Milliken’s tree theorem and computability theory

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The computability-theoretic perspective

We are interested in statements of the form

$$\forall X\,[\Phi(X) \to \exists Y\,\Psi(X, Y)],$$

where Φ and Ψ are some kind of properties of X and Y.

We think of this as a problem, "given X satisfying Φ, find Y satisfying Ψ".

We call the X such that $\Phi(X)$ holds the instances of the problem,
and the Y such that $\Psi(X, Y)$ holds the solutions to X for this problem.

Typically, we look at problems whose instances and solutions are subsets of $\mathbb{N}$,
and where the properties Φ and Ψ are arithmetical.

Basic question. Given an instance of a problem, how complex are its solutions?

Measuring complexity

Computability theory:

- Does every instance compute a solution to itself?
- Does every instance have an arithmetically-definable solution?
- Is there a computable instance all of whose solutions compute $\emptyset'$?

Reverse mathematics/proof theory:

- We look at subsystems of second-order arithmetic, RCA_0, WKL, ACA_0, ...
- Is the theorem provable in RCA_0?
- Is the theorem provable in ACA_0?
- Does the theorem imply ACA over RCA_0?

There is well-understood interplay between these viewpoints.

Ramsey's theorem

Definition. Fix $X \subseteq \mathbb{N}$ and $n, k \geq 1$.

- $[X]^n = \{F \subseteq X : |F| = n\}$.
- A k-coloring of $[X]^n$ is a map $c : [X]^n \to k$.
- A set $Y \subseteq X$ is homogeneous for c if c is constant on $[Y]^n$.

Ramsey's theorem. For all $n, k \geq 1$, every $c : [\mathbb{N}]^n \to k$ has an infinite homogeneous set.

We let RT^n_k denote Ramsey's theorem restricted to k-colorings of $[\mathbb{N}]^n$.

As a problem: The instances of RT^n_k are all $c : [\mathbb{N}]^n \to k$.
The solutions to any specific such c are all its infinite homogeneous sets.

Other examples

Chain/antichain principle. Every partial ordering of $\mathbb{N}$ contains either an infinite chain or an infinite antichain.

Ascending/descending sequence principle. Every infinite linear ordering of $\mathbb{N}$ contains either an infinite ascending or an infinite descending sequence.

Erdős-Moser theorem. Every tournament on $\mathbb{N}$ has an infinite transitive subtournament.

Rainbow Ramsey's theorem. For all $n, k \geq 1$ and all $f : [\mathbb{N}]^n \to \mathbb{N}$ such that $|f^{-1}(n)| < k$ for all n there is an infinite $Y \subseteq \mathbb{N}$ such that f is injective on $[Y]^n$.

The atomic model theorem. Every complete atomic theory has an atomic model.

Chubb-Hirst-McNicholl tree theorem

For $X \subseteq 2^{<\omega}$, write $X \cong 2^{\omega}$ if $(X, \preceq)$ and $(2^{<\omega}, \preceq)$ are isomorphic structures.

Such an X need not be closed under initial segments ($\preceq$) or under meets ($\wedge$).

Definition. Fix $X \subseteq 2^{<\omega}$ and $n, k \geq 1$.

- $[X]^n = \{F \subseteq X : |F| = n \wedge (\forall \sigma, \tau \in F)[\sigma \preceq \tau \vee \tau \preceq \sigma]\}$.
- A k-coloring of $[X]^n$ is a map $c : [X]^n \to k$.
- A set $Y \subseteq X$ is homogeneous for c if c is constant on $[Y]^n$.

Chubb-Hirst-McNicholl tree theorem. For all $n, k \geq 1$ and all $c : [2^{<\omega}]^n \to k$ there exists a $Y \cong 2^{<\omega}$ which is homogeneous for c.

We let TT^n_k denote the CHM tree theorem restricted to k-colorings of $[2^{<\omega}]^n$.

