# New directions in reverse mathematics

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February 20, 2013

# A foundational motivation.

Reverse mathematics is motivated by a foundational question:

Question. Which axioms do we really need to prove a given theorem?

The question leads to the idea of the strength of a theorem. Which theorems does it imply? Which imply it? Which is it equivalent to?

Example. Over ZF, the axiom of choice is equivalent to Zorn's lemma.

Set theory is much too powerful to accurately calibrate strength.

We would like results of the form

Over the theory $T$, theorem $P$ implies/is equivalent to theorem $Q$,

where $T$ is relatively weak, but also expressive enough to accommodate a decent amount of mathematics.

# Subsystems of second-order arithmetic.

Second-order arithmetic, $Z_2$, is a two-sorted theory with variables for numbers and sets of numbers, and the usual symbols of arithmetic.

The axioms of $Z_2$ are those of Peano arithmetic, and the comprehension scheme: if $\varphi$ is a formula (with set parameters) then $\{x : \varphi(x)\}$ exists.

Example. If $G_0, G_1, \ldots$ and $G$ are groups, then $\{i \in \mathbb{N} : G_i \cong G\}$ exists.

We restrict which formulas $\varphi$ we allow to obtain various subsystems:

| | |
|---|---|
| $\mathrm{RCA}_0$ | $\Delta^0_1$ (computable) formulas. |
| $\mathrm{WKL}_0$ | Formulas defining paths through infinite binary trees. |
| $\mathrm{ACA}_0$ | Arithmetical formulas. |

Two other subsystems, $\mathrm{ATR}_0$ and $\Pi^1_1$-$\mathrm{CA}_0$, complete the "big five".

# Subsystems of second-order arithmetic.

$$
\begin{array}{c}
Z_2 \\
\Downarrow \\
\Pi^1_1\text{-CA}_0 \\
\Downarrow \\
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\text{ACA}_0 \\
\Downarrow \\
\text{WKL}_0 \\
\Downarrow \\
\text{RCA}_0
\end{array}
$$

# Connections with computability theory.

A set $A$ is computable from $B$, written $A \leqslant_T B$, if $A = \Phi(B)$ for some effective procedure $\Phi$; i.e., knowing $B$, we can computably figure out $A$.

The halting set of $A$ is $A' = \{i \in \mathbb{N} : \text{the } i\text{th program with oracle } A \text{ halts}\}$.

A model of $Z_2$ consists of (a possibly nonstandard version of) $\mathbb{N}$ and a subset of $\mathcal{P}(\mathbb{N})$ closed under certain computability-theoretic operations.

| $RCA_0$ | Models closed under $\leqslant_T$ and $\oplus$ (disjoint union). |
|---|---|
| $ACA_0$ | Models closed under the jump operator, $A \mapsto A'$. |

The strength of a theorem corresponds to how effectively solutions can be found to a given computable instance of that theorem.

Example. Does every computable commutative unit ring have a computable maximal ideal? Do all ideals compute $\emptyset'$?

# Reverse mathematics.

Most (countable) classical mathematics can be developed within $Z_2$.

Theorem. The following are provable in $RCA_0$.

1 (Simpson). Baire category theorem, intermediate value theorem.

2 (Brown; Simpson). Urysohn's lemma, Tietze extension theorem.

3 (Rabin). Existence of algebraic closures of countable fields.

Work in reverse mathematics has revealed a striking fact: most theorems are provable in $RCA_0$, or equivalent to one of the other four systems.

Theorem. The following are equivalent to $WKL_0$ over $RCA_0$.

1 (Brown; Friedman). Heine-Borel theorem for $[0, 1]$.

2 (Orevkov; Shoji and Tanaka). Brouwer fixed-point theorem.

3 (Friedman, Simpson, and Smith). Prime ideal theorem.

# Reverse mathematics.

Theorem. The following are equivalent to $ACA_0$ over $RCA_0$.

1 (Friedman). Bolzano-Weierstrass theorem.

2 (Dekker). Existence of bases in vector spaces.

3 (Friedman, Simpson, and Smith). Maximal ideal theorem.

The maximal ideal theorem is stronger than the prime ideal theorem.

Question. Are there irregular principles, lying outside the big five?

