Reverse mathematics and equivalents of the axiom of choice

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Reverse mathematics and equivalents of the axiom of choice,
joint work with Carl Mummert (submitted).

Axiom of choice

A well-known principle (Zermelo, 1904):

(AC) Every family of nonempty sets admits a choice function.

History documented by G. H. Moore, *Zermelo's axiom of choice* (1982).

Axiom of choice

Many interesting equivalents:

- Well ordering principle.
- Zorn's lemma.
- The principle that every vector space has a basis.
- The principle that every nontrivial unital ring has a maximal ideal.

Many more in Rubin and Rubin, *Equivalents of the axiom of choice* (1970/1985).

Axiom of choice

We study the proof-theoretic strength of (countable analogues) of these principles using reverse mathematics.

Some previous work along these lines:

- Simpson (1999/2009): direct formalizations of the statement of AC.
- Friedman and Hirst (1990), Hirst (2005): principles concerning countable well-orderings.
- Various: algebraic forms.

Axiom of choice

Our interest is on statements related to the following two equivalents of the axiom of choice:

- Finite intersection principle. Every family of sets has a $\subseteq$-maximal subfamily with the finite intersection property.
- Finite character principle. If P is a property of finite character and A is any set, there is a $\subseteq$-maximal subset B of A such that B has property P.

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Theorem (Klimovsky; Rubin and Rubin)**.** ZF proves AC $\leftrightarrow$ FIP.

Formalizing FIP in RCA_0

Let $A = \langle A_i : i \in \mathbb{N} \rangle$ and $B = \langle B_i : i \in \mathbb{N} \rangle$ be families of sets.

A is nontrivial if $(\exists i)[A_i \neq \emptyset]$.

B is a subfamily of A, written $B \leq A$, if $(\forall i)(\exists j)[B_i = A_j]$.

$B \leq A$ is maximal (among subfamilies with some property) if for every $C \leq A$ (with that property), if $B \leq C$ then $C \leq B$.

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In RCA_0, we formulate FIP for nontrivial families of sets.

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By contrast, with our weak notion we have:

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Theorem (Dzhafarov and Mummert)**.** There is an ω-model of FIP consisting entirely of low sets. Hence, FIP does not reverse to ACA_0.

The proof is a forcing argument; more on this later.

FIP is computably false

Question. If $A = \langle A_i : i \in \omega \rangle$ is a computable nontrivial family, must it have a computable maximal subfamily with the F intersection property?

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Idea. We might try to build such a subfamily $B = \langle B_i : i \in \omega \rangle$ as follows:

- Search through the members of A in some effective fashion. Let B_0 be the first nonempty member of A found.
- Having defined $B_0, \ldots, B_n$ for some $n \geq 0$, search through A again. Let B_{n+1} be the first member of A that is not among $B_0, \ldots, B_n$ and intersects $B_0 \cap \cdots \cap B_n$.

FIP is computably false

Obstacle. But it could, for example, happen that:

- The first nonempty set we discover is A_1, so that we set $B_0 = A_1$.
- A_0 intersects A_1, but we discover this only after discovering that A_2 intersects A_1, so we set $B_1 = A_2$.
- A_0 intersects $A_1 \cap A_2$, but we discover this only after discovering that A_3 intersects $A_1 \cap A_2$, so we set $B_2 = A_3$.

$$\vdots$$

Then the intersection of A_0 with any finite number of members of B is nonempty, yet A_0 does not belong to B.

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We can exploit this problem to get a negative answer:

