# Degrees of Mathias generics

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Cholak, D., Hirst, and Slaman.
Genericity for computable Mathias forcing.
Annals of Pure and Applid Logic, 165, 2014.

Cholak, D., and Soskova.
Mathias generics in different ideals.
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# Mathias conditions.

Recall that a (Turing) ideal is a subset $\mathcal{I}$ of $\mathcal{P}(\omega)$ closed under $\oplus$ and $\leq_T$.

**Definition.**

1. A (Mathias) $\mathcal{I}$-condition is a pair $(D, E)$, where $D$ is a finite set, $E$ is an element of $\mathcal{I}$, and $\max D < \min E$.
2. An $\mathcal{I}$-condition $(D', E')$ extends $(D, E)$ if $D \subseteq D' \subseteq D \cup E$ and $E' \subseteq E$.

Named after Mathias's use of it in set theory, but used earlier by Soare and others in computability theory.

Useful in studying Ramsey's theorem and related principles.

Of particular interest in computability when $\mathcal{I}$ is COMP or a Scott set.

# Mathias genericity.

We restrict to countable ideals $\mathcal{I}$, which we identify with some convenient representations (e.g., an enumeration, or by Kleene and Spector, an exact pair).

**Definition.** Fix an ideal $\mathcal{I}$, a class of $\mathcal{I}$-conditions $\mathcal{C}$, and a set $G$.

1. $G$ meets $\mathcal{C}$ if $D \subseteq G \subseteq D \cup E$ for some $(D, E) \in \mathcal{C}$.
2. $G$ avoids $\mathcal{C}$ if $D \subseteq G \subseteq D \cup E$ for some $(D, E)$ having no extension in $\mathcal{C}$.
3. $G$ is $n$-$\mathcal{I}$-generic if it meets or avoids every $\Sigma^0_n(\mathcal{I})$ class of $\mathcal{I}$-conditions.

We represent an $\mathcal{I}$-condition $(D, E)$ by a code $\langle d, e \rangle$, where $d$ is the canonical index of $D$, and $e$ is a $\Delta^0_1(\mathcal{I})$ index of the characteristic function of $E$.

Then the set of conditions is $\Pi^0_2(\mathcal{I})$, so we only look at $n$-generics for $n \geq 3$.

**Problem.** Study the complexity of Mathias generics for different ideals.

# Comparison with Cohen genericity.

The complexity of Cohen generics: Jockusch, Kurtz, and many others.

Similarities.

1. Implications: $n$-generic $\implies$ weakly $n$-generic $\implies$ $(n-1)$-generic.

2. There exists an $n$-generic $G \leq_T \emptyset^{(n)}$.

3. Every weakly $n$-generic set is hyperimmune relative to $\emptyset^{(n-1)}$.

Dissimilarities.

1. Every Mathias $n$-generic set $G$ is cohesive. (So $G^{[0]} =^* \emptyset$ or $G^{[1]} =^* \emptyset$.)

2. If $G$ is Mathias 3-generic then $G' \geq \emptyset''$.

3 (Cholak, D., Hirst, and Slaman). Forcing $\Sigma^0_n/\Pi^0_n$ statements is $\Sigma^0_{n+1}/\Pi^0_{n+1}$.

# Degrees of Mathias genericity.

**Corollary.** No Mathias 3-generic can be Cohen 1-generic.
No Mathias 3-generic can be computed by a Cohen 2-generic.

It is a well-known result of Jockusch that if $G$ is Cohen $n$-generic then $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$. In particular, every Cohen generic set has $\mathbf{GL}_1$ degree.

**Theorem** (Cholak, D., Hirst, and Slaman). If $G$ is Mathias $n$-generic, then:

1. $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$;
2. $G$ has $\mathbf{GH}_1$ degree, i.e., $G' \equiv_T (G \oplus \emptyset')'$.

**Corollary.** No Mathias $n$-generic can have Cohen 1-generic degree, but every Mathias $n$-generic computes a Cohen 1-generic.

Proof. By Jockusch and Posner, every $\overline{\mathbf{GL}_2}$ degree bounds a 1-generic degree.

# A coding theorem.

In fact, more is true:

**Theorem** (Cholak, D., Hirst, and Slaman). Every Mathias $n$-generic computes a Cohen $n$-generic.

$\neg$ Proof. Map Cohen conditions to Mathias conditions by

$$\sigma \in 2^{<\omega} \mapsto (\{x < |\sigma| : \sigma(x) = 1\}, \{x \in \omega : x > |\sigma|\}).$$

The conditions on the right are not dense in the Mathias conditions.

Actual proof uses a mechanism to encode $\emptyset', \emptyset'', \ldots, \emptyset^{(n)}$ into densely many gaps between successive elements of a Mathias generic.

**Note.** The analog of this result in set theory fails. A. Miller showed that adding a Mathias real to a transitive model of ZFC does not add a Cohen real.

# Generics for different ideals.

Many results can be obtained about Mathias $\mathcal{I}$-generics just by relativizing from the case of $\mathcal{I} = \mathrm{COMP}$. For instance, we have the following.

**Fact.** If $G$ is 3-$\mathcal{I}$-generic, then $p_G$ dominates every function in $\mathcal{I}$.

Other results are less straightforward to generalize.