Remark. Induction in $RCA_0$ is limited to $I\Sigma^0_1$, i.e., to

$$(\varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(x+1)]) \rightarrow (\forall x)\varphi(x)$$

for $\Sigma^0_1$ formulas $\varphi$. $RCA_0$ cannot prove $I\Sigma^0_n$ for $n > 1$.

# Ramsey's theorem.

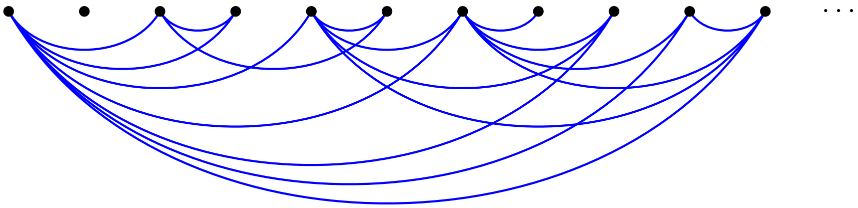
Definition. Let $[\mathbb{N}]^n$ denote the set of all $n$-element subsets of $\mathbb{N}$.

A $k$-coloring of $[\mathbb{N}]^n$ is a map $f : [\mathbb{N}]^n \to k = \{0, 1, \ldots, k-1\}$.

RT$(n, k)$. Every $k$-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

So RT$(1, k)$ is just the pigeonhole principle.

RT$(2, k)$ asserts the existence of infinite cliques or anticliques.



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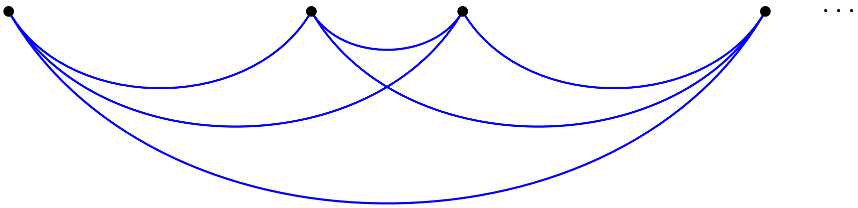

# Ramsey's theorem.

RT$(1,k)$ is proved in $RCA_0$. For $n \geqslant 2$, RT$(n,k)$ is provable in $ACA_0$.

Theorem (Jockusch). For $n \geqslant 3$, RT$(n,k)$ is equivalent to $ACA_0$.

How strong is RT$(2,k)$? It can be shown that it is not provable in $WKL_0$.

Theorem. Let $f$ be a computable $k$-coloring of $[\mathbb{N}]^2$.

1 (Seetapun). For any $C \not\leqslant_T \emptyset$, $f$ has an infinite homogeneous set $H \not\geqslant_T C$.

2 (Cholak, Jockusch, and Slaman). $f$ has a $\text{low}_2$ infinite homogeneous set.

3 (Dzhafarov and Jockusch). $f$ has two $\text{low}_2$ infinite homogeneous sets $H_0$ and $H_1$ whose degree form a minimal pair.

This allows us to build a model of $RCA_0 + \text{RT}(2,k)$ not computing $\emptyset'$.

Corollary. RT$(2,k)$ does not imply $ACA_0$ over $RCA_0$.

# Ramsey's theorem.

$$\Pi^1_1\text{-CA}_0$$

$$\Downarrow$$

$$\text{ATR}_0$$

$$\Downarrow$$

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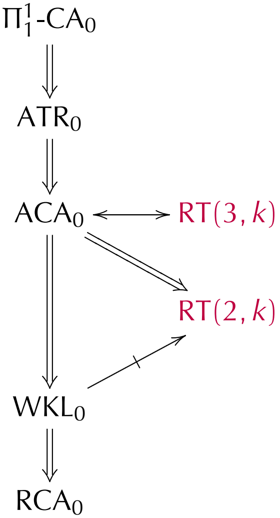
$$\text{WKL}_0$$

$$\Downarrow$$

$$\text{RCA}_0$$

Ramsey's theorem.



# A zoo of combinatorial principles.

Elucidating the strength of $\mathrm{RT}(2,k)$ has given rise to an industry of research into the reverse mathematics of combinatorial principles.

