Counting applications: when implications count

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Classical reverse mathematics.

The goal is to calibrate the strength of (countable analogues of) theorems according to which set-existence axioms are required for their proof.

We work in second-order arithmetic, and use various subsystems of this theory as benchmark systems against which to compare theorems.

In practice, five subsystems turn out to be especially ubiquitous:

RCA_0. basic arithmetic + Δ^0_1 comprehension + Σ^0_1 induction.

WKL_0. RCA_0 + existence of paths through infinite binary trees.

ACA_0. RCA_0 + arithmetical comprehension.

ATR_0. RCA_0 + iterability of arithmetical operators along any well-order.

Π^1_1-CA_0. RCA_0 + Π^1_1 comprehension.

The “big five” subsystems.

$$
\begin{array}{c}
Z_2 \\
\Downarrow \\
\vdots \\
\Downarrow \\
\Pi^1_1\text{-}\mathsf{CA}_0 \\
\Downarrow \\
\mathsf{ATR}_0 \\
\Downarrow \\
\mathsf{ACA}_0 \\
\Downarrow \\
\mathsf{WKL}_0 \\
\Downarrow \\
\mathsf{RCA}_0
\end{array}
$$

Regularity of most of mathematics.

Most “ordinary” theorems are either provable in RCA_0, or equivalent over RCA_0 to one of WKL_0, ACA_0, ATR_0 or Π^1_1-CA_0.

RCA_0. Baire category theorem, intermediate value theorem, Urysohn’s lemma, Tietze extension theorem, soundness theorem.

WKL_0. Prime ideal theorem, Gödel’s compactness theorem, separable Hahn/Banach theorem, Heine/Borel theorem.

ACA_0. Maximal ideal theorem, Ascoli lemma, Bolzano/Weierstrass theorem, ranges of functions exist, Turing jumps of sets exist.

ATR_0. Comparability of well-orders, perfect set theorem for complete separable metric spaces, Lusin’s separation theorem.

Π^1_1-CA_0. Cantor-Bendixson theorem, ability to discern which trees in a sequence of subtrees of $\mathbb{N}^{<\mathbb{N}}$ are infinite.

Weak irregular principles.

Over the past decade, a growing number of principles have been identified that lie outside the “big five”.

Cholak, Dzhafarov, Hirschfeldt, Jockusch, Kjos-Hanssen, Liu, Lempp, Mileti, Seetapun, Slaman. Ramsey’s theorem for pairs, RT^2_2.

Cholak, Friedman, Giusto, Hirst, Jockusch. Free set theorem and thin set theorem for pairs, TS^2 and FS^2.

Hirschfeldt and Shore. Chain/anti-chain principle (Dilworth’s theorem) and ascending/descending sequence principle, CAC and ADS.

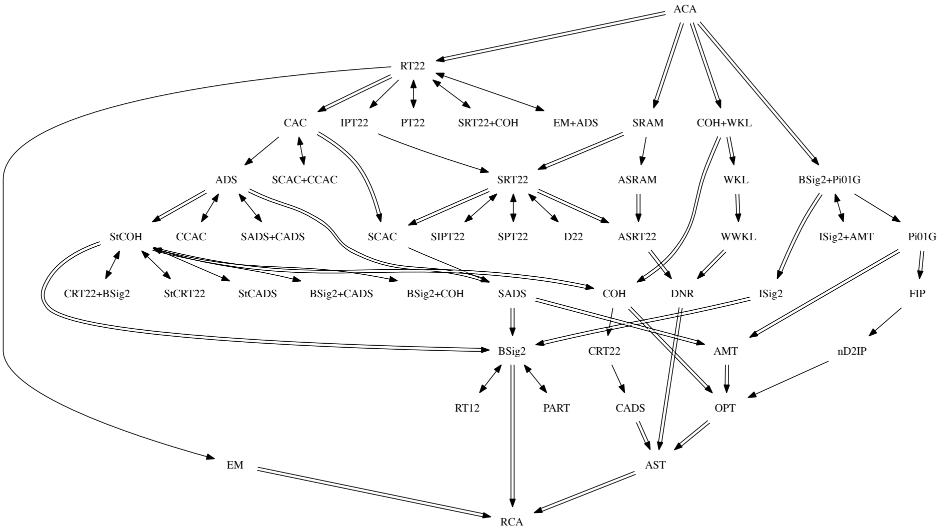
Dzhafarov and Hirst. Polarized Ramsey’s theorem for pairs, PT^2_2.

Hirschfeldt, Shore, and Slaman. Atomic model theorem, AMT.