**Theorem** (Cholak, D., and Soskova). Let $\mathcal{I} \subseteq \mathcal{J}$ be ideals.

1. If $\mathcal{I} \neq \mathcal{J}$, there exists an $\mathcal{I}$-generic set that computes no 3-$\mathcal{J}$-generic set.
2. If $\mathcal{I}, \mathcal{J}$ are arithmetic and $\Sigma^0_m(\mathcal{I}) \subseteq \Sigma^0_n(\mathcal{J})$, then every $n$-$\mathcal{J}$-generic set computes an $m$-$\mathcal{I}$-generic.

It is natural to ask which sets can/must be computed by a Mathias $\mathcal{I}$-generic.

**Fact.** If $A \notin \mathcal{I}$, there is a Mathias $\mathcal{I}$-generic that does not compute $A$.

# Coding ideals into generics.

**Observation** (Cholak, D., and Soskova). Let $\mathcal{I}$ be a $\Delta^0_2$ ideal, and let $G$ be any 3-$\mathcal{I}$-generic set. Then $G$ computes every element of $\mathcal{I}$.

Proof. Every $\Delta^0_2$ set $A$ has a self-modulus (i.e., a function $f \equiv_T A$ such that $A \leq_T g$ for every $g$ that dominates $f$).

(In fact, if $\emptyset^{(n-2)} \in \mathcal{I}$, then every 3-$\mathcal{I}$-generic computes every $\Delta^0_n$ set in $\mathcal{I}$.)

This motivates the following definition.

**Definition.** An ideal $\mathcal{I}$ is $n$-generically coded if every $n$-$\mathcal{I}$-generic set computes every member of $\mathcal{I}$.

Thus, every $\Delta^0_2$ ideal is 3-$\mathcal{I}$-generically coded.

**Question.** Is every arithmetical ideal $n$-generically coded for some $n$?

# A non-coding theorem.

The answer to the question above is no for arbitrary countable ideals. Soare built an infinite set $S$ not computable from any of its infinite, co-infinite subsets.

Let $\mathcal{I}$ be the principal ideal ${}^{\leq_T}S = \{X : X \leq_T S\}$.

Let $G$ be any $\mathcal{I}$-generic satisfying the $\mathcal{I}$-condition $(\emptyset, S)$. Then $G$ is an infinite subset of $S$, and by genericity, $S - G$ is also infinite. So $S \not\leq_T G$.

This ideal is very complicated, as $S$ above cannot even be arithmetical.

(The subsets of an infinite arithmetical set are upwards closed in degree.)

We obtain the following improvement and sharp bound.

**Theorem** (Cholak, D., and Soskova). There is a $\Delta^0_3$ ideal which is not $n$-generically coded for any $n$.

# A non-coding theorem.

**Theorem** (Cholak, D., and Soskova). There is a $\Delta^0_3$ ideal which is not $n$-generically coded for any $n$.

**Definition.** A set $S$ is $\mathcal{I}$-hereditarily uniformly $A$-computing if every infinite subset of $S$ in $\mathcal{I}$ uniformly computes $A$.

**Lemma.** If $A$ is a $\Delta^0_n(\mathcal{I})$ set, then an $n$-$\mathcal{I}$-generic $G$ computes $A$ only if $G$ is contained in an $\mathcal{I}$-hereditarily uniformly $A$-computing member of $\mathcal{I}$.

Outline of proof of Theorem.

Build a $\Delta^0_3$ set $G$ such that for every reduction $\Gamma$ and every infinite $S \leq_T G$, there is an infinite $S' \subseteq S$ with $S' \leq_T S$ and $\Gamma^{S'} \neq G$.

Let $\mathcal{I} = {}^{\leq_T}G$. Then no member of $\mathcal{I}$ is $\mathcal{I}$-hereditarily uniformly $G$-computing.

Thus, no 3-$\mathcal{I}$-generic set can compute $G$.

# Characterization.

**Proposition** (Cholak, D., and Soskova). TFAE for any ideal $\mathcal{I}$ and set $A$.

1. Every $n$-$\mathcal{I}$-generic set computes $A$.
2. Every set in $\mathcal{I}$ has an $\mathcal{I}$-hereditarily uniformly $A$-computing subset in $\mathcal{I}$.

Recall that a set is introreducible if it is computable from all its infinite subsets. Every degree contains an introreducible set.

If a set has a self-modulus, it has an introreducible subset of the same degree. Slaman and Groszek constructed an infinite $\Delta^0_3$ set with no self-modulus.

Using methods from the proof of the proposition, we extend this as follows.

**Theorem** (Cholak, D., and Soskova). There is an infinite $\Delta^0_3$ set with no infinite introreducible subset of the same degree.

# Questions.

1. If $\mathcal{I} \neq \mathcal{J}$ are ideals, is there always an $\mathcal{I}$-generic that computes no $n$-$\mathcal{I}$-generic, for some $n$? (Cholak, D., and Soskova: yes if $\mathcal{I} \subseteq \mathcal{J}$.)

2. Every Turing degree contains an infinite introreducible set. Is there a non-$\Delta^0_2$-degree, every member of which has an infinite introreducible subset of the same degree?

3. Let $n$-MG be the statement of second-order arithmetic asserting the existence of a Mathias $n$-generic. Let $n$-CG be the statement asserting the existence of a Cohen $n$-generic. In $\mathsf{RCA}_0$, we have the strict implications

$$\cdots \implies (n+1)\text{-CG} \implies n\text{-CG} \implies \cdots \implies 1\text{-CG}.$$

Is the same true for the $n$-MG? Could 3-MG imply $n$-MG for some $n > 3$?

Thank you.