Downey, Hirschfeldt, Lempp, and Solomon; Mileti; Liu. Further investigation of the information content of $\mathrm{RT}(2,k)$.

Cholak, Jockusch, and Slaman. Stable Ramsey's theorem, cohesive principle.

Hirschfeldt and Shore. Principles about partial and linear orders.

Dzhafarov and Hirst. Polarized versions of Ramsey's theorem.

Bovykin and Weiermann; Lerman, Solomon, and Towsner. Tournaments.

Ambos-Spies et al.; Dzhafarov. Combinatorial principles related to notions of algorithmic randomness.

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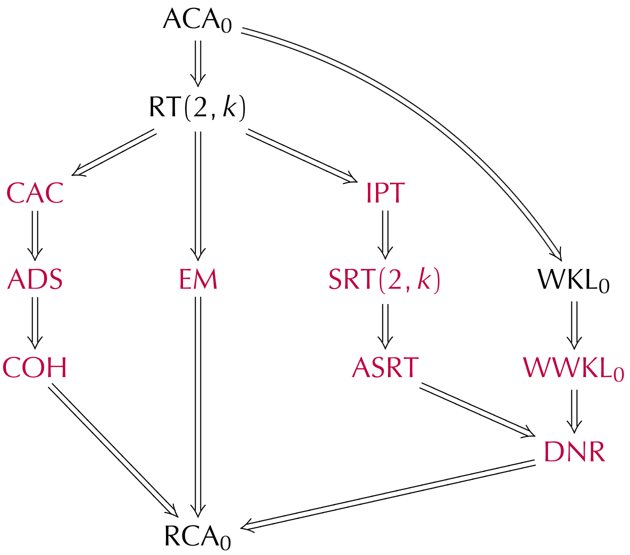


# Beyond combinatorics: the atomic model theorem.

Hirschfeldt, Shore, and Slaman studied model-theoretic theorems concerning when a theory has a an atomic model, one as small as possible. (Atomic models are initial objects in the category of models.)

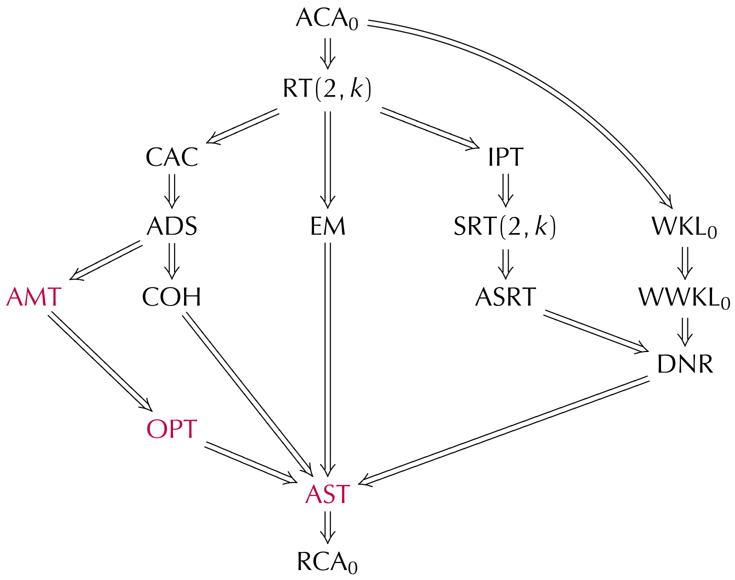
AMT. Every atomic theory has an atomic model.

There are two variants, OPT and AST, which are special cases of AMT.

Theorem (Hirschfeldt, Shore, and Slaman.)

1. AMT is not provable in $\mathrm{RCA}_0$, but it is extremely weak: it is implied over $\mathrm{RCA}_0$ by virtually every combinatorial principle below $\mathrm{RT}(2,k)$.
2. OPT is equivalent to the existence of hyperimmune sets, i.e., it can be characterized in terms of growth rates of computable functions.
3. AST is equivalent to the existence of noncomputable sets.

# Beyond combinatorics: the atomic model theorem.



# Intersection principles.

A family of sets is said to have the finite intersection property (f.i.p.) if the intersection of any finitely many of its members is non-empty.

FIP. Every family of sets has a maximal subfamily with f.i.p.