Dzhafarov and Mummert. Finite intersection principle, FIP.

Flood. Ramsey-type König’s lemma, RKL.

The reverse mathematics zoo.



Relationship with computability theory.

Much of reverse mathematics focuses on Π^1_2 principles.

Say a principle P has the form

$$(\forall X)[\varphi(X) \implies (\exists Y)\psi(X, Y)],$$

where φ and ψ are arithmetical.

- Any X satisfying $\varphi(X)$ is called an instance of P.
- Any Y satisfying $\psi(X, Y)$ is called a solution to the instance X.

A typical implication $\mathsf{P} \to \mathsf{Q}$ of Π^1_2 principles in RCA_0 has the form:

For every instance A of Q, there exists an instance X of P computable from A, such that every solution Y to X computes a solution B to A.

In many cases, $B = Y$.

A simple example.

A k-coloring of exponent n is a map $f\colon [\mathbb{N}]^n \to k = \{0, \ldots, k-1\}$.

A set H is homogeneous for f if $f \upharpoonright [H]^n$ is a constant.

RT^n_k. Every $f\colon [\mathbb{N}]^n \to k$ has an infinite homogeneous set.

CAC. Every partial order has an infinite chain or anti-chain.

$\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \to \mathrm{CAC}$. Let $\leqslant_P$ partially order $\mathbb{N}$. Define $f\colon [\mathbb{N}]^2 \to 2$ by

$$f(x, y) = \begin{cases} 0 & x \leqslant_P y \vee y \leqslant_P x, \\ 1 & \text{else.} \end{cases}$$

A 0-homogeneous set for f is a chain for $\leqslant_P$; a 1-set is an anti-chain.

A more complicated implication.

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_3$. Fix $f : [\mathbb{N}]^n \to 3$. Define $g : [\mathbb{N}]^n \to 2$ by

$$g(x, y) = \begin{cases} 0 & f(x, y) = 0, \\ 1 & \text{else.} \end{cases}$$

A 0-homogeneous set for g is 0-homogeneous for f. Suppose H is an infinite 1-homogeneous set for g. Define $h : [H]^n \to 2$ by

$$h(x, y) = \begin{cases} 0 & f(x, y) = 1, \\ 1 & f(x, y) = 2. \end{cases}$$

A 0-homogeneous set for h is 1-homogeneous for f; a 1-homogeneous set for h is 2-homogeneous for f.

Uniform reductions.

Is the non-uniformity in the above proof necessary?

Question. Does RT^2_2 imply RT^2_3 by a single application?

Definition. Say m instances of RT^2_k are uniformly reducible to RT^2_j if there are Turing reductions Φ and Ψ such that:

- given $f_0, \ldots, f_{m-1} : [\mathbb{N}]^2 \to k$, $\Phi[f_0, \ldots, f_{m-1}]$ is a coloring $[\mathbb{N}]^2 \to j$;
- if H is any infinite homogeneous set for $\Phi[f_0, \ldots, f_{m-1}]$, $\Psi[H] = \langle H_0, \ldots, H_{m-1} \rangle$ where H_i is infinite and homogeneous for f_i.

Example. Two instances of RT^2_2 are uniformly reducible to RT^2_4.

Question. For $j < k$, is (one instance of) RT^2_k uniformly reducible to RT^2_j?

The squashing lemma.

Lemma (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). For $m \geqslant 2$, m instances of RT^2_k are not uniformly reducible to RT^2_k.

Suppose not, and let Φ and Ψ witness the uniform reduction. Say $m = 2$.

By repeatedly applying Φ, we can uniformly reduce n instances of RT^2_k to RT^2_k, for any $n \geqslant m$:

- given $f_0, f_1, f_2 : [\mathbb{N}]^2 \to k$, let $f = \Phi[f_0, \Phi[f_1, f_2]]$;
- given $f_0, f_1, f_2, f_3 : [\mathbb{N}]^2 \to k$, let $f = \Phi[f_0, \Phi[f_1, \Phi[f_2, f_3]]]$;

$\vdots$

The main idea of the proof is that this can be extended to ω instances.

Proof of the squashing lemma.

Fix $f_0, f_1, \ldots : [\mathbb{N}]^2 \to k$. We cannot just set $f = \Phi[f_0, \Phi[f_1, \cdots] \cdots]$.

Instead, we construct $h_0, h_1, \ldots : [\mathbb{N}]^2 \to k$ and $b_0, b_1, \ldots \in \mathbb{N}$ such that $h_i = \Phi[f_{i+1}, h_{i+1}]$ on all numbers beyond b_i.

