Calling a few good combinatorialists

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...or set theorists ... or proof theorists ... or descriptive set theorists ... or...

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I. Prelude

Ramsey's theorem, for us

Throughout, **set** will refer to a subset of $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Syncing notation:

- Given a set X and $n \geq 1$, let $[X]^n = \{(x_0, \ldots, x_{n-1}) \in X^n : x_0 < \cdots < x_{n-1}\}$.
- A **k-coloring of $[X]^n$** is a map $c : [X]^n \to k = \{0, 1, \ldots, k-1\}$.
- $Y \subseteq X$ is **homogeneous** for such a c if $c \restriction [Y]^n$ is constant.

Ramsey's theorem for n-tuples and k colors (RT^n_k).

Every k-coloring of $[\mathbb{N}]^n \to k$ has an infinite homogeneous set.

Thus, RT^n_k is the same as $\aleph_0 \to (\aleph_0)^n_k$.

A curious variant

Say a coloring $s : [\mathbb{N}]^2 \to 2$ is stable if for every $x \in \mathbb{N}$, there exists a $z > x$ and an $i \in \{0, 1\}$ such that $c(x, y) = i$ for all $y > z$.

That is, a coloring s is stable if for every x, the limit $\lim_y s(x, y)$ exists.

Stable Ramsey's theorem for pairs (SRT^2_2).

Every stable 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

On its face, SRT^2_2 is simpler to prove than RT^2_2.

Classically, SRT^2_2 is really just RT^1_2, but from our perspective that isn't quite right. (More on this in a minute.)

Ultimately, we would like to understand the word "simpler" above.

Two perspectives, briefly

Computable mathematics seeks to measure how close a given mathematical problem is to being algorithmically/effectively/computably solvable.

Reverse mathematics is a foundational program that seeks to find the minimal axioms needed to prove a given theorem of ordinary mathematics.

- Set in second-order arithmetic (numbers and sets of numbers).
- Work over a weak system of axioms called RCA_0 that roughly corresponds to constructive mathematics.
- Given a theorem, find the weakest in a hierarchy of benchmark collections of axioms (extending RCA_0) that can prove the theorem.

The first approach is computability-theoretic, the second is proof-theoretic. There is a deep relationship between the two, linking computation with proof.

Classical reverse mathematics

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, *etc.*	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

From *The Gödel Hierarchy and Reverse Mathematics*, by Stephen Simpson.
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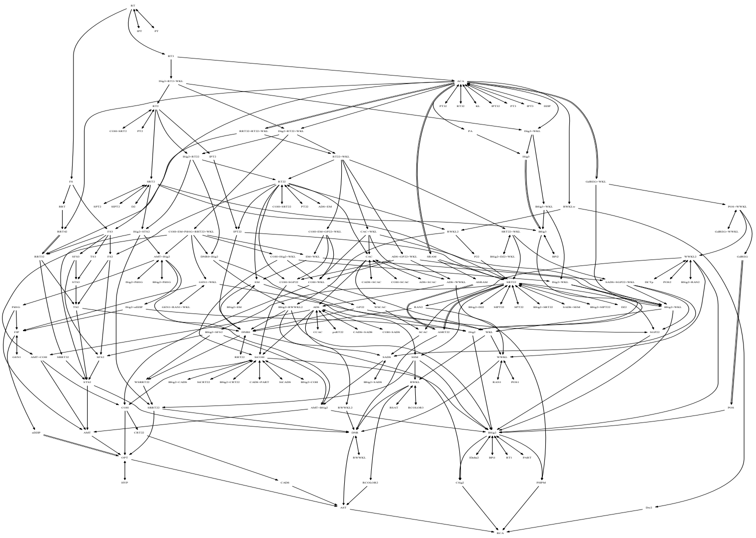
Reverse mathematics of combinatorics



From *The Reverse Mathematics Zoo*.

The SRT^2_2 vs. COH problem

Question. In the base system RCA_0, is it provable that SRT^2_2 implies RT^2_2?

Theorem (Chong, Slaman, and Yang). No.

To prove a separation like this, one needs to build a model satisfying the axioms of RCA_0, satisfying SRT^2_2, but in which RT^2_2 is false. Notably, CSY do this using a nonstandard model.

The SRT^2_2 vs. COH problem. Does every standard model satisfying RCA_0 and SRT^2_2 also satisfy RT^2_2?

Stay tuned for:

- What is COH?
- What does any of this mean?

II. Overview

Theorems as problems

A theorem of the form

For every set I with property A, there is a set S with property B

can be regarded as a problem, namely

Given I satisfying property A, find a set Y satisfying property B.

