Mathematics, backwards and forwards.

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Statements that are proved are called theorems.

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where each of $S_1, S_2 \ldots, S_n$ is either a premise, or follows from some earlier (higher-up) member of $S_1, S_2 \ldots, S_n$ by a logical rule.

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(Have a great weekend!)

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Typically, we adopt as premises the most basic facts we can agree on.

We call these axioms.

Euclid's axioms for geometry.

Euclid of Alexandria (4th century BCE).

Devised a system of five axioms for **geometry in the plane.**

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What mathematicians do.

- Adopt a system of axioms.
- Prove theorems from these axioms.

But which axioms do we *really* need?

Question.

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Key insight.

Above all, all our axioms should be true. So if we **can** drop one of our axioms, then we should be able to prove it from the axioms that are left!

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Reformulating the question.

Are any of our axioms provable from the other axioms?

In other words, are any of our axioms **redundant**?

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Ancient question.

Is Axiom 5 ("the parallel postulate") necessary?

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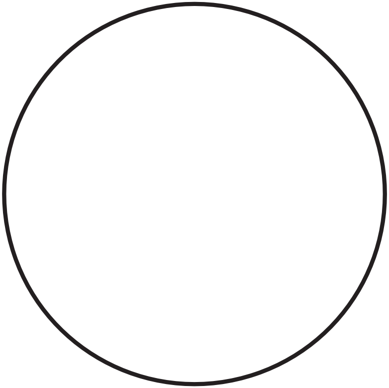

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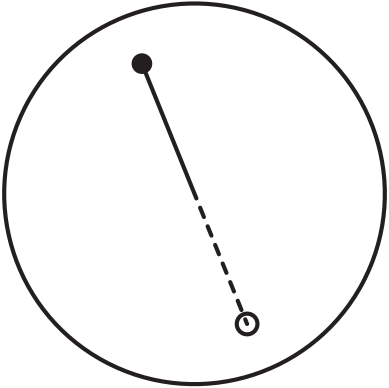

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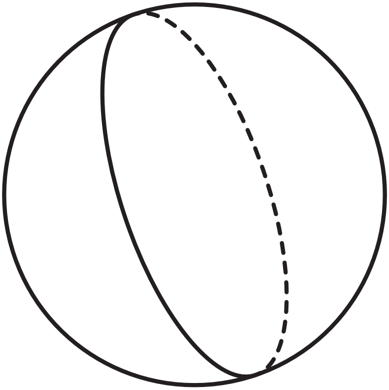

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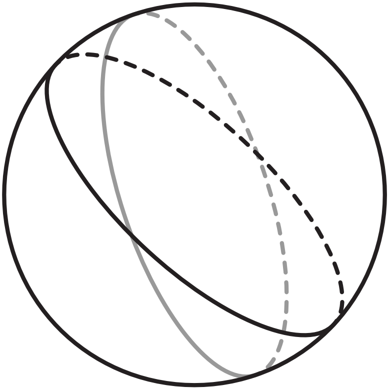

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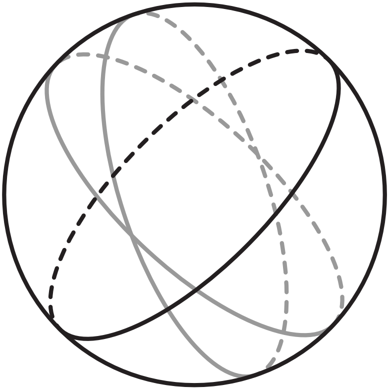

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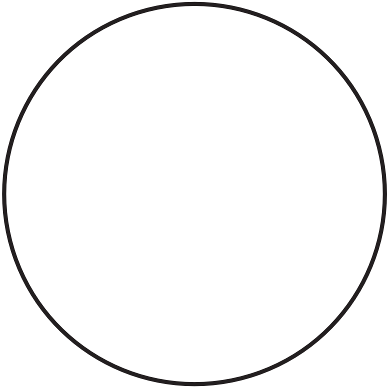

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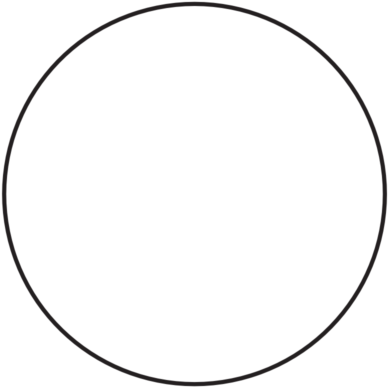