TT and RT

Given $d : [\mathbb{N}]^n \to k$, define $c : [2^{<\omega}]^n \to k$ by

$$c(\sigma_0, \ldots, \sigma_{n-1}) = d(|\sigma_0|, \ldots, |\sigma_{n-1}|).$$

If $Y \cong 2^{<\omega}$ is homogeneous for c and $L \subseteq Y$ is any $\preceq$-chain then

$$H = \{|\sigma| : \sigma \in L\}$$

is homogeneous for d.

Effectivity: c is computable from d, and H can be chosen computable from Y. This can be formalized to show that for all $n, k \geq 1$, $\mathsf{RCA}_0 \vdash \mathsf{TT}^n_k \to \mathsf{RT}^n_k$.

Patey (2016). Over RCA_0, RT^2_2 does not imply TT^2_2.

Effective results about RT and TT

Jockusch (1972).

- For all $n, k \geq 1$, every computable instance of RT^n_k has a Π^0_n solution.
- For all $n \geq 2$, there is a computable instance of RT^n_2 all of whose solutions compute $\emptyset^{(n-2)}$.
- Thus, for all $n \geq 3$ and $k \geq 2$, RT^n_k is equivalent to ACA over RCA_0.

Seetapun (1995). For all $k \geq 1$, every computable instance of RT^2_k has a solution not computing $\emptyset'$. Thus, over RCA_0, RT^2_k does not imply ACA_0.

Chubb, Hirst, and McNicholl (2005). For all $n, k \geq 1$, every computable instance of TT^n_k has a Π^0_n solution. Thus, $\mathrm{ACA}_0 \vdash \mathrm{TT}^n_k$. If $n \geq 3$, equivalent.

Dzhafarov and Patey (2017). For all $k \geq 1$, every computable instance of TT^2_k has a solution not computing $\emptyset'$. Thus, over RCA_0, TT^2_k does not imply ACA_0.

Strong subtrees

Definition. A tree is a subset T of $\omega^{<\omega}$ as follows:

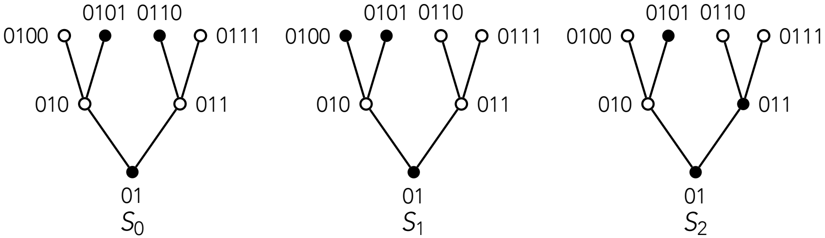
- there exists a root $\rho \in T$ such that $\rho \preceq \sigma$ for all $\sigma \in T$;
- if $\sigma, \tau \in T$ then also $\sigma \wedge \tau \in T$;
- every $\sigma \in T$ there are finitely many $\tau \in T$ such that $\sigma \prec \tau$ and there is no τ' such that $\sigma \prec \tau' \prec \tau$.

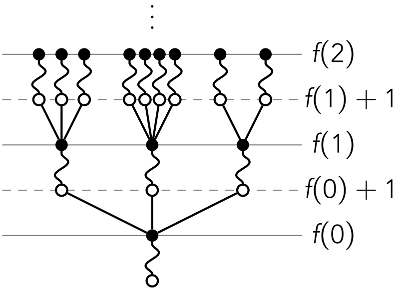
For each $n \in \mathbb{N}$, let $T(n) = \{\sigma \in T : |\tau \in T : \tau \prec \sigma| = n\}$
and $\text{height}(T) = \sup\{n + 1 \in \mathbb{N} : T(n) \neq \emptyset\}$.

Definition Let $U \subseteq T$ be trees. U is a strong subtree of T if:

- there is a level function $f : \text{height}(U) \to textheight(T)$ such that for all $n < \text{height}(U)$, if $\sigma \in U(n)$ then $\sigma \in T(f(n))$.
- a node $\sigma \in U$ is k-branching in U if and only if it is k-branching in T.