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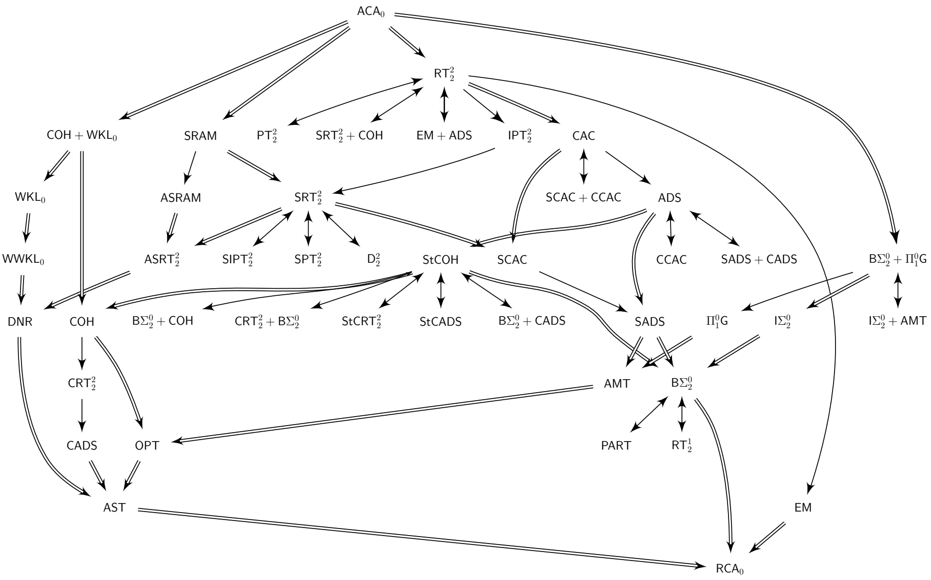
Corollary. FIP is not provable in WKL_0.

Proof is a considerably more complicated argument, but one of the ideas in it is still the exploitation of the failure of the computable strategy.

Principles between RCA_0 and ACA_0



Restricted Π^1_2 conservation

Many of these principles are restricted Π^1_2, i.e., of the form

$$(\forall A)[\varphi(A) \to (\exists B)\psi(A, B)],$$

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Proofs differ only in choice of forcing notion: Mathias forcing for COH, Cohen forcing for AMT and Π^0_1G.

Restricted Π^1_2 conservation

If we choose forcing notion carefully, we can adapt the proof for FIP.

The forcing notion. For every $A = \langle A_i : i \in \omega \rangle$, let $\mathbb{F}_A$ be the notion of forcing whose conditions are strings $\sigma \in \omega^{<\omega}$ such that

- there is an $x < \sigma(|\sigma| - 1)$ that belongs to $A_{\sigma(i)}$ for every $i < |\sigma| - 1$,
- and $\sigma \leq \tau$ iff $\sigma \upharpoonright |\sigma| - 1 \succeq \tau \upharpoonright |\tau| - 1$.

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- there is an $x < \sigma(|\sigma| - 1)$ that belongs to $A_{\sigma(i)}$ for every $i < |\sigma| - 1$,
- and $\sigma \leq \tau$ iff $\sigma \upharpoonright |\sigma| - 1 \succeq \tau \upharpoonright |\tau| - 1$.

Proposition (Dzhafarov and Mummert)**.** FIP is conservative over RCA_0 for restricted Π^1_2 sentences.

Corollary. None of the following are implied by FIP over RCA_0: RT^2_2, SRT^2_2, DNR, CAC, ADS, etc.

The atomic model theorem

Let T be a countable, complete, consistent theory.

A partial type of T is a T-consistent set of formulas in a fixed number of free variables. A (complete) type is a $\subseteq$-maximal partial type.

A model $\mathscr{M}$ of T realizes a partial type p if there is a tuple $\vec{a} \in |\mathscr{M}|$ such that $\mathscr{M} \models \varphi(\vec{a})$ for every $\varphi \in p$. Otherwise, $\mathscr{M}$ omits p.

A partial type p is principal if there is a formula ψ such that $T \vdash \psi \to \varphi$ for every formula $\varphi \in p$. A model $\mathscr{M}$ of T is atomic if every partial type realized in $\mathscr{M}$ is principal.

An atom of T is a formula ψ such that for every formula φ in the same free variables, exactly one of $T \vdash \psi \to \varphi$ or $T \vdash \psi \to \neg\varphi$ holds. T is atomic if for every T-consistent φ, $T \vdash \psi \to \varphi$ for some atom ψ.

The atomic model theorem

Classically, a theory is atomic if and only if it has an atomic model. This was studied by Hirschfeldt, Slaman, and Shore (2009) in the forms:

(AMT) Every complete atomic theory has an atomic model.

(OPT) If S is a set of partial types in a complete theory, then the theory has a model in which all the nonprincipal partial types in S are omitted.

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(Π^0_1G) For any uniformly Π^0_1 collection of sets S_i each of which is dense in $2^{<\mathbb{N}}$ there exists a set G such that $(\forall i)(\exists n)[G \upharpoonright n \in S_i]$.