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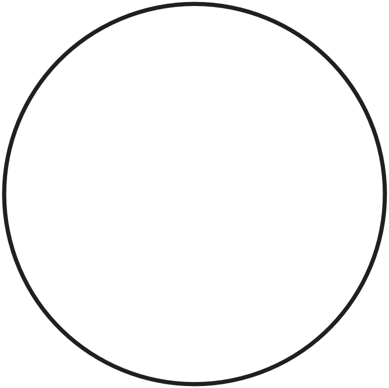

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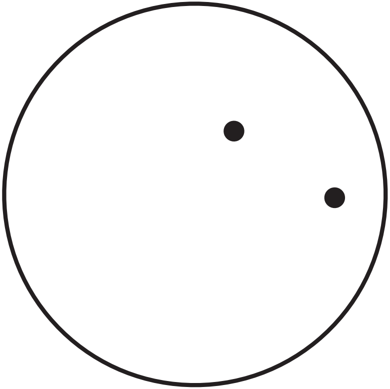

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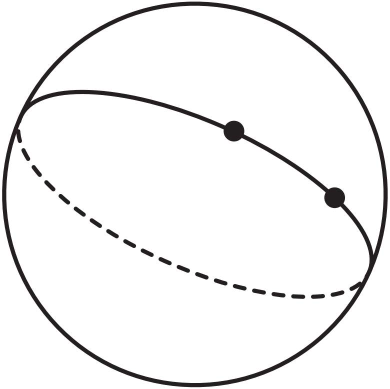

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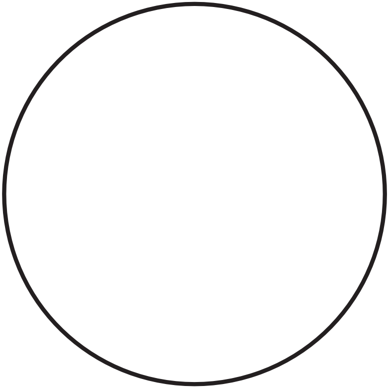

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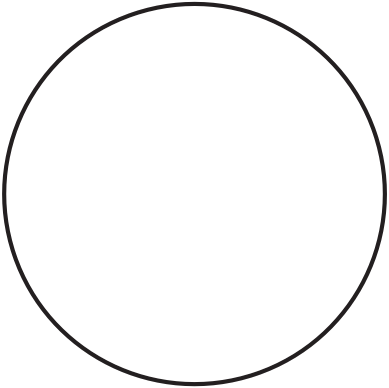

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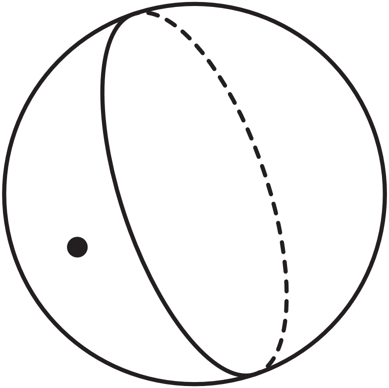

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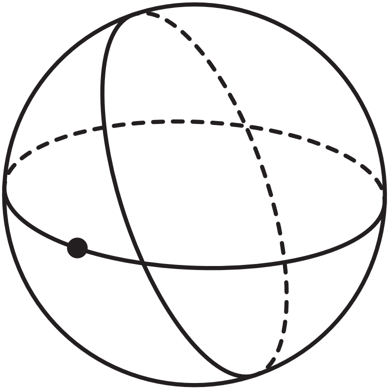

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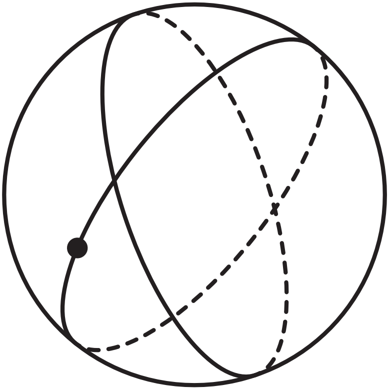

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So axiom 5 is **not redundant**.

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In particular we can't prove the Triangle Theorem just from Axioms 1–4.

(In elliptic geometry, the sum of the angles of a triangle is $> 180^\circ$.)

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Much of the quest to do this is recorded in history as a string of failures (attempts by Russell, Hilbert, Frege, and lots of other smart people).

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The **axioms of set theory** tell us which things are sets, and what we can do with sets to form other sets.

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- If X and Y are sets, so is their union: that is, the set of things in X or in Y or in both.

Set theory (continued).

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If you knew X was the set of U.S. presidents, you might name George Washington or Abraham Lincoln.