Let $f = \Phi[f_0, h_0]$, and let H be infinite and homogeneous for f.

Then $\Psi[H] = \langle H_0, G_0 \rangle$, where H_0 is homogeneous for f_0 and G_0 is homogeneous for h_0. So $G_0 - b_0$ is homogeneous for $\Phi[f_1, h_1]$.

Then $\Psi[G_0 - b_0] = \langle H_1, G_1 \rangle$, where H_1 is homogeneous for f_1 and G_1 is homogeneous for h_1. So $G_1 - b_1$ is homogenous for $\Phi[f_2, h_2]$.

Continuing, we obtain H_i infinite and homogeneous for f_i.

Proof of the squashing lemma.

Theorem (Seetapun). Every $f : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set that does not compute $\emptyset'$.

By contrast, it is easy to construct a sequence $f_0, f_1, \ldots : [\mathbb{N}]^2 \to k$ such that any sequence $\langle H_0, H_1, \ldots \rangle$ in which each H_i is infinite and homogeneous for f_i computes $\emptyset'$. Namely, define

$$f_i(x, y) = \begin{cases} 1 & i \in \emptyset' \text{ by stage } x, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, there can be no uniform reduction of ω instances of RT^n_k to RT^n_k.

This is a contradiction, so the lemma is proved.

Multiple instances.

We can extend the definition of uniform reduction to define a uniform reduction of m instances of RT^2_k and n instance of RT^2_l to RT^2_j.

Example. One instance of RT^2_k and one instance of RT^2_l are uniformly reducible to RT^2_{kl}.

We obtain the following analogue of the squashing lemma:

Lemma (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). For $m \geqslant 2$, m instances of RT^2_k and one of RT^2_2 are not uniformly reducible to RT^2_k.

Corollary. There is no uniform reduction of RT^2_{2k} to RT^2_k.

No uniform reduction of colors.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). For $j < k$, there is no uniform reduction of RT^2_k to RT^2_j.

Proof. Suppose not, and let Φ and Ψ witness the reduction.

Fix n and m such that $j^n < 2^m < 2^{m+1} < k^n$, and $f\colon [\mathbb{N}]^2 \to 2^{m+1}$.

View f as a k^n-coloring. There exist $f_0, \dots, f_{n-1} : [\mathbb{N}]^2 \to k$ such that any set homogeneous for each of them is homogeneous for f.

Let $g_i = \Phi[f_i] : [\mathbb{N}]^2 \to j$, and let $g : [\mathbb{N}]^2 \to j^n$ be such that any set homogeneous for g is homogeneous for each g_i.

View g as a 2^m-coloring. Given any H homogenous for g, $\Psi[H]$ is homogenous for f. Thus, there is a uniform reduction of $\mathsf{RT}^2_{2^{m+1}}$ to $\mathsf{RT}^2_{2^m}$.

Extensions.

In the form “if n instances are reducible to one instance, then ω instances are reducible to one instance”, the squashing lemma can be formulated for various other combinatorial principles, including:

- RT^n_k, for any n;
- TS^n_k, for any n and k, asserting that for every $f : [\mathbb{N}]^n \to k$ there exists an infinite set H and $c < k$ such that $f(\bar{x}) \neq c$ for all $\bar{x} \in [H]^n$.
- RRT^n_k, for any n and k, asserting that for every k-to-one $f : [\mathbb{N}]^n \to \mathbb{N}$ there exists an infinite H such that f is injective on $[H]^n$.

For each of these examples, a degree-theoretic difference between finitely many and infinitely many instances can be found, thus showing there is no uniform reduction of finitely many instances to one instance.

Extensions.

The squashing lemma does not apply to WKL or WWKL, because paths through binary trees need not be closed under subset. Indeed, it is not difficult to see that there is a uniform reduction of ω instances of WKL to one instance of WKL.

Proposition (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). There exists a uniformly computable sequence $T_0, T_1, \ldots$ of binary trees of positive measure such that any sequence $\langle f_0, f_1, \ldots \rangle$ of paths has PA degree.

Corollary. There is no uniform reduction of ω many instances of WWKL to one instance of WWKL.

Corollary. There are no procedures Φ and Ψ such that if i is an index of a tree of positive measure and α is a rational, then $\Phi(i, \alpha)$ is a tree of measure $\geqslant \alpha$, such that $\Psi[f] \in [T]$ for all $f \in [\Phi(i, \alpha)]$.

Restricting access to the problem.

