Reverse mathematics and the finite intersection principle

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Reverse mathematics and equivalents of the axiom of choice,
joint work with Carl Mummert (submitted).

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Theorem (Klimovsky; Rubin and Rubin)**.** Over ZF, AC $\leftrightarrow$ FIP.

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A family $B = \langle B_i : i \in \mathbb{N} \rangle$ is a subfamily of A if $(\forall i)(\exists j)[B_i = A_j]$.

A subfamily B of A is maximal among subfamilies with some property if for every subfamily C of A with that property, if B is a subfamily of C then C is a subfamily of B.

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(FIP). Every nontrivial family of sets has a maximal subfamily with the finite intersection property.

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The proof is a forcing argument that exploits the weak notion of "subfamily".

Stronger notions of "subfamily" result in FIP reversing to ACA_0.

Between RCA_0 and ACA_0

Theorem (Dzhafarov and Mummert)**.** There is a computable nontrivial family that has no computable maximal subfamily with the finite intersection property. Hence, $F\mathsf{IP}$ is not provable in RCA_0

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Continue. Either Φ_e will never output i, and then B will not be maximal, or it will, and then B will not have the finite intersection property.

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In fact, more is true:

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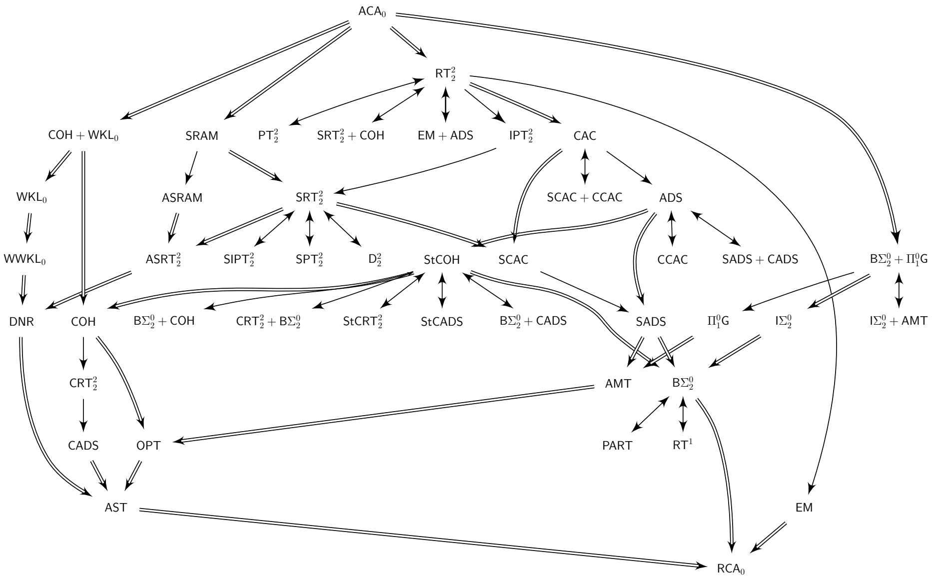
Proof is a considerably more complicated argument because we no longer have computable approximations to the potential maximal subfamilies.

Principles between RCA_0 and ACA_0



The atomic model theorem

Let T be a countable, complete, consistent theory.

A model $\mathscr{M}$ of T realizes a partial type p if there is a tuple $\vec{a} \in |\mathscr{M}|$ such that $\mathscr{M} \models \varphi(\vec{a})$ for every $\varphi \in p$. Otherwise, $\mathscr{M}$ omits p.

A partial type p is principal if there is a formula ψ such that $T \vdash \psi \to \varphi$ for every formula $\varphi \in p$. A model $\mathscr{M}$ of T is atomic if every type realized in $\mathscr{M}$ is principal.

An atom of T is a formula ψ such that for every formula φ in the same free variables, exactly one of $T \vdash \psi \to \varphi$ or $T \vdash \psi \to \neg\varphi$ holds. T is atomic if for every T-consistent φ, $T \vdash \psi \to \varphi$ for some atom ψ.

The atomic model theorem

Classically, a theory is atomic if and only if it has an atomic model. This was studied by Hirschfeldt, Slaman, and Shore (2009) in the forms:

Atomic model theorem (AMT)**.** Every complete atomic theory has an atomic model.

Omitting partial types principle (OPT)**.** For any collection S of partial types of a complete theory T, there is a model of T that omits all the the nonprincipal partial types in S.

Π^0_1 **genercity principle** (Π^0_1G)**.** For any uniformly Π^0_1 collection of dense subsets of $2^{<\mathbb{N}}$ $\langle S_i : i \in \mathbb{N}\rangle$ there exists G such that $(\forall i)(\exists n)[G \upharpoonright n \in S_i]$.

The atomic model theorem

Theorem (Hirschfeldt, Slaman, and Shore)**.** Over RCA_0,

$$\Pi^0_1\mathsf{G} \rightarrow \mathsf{AMT} \rightarrow \mathsf{OPT}$$

and the implications are strict. The principles all lie strictly in-between RCA_0 and ACA_0 and are incomparable with WKL_0.

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These principles are some of the weakest to have been studied that are not computably true.

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and the first implication is strict.

The second implication follows by formalizing our proof that there is a computable instance of FIP with all solutions of hyperimmune degree, and a result of Hirschfeldt, Shore, and Slaman that OPT is equivalent to the existence of a hyperimmune set.

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Open question. Does OPT imply FIP?

Thank you for your attention!