The atomic model theorem

Theorem (Hirschfeldt, Slaman, and Shore)**.** Over RCA_0,

$$\Pi^0_1\mathsf{G} \to \mathsf{AMT} \to \mathsf{OPT}$$

and the implications are strict. The principles all lie strictly in-between RCA_0 and ACA_0 and are incomparable with WKL_0.

Theorem (Conidis; Hirschfeldt, Slaman, and Shore)**.** Over RCA_0, $\mathsf{AMT} + \mathsf{I}\Sigma^0_2 \to \Pi^0_1\mathsf{G}$.

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These principles are some of the weakest to have been studied that do not hold in the ω-model REC.

FIP and model-theoretic principles

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Theorem (Dzhafarov and Mummert)**.** Over RCA_0, $\Pi^0_1\mathsf{G} \to F\mathsf{IP}$.

So modulo Σ^0_2 induction, FIP follows from AMT.

Proof uses the forcing notion $\mathbb{F}_A$ discussed earlier to define an appropriate uniformly Π^0_1 collection of dense subsets of $2^{<\mathbb{N}}$.

FIP and model-theoretic principles

Theorem (Hirschfeldt, Shore, and Slaman)**.** Over RCA_0, the following are equivalent:.

- OPT.
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- For every set X, there is a set of degree hyperimmune relative to X.

Our proof that there is a computable instance of FIP with all solutions of hyperimmune degree formalizes in RCA_0. Hence, we have:

Theorem (Dzhafarov and Mummert)**.** Over RCA_0, $F\mathsf{IP} \to \mathsf{OPT}$.

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Csima, Hirschfeldt, Knight, and Soare (2004) showed no low$_2$ Δ^0_2 set computes an atomic model of every complete atomic decidable theory.

Corollary. There is an ω-model of FIP consisting entirely of sets Turing below a low$_2$ c.e. set. Hence, FIP does not imply Π^0_1G or even AMT.

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Question. Does OPT reverse to $\mathcal{F}$IP over RCA_0?

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Curiously, this does not seem to work. The question is open.

Other intersection principles

Fix $n \geq 2$. A family of sets A has the

- D_n intersection property if $\bigcap F = \emptyset$ for all n-element $F \subseteq A$.
- $\overline{D}_n$ intersection property if $\bigcap F \neq \emptyset$ for all n-element $F \subseteq A$.

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Theorem (Chang; Kurepa; Lévy; Vaught)**.** For every $n \geq 2$, ZF proves $\mathrm{AC} \leftrightarrow D_n\mathrm{IP} \leftrightarrow \overline{D}_n\mathrm{IP}$.

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The principles $\overline{D}_n$IP behave very similarly to FIP:

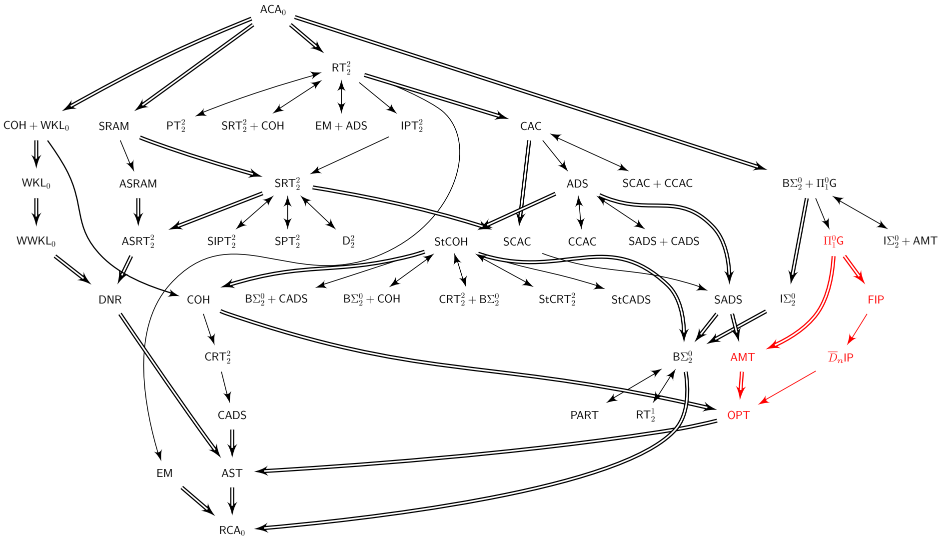
Theorem (Dzhafarov and Mummert)**.** For every $n \geq 2$, RCA_0 proves

$$F\mathsf{IP} \rightarrow \overline{D}_{n+1}\mathsf{IP} \rightarrow \overline{D}_n\mathsf{IP},$$

and all our other results about FIP also hold for $\overline{D}_n$IP.

