{WKL} = \mathsf{WKL}$.

- In general, if $\mathsf{P} \times \mathsf{P} \leq_W \mathsf{P}$, then $\mathsf{P}/\mathsf{P} = \mathsf{P}$.

Question.

What are some other examples of Weihrauch quotients?

D^2_2 and LPO.

(D^2_k) For very stable coloring $c : [\omega]^2 \to k$, there exists an infinite set L and an $i < k$ such that $\lim_y c(x, y) = i$ for all $x \in L$.

(Chong, Lempp, and Yang.) Over RCA_0, $\mathsf{SRT}^2_k \leftrightarrow \mathsf{D}^2_k$.

(Dzhafarov.) $\mathsf{SRT}^2_k \not\leq_W \mathsf{D}^2_k$.

(Hirschfeldt and Jockusch.) $\mathsf{SRT}^2_2 \leq_W \mathsf{D}^2_k * \mathsf{D}^2_k$.

Theorem (Dzhafarov, Hirschfeldt, Patey, and Pauly).

$\mathsf{D}^2_k/\mathsf{LPO}$ is equal to D^2_k restricted stable colorings $c : [\omega]^2 \to k$ such that there is an $i < k$ such that c has no infinite homogeneous set of color i, and there is a computable $\ell : \omega \to k$ with $i = \lim_s \ell(s)$.

Question. Does $\mathsf{SRT}^2_k/\mathsf{LPO}$ exist?

Thank you for your attention.

Happy birthday, Carl!