To show one Π^1_2 statement, P, implies another, Q, in RCA_0, we computably transform each instance A of Q into an instance X of P, such that for any solution Y to X, there is a solution B to A with $B \leqslant_T X \oplus Y$.

As mentioned above, often we have $B = Y$, and in many other cases we have $B = \Psi[Y]$ for some functional Ψ that does not depend on X.

Question. What computability-theoretic relationships hold between principles if we look for solutions B computable merely from the solution Y, and not from Y together with the instance X?

The stable Ramsey's theorem and cohesiveness.

A coloring $f : [\omega]^2 \to k$ is stable if for all x, $\lim_y f(x, y)$ exists.

A set S is cohesive for a sequence $A_0, A_1, \ldots$ if for each i, either $S \subseteq^* A_i$ or $S \subseteq^* \overline{A_i}$.

SRT^2_k. Every stable $f : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.

COH. Every sequence of sets admits a cohesive set.

The restriction of a computable coloring to a set cohesive for the recursive sets yields a stable coloring. This observation motivates the following result:

Theorem (Cholak, Jockusch, and Slaman; Mileti). Over RCA_0, $\mathsf{RT}^2_k \leftrightarrow \mathsf{SRT}^2_k + \mathsf{COH}$.

A question about ω models.

Whether SRT^2_2 and RT^2_2 are equivalent was, until very recently, a notoriously difficult, open question in reverse mathematics.

Theorem (Chong, Slaman, and Yue). In RCA_0, SRT^2_2 does not imply RT^2_2.

The proof of this result builds a non-standard model of SRT^2_2 in which the number theory has been highly customized to make RT^2_2 fail. Indeed, in this model, all sets are low!

In some sense, then, the question remains open. Recall that a model of second-order arithmetic is an ω-model if its integers are standard, and its sets are actual sets of integers. (These are also the Turing ideals.)

Question. Does SRT^2_2 imply RT^2_2 in ω-models? Equivalently, does SRT^2_2 imply COH in ω-models?

No Muchnik reduction without access to the problem.

Theorem (Dzhafarov). There exist sets A_0 and A_1 such that for every stable 2-coloring $f \leqslant_T A_0 \oplus A_1$, there exists an infinite subset of $\{x : \lim_y f(x, y) = 0\}$ or $\{x : \lim_y f(x, y) = 1\}$ that computes no set cohesive for both A_0 and A_1.

By contrast, consider the 4-coloring $f \leqslant_T A_0 \oplus A_1$ defined by

$$f(x, y) = \begin{cases} 0 & x \in A_0 - A_1, \\ 1 & x \in A_1 - A_0, \\ 2 & x \in A_0 \cap A_1, \\ 3 & x \notin A_0 \cup A_1. \end{cases}$$

Then any infinite homogeneous set for f is cohesive for A_0 and A_1.

No Muchnik reduction without access to the problem.

The theorem generalizes from 2-colorings to arbitrary k-colorings, by increasing the sequence from A_0, A_1 to $A_0, \ldots, A_{k-1}$.

By combining all these sequences into one, the conclusion holds for all stable colorings, regardless of the number of colors.

In the parlance of Muchnik degrees, this can be stated as follows:

- for a sequence of sets $\vec{A}$, let $\mathcal{C}_{\vec{A}}$ be the set of all $\vec{A}$-cohesive sets;
- for a coloring f, let $\mathcal{S}_f$ be the set of all sets of eventually like-colored numbers;

The theorem asserts there is an instance $\vec{A}$ of COH such that for every k, and every instance $f \leqslant_T \vec{A}$ of SRT^2_k, $\mathcal{C}_{\vec{A}} \nleqslant_w \mathcal{S}_f$.

Further extensions.

The sets A_0 and A_1 are obtained generically for a suitable forcing notion. The proof can be extended to show the following general fact about subsets of ω:

Theorem (Dzhafarov). There exist sets A_0 and A_1 such that for every arithmetical operator Γ, there is an infinite subset of $\Gamma^{A_0 \oplus A_1}$ or $\overline{\Gamma^{A_0 \oplus A_1}}$ that computes no set cohesive for both A_0 and A_1.

Thus, however complicated a set one arithmetically makes out of $A_0 \oplus A_1$, it or its complement will have a computationally feeble subset. This complements some previous results along these lines:

Theorem (Dzhafarov and Jockusch). Every set A has an infinite subset or co-subset that does not compute a given non-computable set.

Theorem (Soare). There is a set with no infinite subset of higher degree.

Thank you for your attention.