Stable Ramsey's theorem and measure

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Stable Ramsey's theorem

Definition

Let $X \subseteq \omega$ be an infinite set and let $n, k \in \omega$.

- $[X]^n := \{Y \subset X : |Y| = n\}$.
- A *coloring on X* is a function $f : [X]^n \to k = \{0, \dots, k-1\}$.
- A set $H \subseteq X$ is *homogeneous for f* if $f \upharpoonright [H]^n$ is constant.
- A coloring $f : [X]^2 \to k$ is *stable* if for all $x \in X$, $\lim_{y \in X} f(x, y)$ exists.

Stable Ramsey's theorem

(RT^n_k) Every $f : [\omega]^n \to k$ has an infinite homogeneous set.

(SRT^2_k) Every stable $f : [\omega]^2 \to k$ has an infinite homogeneous set.

We deal only with stable colorings, and only with $k = 2$.

Stable Ramsey's theorem

Fact

- *For every computable stable $f : [\omega]^2 \to 2$, there is a Δ^0_2 set every infinite subset or cosubset of which computes an infinite homogeneous set for f.*
- *For every Δ^0_2 set A, there is a computable stable $f : [\omega]^2 \to 2$ such that every infinite homogeneous set of f is a subset or cosubset of A.*

Stable Ramsey's theorem

Theorem (Hirschfeldt, 2006)

Every Δ^0_2 set has an infinite subset or cosubset $H <_T \emptyset'$.

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Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There is a Δ^0_2 set with no low infinite subset or cosubset.

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Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There is a Δ^0_2 set with no low infinite subset or cosubset.

Theorem (Cholak, Jockusch, and Slaman, 2001)

Every Δ^0_2 set has a low_2 infinite subset or cosubset.

Stable Ramsey's theorem

Definition (Mileti, 2005)

Let $\mathbf{d}$ be a degree.

- Let $\mathscr{C}_{\mathbf{d}}$ be the class of all Δ^0_2 sets with an infinite subset or cosubset of degree $\leq \mathbf{d}$.
- Say $\mathbf{d}$ is *s-Ramsey* if $\mathscr{C}_{\mathbf{d}} = \Delta^0_2$.

Stable Ramsey's theorem

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Theorem (Mileti, 2005)

- *There is no s-Ramsey degree* $\mathbf{d} < \mathbf{0}'$.
- *There is no low$_2$ s-Ramsey degree.*

Measure

Definition

- A *martingale* is a map $M : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$ such that for all $\sigma \in 2^{<\omega}$,

$$M(\sigma) = \frac{M(\sigma 0) + M(\sigma 1)}{2}$$

- A martingale *succeeds* on a set X if $\limsup_{n \to \infty} M(X \upharpoonright n) = \infty$.
- A class $\mathscr{C}$ of Δ^0_2 sets is Δ^0_2 *null* if there is a martingale $M \leq_T \emptyset'$ that succeeds on every $X \in \mathscr{C}$.

Measure

Theorem (Hirschfeldt and Terwijn, 2008)

The class of low sets is not Δ^0_2 null.

Corollary

The class of Δ^0_2 sets having a low infinite subset or cosubset is not Δ^0_2 null.

Measure

Definition

A degree $\mathbf{d}$ is *almost s-Ramsey* if $\mathscr{C}_{\mathbf{d}}$ is not Δ^0_2 null.

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A degree $\mathbf{d}$ is *almost s-Ramsey* if $\mathscr{C}_{\mathbf{d}}$ is not Δ^0_2 null.

Theorem (Dzhafarov)

There is no almost s-Ramsey degree $\mathbf{d} < \mathbf{0}'$.

Measure

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There is an almost s-Ramsey degree that is not s-Ramsey.

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Proof idea.

- Fix $A \in \Delta^0_2$ with no low infinite subset or cosubset.
- Let $M_0, M_1, \ldots$ list all $\emptyset'$-computable martingales.
- For all i, fix L_i on which M_i does not succeed and $\bigoplus_{j \leq i} L_i$ is low.
- Let $D^{[0]} = L_0$.
- If $(\exists x \notin A)(\exists F \text{ finite})[F^{[0]} \upharpoonright \max F = D^{[0]} \upharpoonright \max F \wedge \Phi^F_0(x) \downarrow = 1]$: let $r_1 = \varphi^F_0(x)$, make $F \subset D$, and let $D^{[1]} = F^{[1]} \cup \{x \in L_1 : x > r_1\}$.
- If not, let $D^{[1]} = L_1$.
- Continue.

□

Measure

Theorem (Dzhafarov)

There is an almost s-Ramsey degree $\mathbf{d} < \mathbf{0}''$ *which is not s-Ramsey.*

Measure

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Question

Is there a low_2 almost s-Ramsey degree?

Reverse mathematics

Recall the following principles:

(COH) For every family $\langle X_i : i \in \mathbb{N} \rangle$ there is a set C such that for all i, $C \subseteq^* X_i$ or $C \subseteq^* \overline{X_i}$.

(DNR) For every set X there is an f such that for all e, $\Phi_e^X(e) \neq f(e)$.

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Theorem (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman, 2006)

Over RCA_0, $\mathsf{SRT}^2_2 \implies \mathsf{DNR}$.

Question

Over RCA_0, does $\mathsf{SRT}^2_2 \implies \mathsf{WKL}_0$ or $\mathsf{SRT}^2_2 \implies \mathsf{COH}$?

Reverse mathematics

Define the following principles:

(SRAM) For every set X there is a set Y such that every stable coloring $f \leq_T X$ has an infinite homogeneous set $H \leq_T Y$.

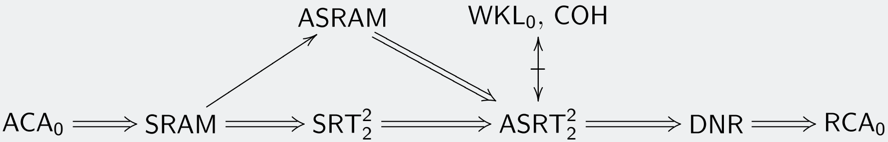
(ASRAM) For every set X there is a set Y such that for every X-computable approximation to a martingale M there is a stable coloring $f \leq_T X$ on which M does not succeed and which has an infinite homogeneous set $H \leq_T Y$.

(ASRT^2_2) For every approximation M_s to a martingale M there is a stable coloring $f \leq_T M_s$ on which M does not succeed and which has an infinite homogeneous set.

Reverse mathematics

Theorem (Dzhafarov)

Over RCA_0*, the following implications hold:*



(Double arrows are not reversible in RCA_0*.)*

Thank you for your attention.