Mathematics, backwards and forwards.

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What do mathematicians do?

Mathematicians don’t care about:

- simplifying fractions
- avoiding square roots in denominators
- $\sin^2 x + \cos^2 x = 1$ (any more than any other formula, at least)

Mathematicians do care about:

- making precise statements
- proving those statements beyond reasonable doubt
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Statements that are proved are called theorems.
Proofs.

A proof of a theorem $T$ consists of a finite sequence

$S_1, S_2, \ldots, S_n$ where each of $S_1; S_2; \ldots, S_n$ is either a premise, or follows from some earlier (higher-up) member of $S_1, S_2; \ldots, S_n$ by a logical rule.
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$S_1$

$S_2$

\vdots

$S_n$

\hline

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A silly example.
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**Theorem.**

If $1 = 2$, then Tuesday is Friday.

(Have a great weekend!)
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Suppose $1 = 2$. 

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Axioms.

We want our premises to be true.
Axioms.

We want our premises to be **true**.

But remember: every premise is a theorem!

If $P$ is a premise, here is its proof:

\[
\begin{align*}
P \\
\hline
P
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So we don’t want too many premises!
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$\quad$

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Typically, we adopt as premises the most basic facts we can agree on.

We call these axioms.
Euclid’s axioms for geometry.

Euclid of Alexandria (4th century BCE).

Devised a system of five axioms for geometry in the plane.
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1. Two points determine a line.
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**Euclid’s axioms.**

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2. Every finite line segment can be extended to an infinite line.
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5. Given a line and a point not on the line, there is a unique line through the given point that is parallel to the given line.
Mathematics: forwards.

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▶ Adopt a system of axioms.
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What mathematicians do.

► Adopt a system of axioms.
► Prove theorems from these axioms.
But which axioms do we really need?

Question.

How do we know if our axioms are any good?
But which axioms do we really need?

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More specifically, how do we know we didn’t adopt too many axioms?
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Key insight.

Above all, all our axioms should be true. So if we can drop one of our axioms, then we should be able to prove it from the axioms that are left!
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Reformulating the question.

Are any of our axioms provable from the other axioms?

In other words, are any of our axioms redundant?
Euclid’s axioms, revisited.

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Ancient question.

Is Axiom 5 (“the parallel postulate”) necessary?
Non-euclidean geometry.

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This geometry, called elliptic geometry, is an example of non-Euclidean geometry.

It shows it’s possible to have a “world” where Axioms 1–4 hold, but Axiom 5 is false.
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Since the same logic applies in both worlds, there cannot be a proof of Axiom 5 from the other four axioms. (A proof is a proof is a proof...
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So axiom 5 is not redundant.
Here’s a fact we all know...

**Triangle Theorem.** The sum of the angles of a triangle is $180^\circ$. 
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Or do we...?
The Triangle Theorem.

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It’s a theorem because we can prove this from Euclid’s Axioms 1–5.
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As it turns out, we can prove Axiom 5 from Axioms 1–4 **together with** the Triangle Theorem.

(We say Axiom 5 and the Triangle Theorem are equivalent.)
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In particular we can’t prove the Triangle Theorem just from Axioms 1–4.

(In elliptic geometry, the sum of the angles of a triangle is $>180^\circ$.)
Foundations of mathematics.

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► Make sure these axioms prove everything that’s true.

Much of the quest to do this is recorded in history as a string of failures (attempts by Russell, Hilbert, Frege, and lots of other smart people).
Set theory.

Set theory provides a common language for all of mathematics.

The axioms of set theory tell us which things are sets, and what we can do with sets to form other sets.

A commonly-used set of axioms is called use Zermelo-Fraenkel (ZF) axioms (developed 1908; 1921).
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**Example.**

- The empty set, $\emptyset$, is a set; it has no elements.
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- If $X$ is a set, so is $\{X\}$. 
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- The empty set, $\emptyset$, is a set; it has no elements.
- If $X$ is a set, so is $\{X\}$.
- If $X$ and $Y$ are sets, so is their union: that is, the set of things in $X$ or in $Y$ or in both.
Set theory (continued).