Examples of subtrees, strong and not strong





Milliken's tree theorem

For a tree T, let $\mathcal{S}_\alpha(T)$ be the class of all strong subtrees of T of height $\alpha \leq \omega$.

Milliken's tree theorem. Let T be an infinite tree with no leaves. For all $n, k \geq 1$ and all $c : \mathcal{S}_n(T) \to k$ there is a $U \in \mathcal{S}_\omega(T)$ such that c is contant on $\mathcal{S}_n(U)$.

We let MTT^n_k denote Milliken's tree theorem restricted to k-colorings of $\mathcal{S}_n(T)$.

Proved by **Milliken (1979)**.

For a newer proof, see **Todorcevic (2010)**.

Generalizes many combinatorial results, including Ramsey's theorem.

Dobrinen (2018). What about the effectivity/reverse math of MTT?

MTT and TT

Given $d : [2^{<\omega}]^2 \to k$, define $c : \mathcal{S}_2(2^{<\omega}) \to k$ as follows: for every $\{\sigma, \tau_0, \tau_1\} \in \mathcal{S}_2(2^{<\omega})$ with $\tau_i \succeq \sigma i$, define

$$c(\sigma, \tau_0, \tau_1) = d(\sigma, \tau_0).$$

Let $U \in \mathcal{S}_\omega(2^{<\omega})$ be such that c is constant on $\mathcal{S}_2(U)$. Then U can be thinned out to a set $Y \cong 2^{<\omega}$ that is homogeneous for d.

Effectivity: c is computable from d, and Y can be chosen to be computable from V. This can be formalized in RCA_0.

The argument can be easily extended to arbitrary exponents.

Fact. For all $n, k \geq 1$, $\mathsf{RCA}_0 \vdash \mathsf{MTT}^n_k \to \mathsf{TT}^n_k$.

The case $n = 1$

Ramsey's theorem and the Chubb-Hirst-McNicholl tree theorem can each be proved by induction on the exponent n. The inductive step uses the $n = 1$ case to increase the exponent.

Milliken's tree theorem for $n = 1$. Let T be an infinite tree with no leaves. For all $k \geq 1$ and all $c : T \to k$ there exists $U \in \mathcal{S}_\omega(T)$ such that c is constant on U.

But MTT^1_k is not enough to carry out the induction in the proof of Milliken's tree theorem. Instead, the following stronger result is needed.

Halpern-Laüchli theorem. Fix $d \geq 1$, and let $T_0, \ldots, T_{d-1}$ be infinite trees with no leaves. For all $k \geq 1$ and all $c : \bigcup_n \prod_{i<d} T_i(n) \to k$ there exist $U_0, \ldots, U_{d-1}$ in $\mathcal{S}_\omega(T_0), \ldots, \mathcal{S}_\omega(T_{d-1})$, respectively, with common level function, such that c is constant on $\bigcup_n \prod_{i<d} U_i(n)$.

Product version of Milliken's tree theorem

Given trees $T_0, \ldots, T_{d-1}$, let $\mathcal{S}_\alpha(T_0, \ldots, T_{d-1})$ denote the collection of all tuples $(U_0, \ldots, U_{d-1})$ such that $U_i \in \mathcal{S}_\alpha(T_i)$ and the U_i have a common level function.

Product version of Milliken's tree theorem. Fix $d \geq 1$, and let $T_0, \ldots, T_{d-1}$ be infinite trees with no leaves. For all $k \geq 1$ and all $c : \mathcal{S}_n(T_0, \ldots, T_{d-1}) \to k$ there exists $(U_0, \ldots, U_{d-1}) \in \mathcal{S}_\omega(T_0, \ldots, T_{d-1})$ such that c is constant on $\mathcal{S}_n(U_0, \ldots, U_{d-1})$.