Definition. An instance-solution problem P consists of

- a set of instances,
- for each instance I, a set of solutions to I.

In practice, we only care about theorems where the properties A and B above are arithmetical. These are the typical statements encountered in reverse mathematics and computable mathematics.

Examples of problems

RT^2_2. Instances are colorings $c : [\mathbb{N}]^2 \to \{0, 1\}$, and the solutions to any such c are its infinite homogeneous set.

SRT^2_2. Instances are stable colorings $s : [\mathbb{N}]^2 \to \{0, 1\}$, and the solutions to any such s are its infinite homogeneous set.

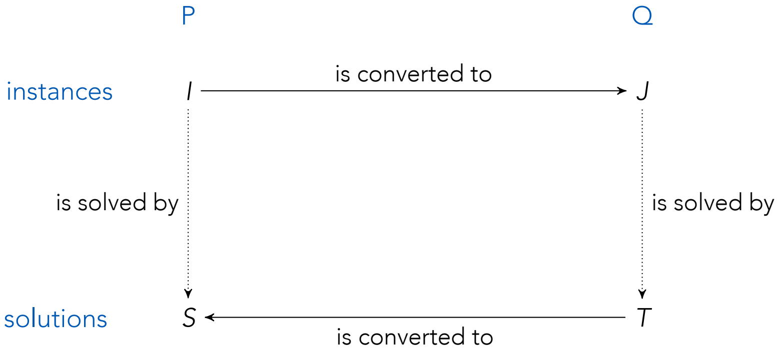
KL. Instances are finite branching trees in $\mathbb{N}^{<\mathbb{N}}$ of infinite height, and the solutions to any such tree are its infinite branches.

CAC. Instances are partial orders on $\mathbb{N}$, and the solutions to any such partial order are its infinite chains and antichains.

Reductions, intuitively

Let P and Q be instance-solution problems.

We can say that **P is reducible to Q** (in some sense) if the following diagram commutes:



Example: a bad reduction

Recall that $s : [\mathbb{N}]^2 \to 2$ is stable if for every x, the limit $\lim_y s(x, y)$ exists.

Given a stable coloring $s : [\mathbb{N}]^2 \to 2$, we can define $c : \mathbb{N} \to 2$ by

$$c(x) = \lim_y s(x, y).$$

An RT^1_2-solution to c is an infinite set G such that for some i, $c(x) = i$ for all $x \in G$. Thus, $\lim_y s(x, y) = i$ for all $x \in G$. Define $H = \{x_0, x_1, x_2, \ldots\}$ as:

- Suppose we have defined $x_0 < \cdots < x_{n-1}$ for some n.
- Find the least $y > x_{n-1}$ in G such that $s(x_0, y) = \cdots = s(x_{n-1}, y) = i$. Let $x_n = y$.

This is a non-computable reduction of D^2_2 to RT^1_2. The conversion of G to H is effective, but the conversion of s to c is not.

Example: a good reduction

SRT^2_2. Instances are stable colorings $s : [\mathbb{N}]^2 \to \{0, 1\}$, and the solutions to any such s are its infinite homogeneous set.

D^2_2. Instances are stable colorings $s : [\mathbb{N}]^2 \to \{0, 1\}$, and the solutions to any such s are the infinite sets L such that $\lim_y s(x, y)$ is the same for all $x \in L$.

Each instance of SRT^2_2, stable coloring $s : [\mathbb{N}]^2 \to 2$, is also an instance of D^2_2.

Given a solution to s as an instance of D^2_2, i.e., an infinite set L such that $\lim_y s(x, y) = i \in 2$ for all $x \in L$, define $H = \{x_0, x_1, x_2, \dots\}$ as follows:

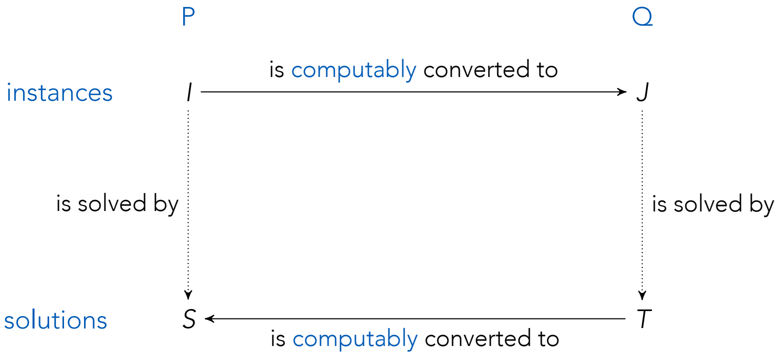
- Suppose we have defined $x_0 < \cdots < x_{n-1}$ for some n.
- Find the least $y > x_{n-1}$ s.t. $s(x_0, y) = \cdots = s(x_{n-1}, y) = i$, and let $x_n = y$.