How exactly do we build up all mathematical objects just from sets?

Example. Consider the set of natural numbers, \( f_0; 1; 2; 3; \ldots g. \)

We need to represent the number 0 as a set. Let's represent it by \( \emptyset. \)

What about the number 1? Let's use \( f\emptyset g. \) Notice that this is \( f0g. \)

For 2, let's use \( f0; 1g = f\emptyset; f\emptyset gg. \) This is the union of \( f\emptyset g \) and \( ff\emptyset gg. \)

Continuing, we get \( f0; 1; 2; 3; \ldots g \) represented by \( f\emptyset; f\emptyset g; f\emptyset; f\emptyset gg; f\emptyset; f\emptyset g; f\emptyset; f\emptyset gg; \ldots g. \)
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- Continuing, we get \(\{0, 1, 2, 3, \ldots\}\) represented by

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The Axiom of Choice.

**Task.** Here’s a non-empty set, $X$. Name me an element of it.
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The **Axiom of Choice** (abbreviated **AC**) says you can always do this.
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AC is funny. It says you can name an element, but doesn’t tell you how to do it. In that sense, it’s rather unusual.

We know that without AC, set theory does not get very far.
The well-ordering principle.

For every set $X$, there is a relation $<$ on $X$ as follows:

- for every $x$ and $y$ in $X$, exactly one of $x < y$ or $x = y$ or $y < x$ holds
- for all $x$, $y$, and $z$ in $X$, if $x < y$ and $y < z$ then $x < z$

there do not exist $x_1, x_2, x_3, \ldots$ in $X$ with $x_1 > x_2 > x_3 > \cdots$.

Theorem (Zermelo).

Using the axioms of ZF, AC is equivalent to the well-ordering principle.

Corollary. We cannot prove the well-ordering principle from ZF.
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The linear-ordering principle.

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**Theorem.**

1. The linear-ordering principle is provable from ZF together with AC.
2. The linear-ordering principle is not provable from ZF.
3. AC is not provable from ZF together with the linear-order principle.
Diagram.

ZF + AC

AC

well-ordering principle

linear-ordering principle

ZF
Relative to some fixed system of axioms,

- theorem $T_0$ is **stronger** than theorem $T_1$ if $T_1$ is provable from the axioms together with $T_0$.
The strength of a theorem.

Relative to some fixed system of axioms,

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Examples.

- Relative to Euclid’s first four axioms, the parallel postulate and the triangle postulate have the same strength.
The strength of a theorem.

Relative to some fixed system of axioms,

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**Examples.**

- Relative to Euclid’s first four axioms, the parallel postulate and the triangle postulate have the same strength.

- Relative to set theory, AC has the same strength as the well-ordering principle, but is (strictly) stronger than the linear-ordering principle.
What mathematicians do.

- Adopt a system of axioms.
- Prove theorems from these axioms.
What reverse mathematicians do.

- Look at axiom systems, and the theorems they prove.
- Prove which of these axioms are necessary to prove a given theorem, and which axioms can be dispensed with.
- Compare the strength of theorems: Which of two given theorems is stronger? Do they have the same strength?
Reverse mathematics.

Systematic study of the strength of mathematical theorems, initiated by Harvey Friedman and Stephen Simpson, starting in the 1970s.
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Led to the development of various new mathematical techniques:

► **Example:** yes, you can make due with fewer axioms to prove a given theorem, but you need a more efficient proof!
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**Extraordinary empirical fact.**

- In the Friedman-Simpson framework, there are **five theorems** that most other theorems end up having the same strength as!
Reverse mathematics.

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► Example: yes, you can make due with fewer axioms to prove a given theorem, but you need a more efficient proof!

Extraordinary empirical fact.

► In the Friedman-Simpson framework, there are five theorems that most other theorems end up having the same strength as!

► Each of the five represents a certain mathematical concept that shows up commonly, and across different areas of mathematics.
The zoo (rmzoo.uconn.edu).
Thanks for your attention!