We let PMTT^n_k denote the product version of Milliken's tree theorem restricted to k-colorings of $\mathcal{S}_n(T_0, \ldots, T_{d-1})$.

So MTT^n_k is just PMTT^n_k for $d = 1$. Notice that the Halpern-Laüchli theorem is exactly PMTT^1_k.

Halpern-Laüchli theorem

Like RT^1_k and TT^1_k, it is easy to see that MTT^1_k is computably true, meaning that each instance computes a solution to itself.

While the Halpern-Laüchli theorem appears on its face as just a kind of parallelized/sequential version of MTT^1_k, this is misleading. It encompasses much of the combinatorial core of the full Milliken's tree theorem.

A careful analysis of the proof reveals it to be basically an effective construction of a solution from a given instance of Halpern-Laüchli, with most of the combinatorial machinery being used merely to verify that the construction succeeds.

Thm (Anglès d'Auriac, Cholak, D., Monin, and Patey). The Halpern-Laüchli theorem is computably true (and uniformly so, in an arithmetical oracle).

Upper bounds on Milliken's tree theorem

Thm (Anglès d'Auriac, Cholak, D., Monin, and Patey). PMTT is arithmetically true: every instance has a solution arithmetically definable in itself.

Corollary. For all $n, k \geq 1$, $\mathsf{ACA}_0 \vdash \mathsf{PMTT}^n_k$.

Since MTT^n_k implies TT^n_k, which implies RT^n_k, we also have:

Corollary. For all $n \geq 3$ and all $k \geq 2$, the following are equivalent over RCA_0:

1. ACA;
2. PMTT^n_k;
3. MTT^n_k;
4. TT^n_k;
5. RT^n_k.

Cone avoidance and strong cone avoidance

Definition. Let P be a problem. Then P satisfies

- cone avoidance if for every A and every $C \not\leq_T A$, every A-computable instance of P has a solution Y such that $C \not\leq_T A \oplus Y$.
- strong cone avoidance if for every A and every $C \not\leq_T A$, every instance of P has a solution Y such that $C \not\leq_T A \oplus Y$.

If P satisfies cone avoidance then there is a model of $\mathsf{RCA}_0 + \mathsf{P}$ in which ACA fails (indeed, a Turing ideal not containing $\emptyset'$).

Being computably true does not necessarily imply strong cone avoidance.

Dzhafarov and Jockusch (2009). RT^1_2 admits strong cone avoidnace.

Dzhafarov and Patey (2017). TT^1_2 admits strong cone avoidance.

Lower bounds on Milliken's tree theorem for height 2

Thm (Anglès d'Auriac, Cholak, D., Monin, and Patey).

- The Halpern-Laüchli theorem satisfies strong cone avoidance.
- The product version of Milliken's theorem for $n = 2$ satisfies cone avoidance.

Corollary. For all $k \geq 1$, PMTT^2_k does not imply ACA over RCA_0.

The proof is an effective forcing argument, following the scheme of **Cholak, Jockusch, and Slaman (2001)** of splitting into a stable part and a cohesive part.

Definition. Fix $d \geq 1$, and let $T_0, \ldots, T_{d-1}$ be infinite trees with no leaves. A coloring $c : \mathcal{S}_2(T_0, \ldots, T_{d-1}) \to k$ is stable if for each $(\sigma_0, \ldots, \sigma_{d-1}) \in \bigcup_n \prod_{i<d} T_i(n)$ there is a $N \in \mathbb{N}$ and $j < k$ such that $c(U_0, \ldots, U_{d-1}) = j$ for all $(U_0, \ldots, U_{d-1}) \in \mathcal{S}_2(T_0, \ldots, T_{d-1})$ with $U_i(0) = \sigma_i$ and $U_i(1) \subseteq \bigcup_{n>N} T_i(n)$ for each $i < d$.

Schematic of proving cone avoidance

Stability. Strong cone avoidance of the $n = 1$ case of a problem lifts to cone avoidance of the restriction to stable instances.