This is a computable reduction of SRT^2_2 to D^2_2.

Computable reductions

Let P and Q be instance-solution problems.

We say that P is computably reducible to Q if the following diagram commutes:



Example: multiple uses

It is easy to prove RT^1_3 from RT^1_2.

Given $c : \mathbb{N} \to 3$, define $d : \mathbb{N} \to 2$ by $d(x) = \min\{c(x), 1\}$.

If H_d is an infinite homogeneous set for d of color 0, then it is also homogeneous for c.

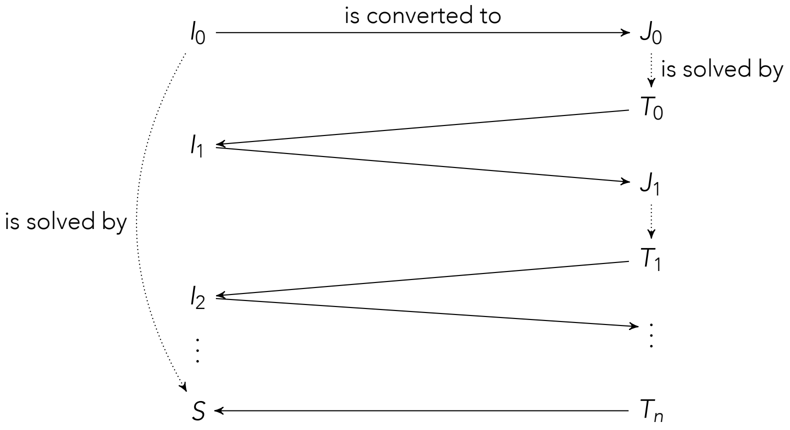
if $H_d = \{x_0 < x_1 < \cdots\}$ is an infinite homogeneous set for d of color 1, define $e : \mathbb{N} \to 2$ by $e(y) = c(x_y)$.

If H_e is an infinite homogeneous set for e, then $H = \{x_y : y \in H\}$ is homogeneous for c.

Theorem (Patey). There is no effective reduction of RT^1_3 to RT^1_2.

Generalized computable reductions

Let P and Q be instance-solution problems. We say that P is generalized computably reducible to Q if for some n, the following diagram commutes:



Connection to reverse mathematics

Commonly, if we have an implication of some theorem P by some theorem Q, it is due to a computable reduction of P to Q.

Theorem (Hirschfeldt and Jockusch).

The following are equivalent for mathematical problems P and Q:

- Every standard model satisfying RCA_0 and Q also satisfies P;
- P is generalized computably reducible to Q.

The SRT^2_2 **vs.** COH **problem.** Does every standard model satisfying RCA_0 and SRT^2_2 also satisfy RT^2_2?

The SRT^2_2 **vs.** COH **problem, restated.** Is RT^2_2 generalized computably reducible to SRT^2_2?

III. An appeal

Finite error

Definition.

Given a family of sets $(A_0, A_1, \ldots)$, a set S is **cohesive** for this family if for each x, either $S \cap A_x$ or $S \cap (\mathbb{N} - A_x)$ is finite.

The cohesive principle (COH)**.**

Every family of sets $(A_0, A_1, \ldots)$ has an infinite cohesive set.

The proof is easy:

- Either A_0 or $\mathbb{N} - A_0$ is infinite. Say A_0 is. Pick $x_0 \in A_0$.
- Either $A_0 \cap A_1$ or $A_0 \cap (\mathbb{N} - A_1)$ is infinite. Say $A_0 \cap (\mathbb{N} - A_1)$ is. Pick $x_1 > x_0$ in $A_0 \cap (\mathbb{N} - A_1)$.
- Continue to build an infinite cohesive set $S = \{x_0, x_1, \ldots\}$.

Stable colorings and cohesiveness

SRT^2_2. Every stable coloring $s : [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set.

COH. Every family of sets $(A_0, A_1, \ldots)$ has an infinite cohesive set.

The significance of COH is that it stabilizes colorings.

Given an arbitrary coloring $c : [\mathbb{N}]^2 \to 2$, define $(A_0, A_1, \ldots)$ by

$$A_x = \{y > x : c(x, y) = 0\}.$$

Now apply COH to get an infinite set S for this family of sets.