Open question. Does $\overline{D}_n$IP imply FIP or at least $\overline{D}_{n+1}$IP for any n?

Updated picture



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Theorem (Rubin and Rubin)**.** Over ZF, AC $\leftrightarrow$ FCP.

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Fix $n \geq 1$. The following are provable in RCA_0:

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- $\Pi^1_n\text{-CA}_0 \leftrightarrow \Pi^1_n\text{-FCP} \leftrightarrow \Sigma^1_n\text{-FCP}$.
- $\mathrm{Z}_2 \leftrightarrow \mathrm{FCP}$.

The strength of FCP

Π^0_1-FCP reverses to ACA_0 because for any any Π^0_1 formula $\psi(x)$,

$$\varphi(X) \equiv (\forall n)[n \in X \to \psi(n)]$$

has finite character, and $\{x : \psi(x)\}$ is a maximal subset of $\mathbb{N}$ satisfying φ.

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Theorem (Dzhafarov and Mummert)**.** Σ^0_1-FCP is provable in RCA_0.

Finitary closure operators

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A nondeterministic finitary closure operator is a collection N of pairs (F, X) where F is finite and $X \neq \emptyset$.

A set A is N-closed if for every $(F, X) \in N$, $F \subseteq A \rightarrow X \cap A \neq \emptyset$.

Finitary closure operators

(CE) If φ is a formula of finite character, D a finitary closure operator, and A a set, then every D-closed subset of A satisfying φ extends to a $\subseteq$-maximal D-closed subset of A satisfying φ.

(NCE) If φ is a formula of finite character, N a nondeterministic finitary closure operator, and A a set, then every N-closed subset of A satisfying φ extends to a $\subseteq$-maximal N-closed subset of A satisfying φ.

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Theorem (Dzik; Rubin and Rubin)**.** Over ZF, AC $\leftrightarrow$ CE $\leftrightarrow$ NCE.

Formalizing CE and NCE in RCA_0

In RCA_0, we formalize finitary closure operators and nondeterministic closure operators as sets of pairs $\langle F, n \rangle$ where F is (a canonical index for) a finite subset of $\mathbb{N}$ and $n \in \mathbb{N}$.

We formalize nondeterministic finitary closure operators as sequences of pairs $\langle F, X \rangle$ respectively, where F is (a canonical index for) a finite subset of $\mathbb{N}$ and $\emptyset \neq X \subseteq \mathbb{N}$.

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We consider restrictions Γ-CE and Γ-NCE as above.

Theorem (Dzhafarov and Mummert)**.** Fix $n \geq 1$. If Γ is Δ^1_n, Σ^1_n, or Π^1_n, then over RCA_0, Γ-$\mathsf{CA}_0 \leftrightarrow \Gamma$-CE $\leftrightarrow \Gamma$-NCE. So $\mathsf{Z}_2 \leftrightarrow$ CE $\leftrightarrow$ NCE.

The strength of CE and NCE

More interesting things happen if Γ is a smaller class.

Theorem (Dzhafarov and Mummert)**.** Fix $n \geq 1$. Over RCA_0,

$$\mathsf{ACA}_0 \leftrightarrow \Pi^0_n\text{-CE} \leftrightarrow \Sigma^0_1\text{-CE} \leftrightarrow \text{CE for quantifier-free formulas.}$$

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$$\mathsf{ACA}_0 \leftrightarrow \Pi^0_n\text{-CE} \leftrightarrow \Sigma^0_1\text{-CE} \leftrightarrow \text{CE for quantifier-free formulas.}$$

Theorem (Dzhafarov and Mummert)**.** Fix $n \geq 1$. Over RCA_0,

$$\Pi^1_1\text{-CA}_0 \leftrightarrow \Pi^0_n\text{-NCE} \leftrightarrow \Sigma^0_1\text{-NCE} \leftrightarrow \text{NCE for quantifier-free formulas.}$$

Thank you for your attention!