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AC is funny. It says you **can** name an element, but doesn't tell you **how** to do it. In that sense, it's rather unusual.

We know that without AC, set theory does not get very far.

The well-ordering principle.

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For every set X, there is a relation $<$ on X as follows:

- for every x and y in X, exactly one of $x < y$ or $x = y$ or $y < x$ holds
- for all x, y, and z in X, if $x < y$ and $y < z$ then $x < z$

there do not exist $x_1, x_2, x_3, \ldots$ in X with $x_1 > x_2 > x_3 > \cdots$.

Theorem (Zermelo).

Using the axioms of ZF, AC is equivalent to the well-ordering principle.

Corollary. We cannot prove the well-ordering principle from ZF.

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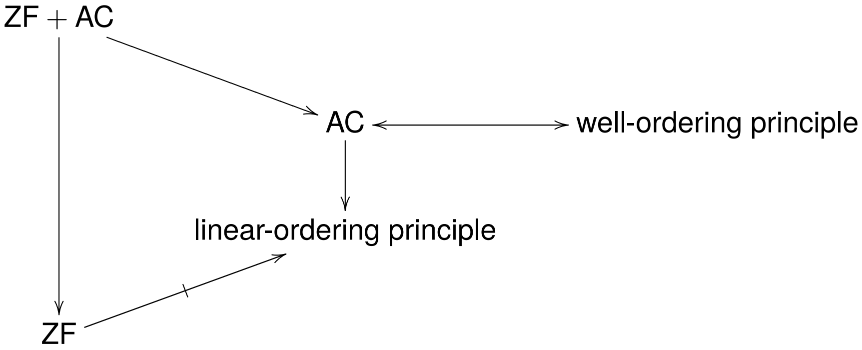
Theorem.

1. The linear-ordering principle is provable from ZF together with AC.
2. The linear-ordering principle is not provable from ZF.
3. AC is not provable from ZF together with the linear-order principle.

Diagram.



ZF + AC
AC
well-ordering principle
linear-ordering principle
ZF

The strength of a theorem.

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Examples.

- Relative to Euclid's first four axioms, the parallel postulate and the triangle postulate have the same strength.
- Relative to set theory, AC has the same strength as the well-ordering principle, but is (strictly) stronger than the linear-ordering principle.

Mathematics: backwards.

What mathematicians do.

- Adopt a system of axioms.
- Prove theorems from these axioms.

Mathematics: backwards.

What *reverse* mathematicians do.

- Look at axiom systems, and the theorems they prove.
- Prove which of these axioms are necessary to prove a given theorem, and which axioms can be dispensed with.
- Compare the strength of theorems: Which of two given theorems is stronger? Do they have the same strength?

Reverse mathematics.

Systematic study of the strength of mathematical theorems, initiated by Harvey Friedman and Stephen Simpson, starting in the 1970s.

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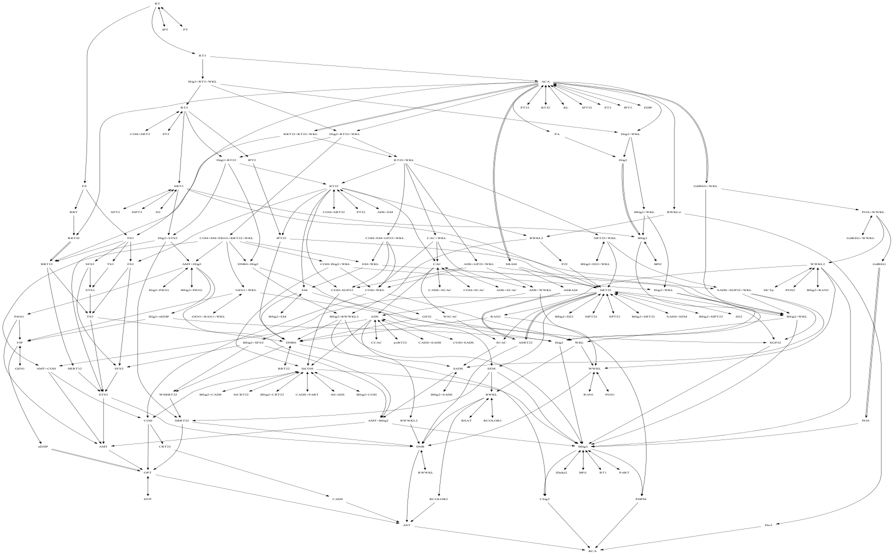
- Each of the five represents a certain mathematical concept that shows up commonly, and across different areas of mathematics.

The zoo (`rmzoo.uconn.edu`).



Thanks for your attention!