Claim. $c \restriction [S]^2$ is stable. Indeed, for each $x \in S$, we have:

- if $S \cap A_x$ is finite then $c(x, y) = 1$ for all but finitely many $y \in S$;
- if $S \cap (\mathbb{N} - A_x)$ is finite then $c(x, y) = 0$ for all but finitely many $y \in S$.

A decomposition theorem

Theorem (Cholak, Jockusch, and Slaman).

Over the base theorey RCA_0, RT^2_2 is equivalent to $SRT^2_2 \wedge COH$.

Exercise. Show that COH is computably reducible to RT^2_2.

(I would be very interested in seeing your proof.)

Is this a proper split? It is known that RT^2_2 is not reducible to COH.

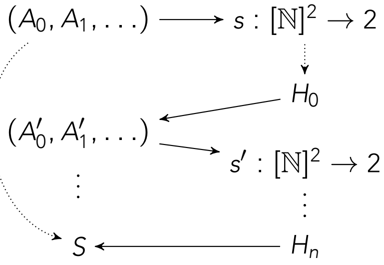
Three equivalent versions of the SRT^2_2 vs. COH problem.

- Is RT^2_2 generalized computably reducible to SRT^2_2?
- Is COH generalized computably reducible to SRT^2_2?
- Is COH generalized computably reducible to D^2_2?

Open questions

The SRT^2_2 **vs.** COH **problem.**

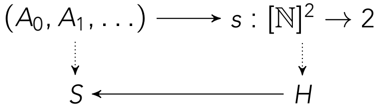
Is COH generalized computably reducible to SRT^2_2?



The entire combinatorial difficulty seems already present in the following special case.

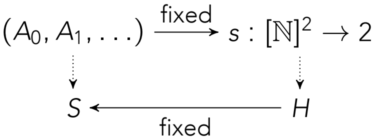
Open question.

Is COH computably reducible to SRT^2_2?



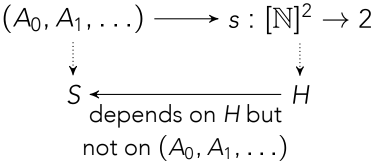
Partial results towards a negative answer

Theorem (D.) COH is not Weihrauch (uniformly) computably reducible to SRT^2_2.



Theorem (D., Patey, Solomon, and Westrick).

COH is not strongly computably reducible to SRT^2_2.



A step towards a positive answer

Fix a family of sets, $(A_0, A_1, \ldots)$.

Define $s : [\mathbb{N}]^2 \to 2$ by

$$s(x, y) = \begin{cases} 0 & \text{if some intersection of } A_0, \ldots, A_x, \mathbb{N} - A_0, \ldots, \mathbb{N} - A_x \\ & \text{is finite but contains an element } z > y. \\ 1 & \text{otherwise.} \end{cases}$$

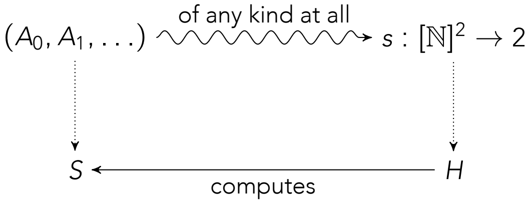
Let $H = \{x_0 < x_1 < \cdots\}$ be a homogeneous set for s, necessarily of color 1. We can now compute from H an infinite cohesive set for $(A_0, A_1, \ldots)$.

Corollary.

For every family of sets $(A_0, A_1, \ldots)$ there is some stable coloring $s : [\mathbb{N}]^2 \to 2$, each of whose infinite homogeneous sets computes a cohesive set.

Removing computability

Question. Given a family $(A_0, A_1, \ldots)$, is there some coloring $c : \mathbb{N} \to 2$, every infinite homogeneous set for which computes an infinite cohesive set?



Turing computations are continuous maps $2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Question. Given a family $(A_0, A_1, \ldots)$, is there some coloring $c : \mathbb{N} \to 2$, every infinite hom. set for which continously maps onto an infinite cohesive set?

IV. An invitation

What can computable combinatorics do for you?

It can provide something for your students to work on...

Computable combinatorics is deeply combinatorial.

The problems encountered in this investigation tend to rely on intricate combinatorial ideas that computability theorists have to develop from scratch.

Yet ideas from (pure) combinatorics increasingly lead to new insights, including to previously inaccessible questions.

There is no shortage of open problems to which this could be applied.

Thanks for your attention!