Strong cone avoidance of HL gives cone avoidance of stable PMTT^2_k.

Cohesiveness. Reducing from a general instance to a stable one on some domain satisfies cone avoidance

For every instance $(T_0, \ldots, T_{d-1})$ and c of PMTT^2_k there is $(U_0, \ldots, U_{d-1})$ in $\mathcal{S}_\omega(T_0, \ldots, T_{d-1})$ not computing a given $C \not\leq_T \emptyset$ on which c is stable.

In our case, this requires the construction of certain very fast growing functions that dominate the levels of all possible strong subtrees of height 2 with nice computational properties. This values are analogous to Ramsey numbers.

Putting it together. The two results combine to give cone avoidance of PMTT^2_k.

Applications

Milliken's tree theorem is false if, instead of coloring, say, $\mathcal{S}_2(T)$, we color $[T]^2$.

Example. Let $T = 2^{<\omega}$ and set $c(\sigma, \tau) = i$ if $\tau \succeq \sigma i$, for $i < 2$.

We cannot tell apart two pairs of comparable strings structurally. By enriching the structure we can find distinct embedding types, e.g., $\{\sigma, \sigma 0\}$ and $\{\sigma, \sigma 1\}$.

Definition. The big Ramsey degree of a structure $\mathcal{A}$ in a structure $\mathcal{B}$ is the least number ℓ, if it exists, such that for all $k \geq 1$ and all $c : \binom{\mathcal{B}}{\mathcal{A}} \to k$ there exists $\mathcal{B}' \cong \mathcal{B}$ such that $|c'' \binom{\mathcal{B}'}{\mathcal{A}}| \leq \ell$.

See **Zucker (2019)** for a chracterization in terms of additional/enriched structure (big Ramsey structures).

Many "existence of big Ramsey degrees" results follow from MTT.

The Rado graph theorem

Let $\mathcal{R}$ be the Rado graph (or random graph).

The Rado graph theorem. For every finite graph F there is an $\ell \in \mathbb{N}$ such that for every $k \geq 1$ and every $c : \binom{\mathcal{R}}{F} \to k$ there is a Random subgraph $\mathcal{R}'$ of $\mathcal{R}$ such that $|c''\binom{\mathcal{R}}{F}| \leq \ell$.

For each finite graph F, let ℓ_F be the least ℓ as above. Computed by **Sauer, Laflamme, Vuksanovic (2006)**; also for various F by **Larson (2008)**.

Thm (Anglès d'Auriac, Cholak, D., Monin, and Patey).

- For every finite graph F of size 2, the Random graph theorem for F and $\ell = \ell_F$ satisfies cone avoidance.
- For every finite graph F of size 3, there is a computable instance of the Random graph theorem for F and $\ell = \ell_F$ s.t. every solution computes $\emptyset'$.

Devlin's theorem

Devlin's theorem. For every $n \geq 1$ there is an $\ell \in \mathbb{N}$ such that for every $k \geq 1$ and every $c : [\mathbb{Q}]^n \to k$ there is a DLO $S \subseteq \mathbb{Q}$ such that $|c''[S]^n|| \leq \ell$.

Devlin (1980) computed for each n the least $\ell = \ell_n$ satisfying the conclusion of Devlin's theorem. These turn out to be the number of Joyce trees of size n.

The rationals admit a representation in terms of $2^{<\omega}$ that allows Devlin's theorem to be derived from Milliken's tree theorem. A pair of rationals thus specifies three strings: one for each rational, and their meet.

Thm (Anglès d'Auriac, Cholak, D., Monin, and Patey).

- Devlin's theorem for $n = 1$ is computably true.
- Devlin's theorem for $n = 2$ and $\ell = 3$ implies ACA_0 over RCA_0.
- Devlin's theorem for $n = 2$ and $\ell = 4$ satisfies cone avoidance.

Thanks for your attention!