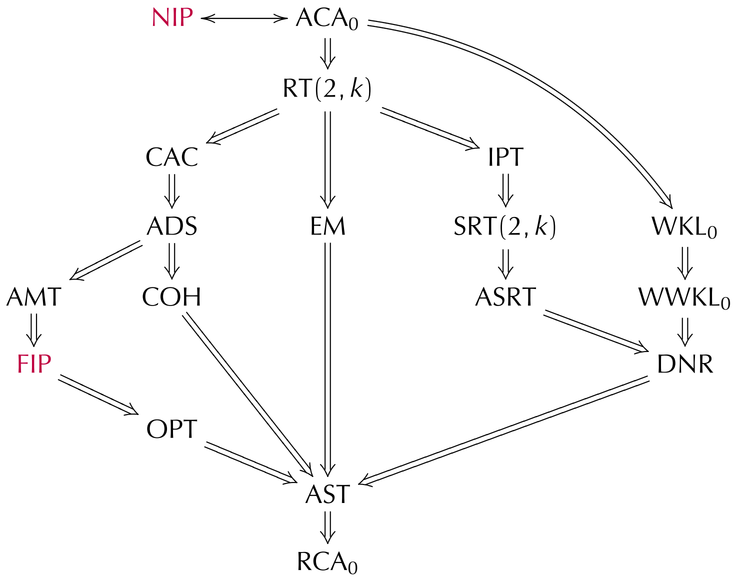
NIP. Every family of sets has a maximal disjoint subfamily.

Over ZF, these principles are equivalent to the axiom of choice.

Theorem (Dzhafarov and Mummert).

1. Over $RCA_0$, NIP is equivalent to $ACA_0$.
2. FIP is not provable in $RCA_0$, but it has solutions computable from any noncomputable c.e. set, as well as from any Cohen generic set.
3. Over $RCA_0$, AMT implies FIP, which implies OPT.

# Intersection principles.



# Computable and uniform reductions.

An implication $P \to Q$ in $RCA_0$ may use P several times to obtain Q, or involve non-uniform decisions about how to proceed in a construction.

But in most cases, the implication is actually a computable reduction.

Example. P = “every infinite binary tree has an infinite path”.
Q = “every commutative ring with unity has a prime ideal”.

| | P | Q |
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| Instances | | |
| Solutions | | |

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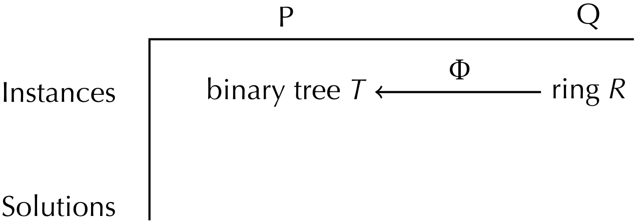

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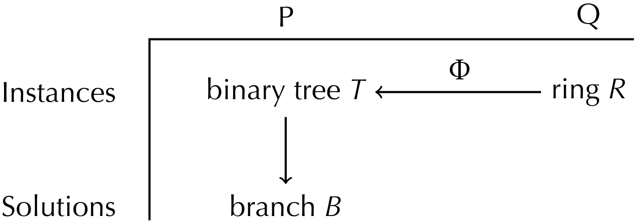

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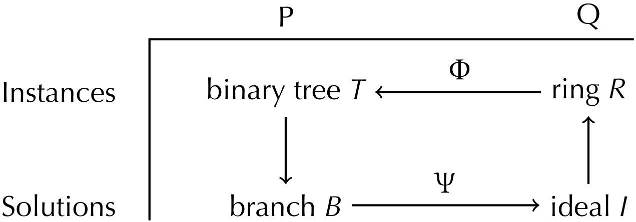


Here, the reduction is also uniform: $\Phi$ and $\Psi$ do not depend on $R$.

# Stable Ramsey's theorem.

Two versions of $\mathsf{RT}(2,k)$ aimed at simplifying the problem:

SRT(2, $k$). $\mathsf{RT}(2,k)$ for stable colorings: $f(x,y) = f(x,z)$ for almost all $y, z$.

COH. Sequential form of $\mathsf{RT}(1,k)$, but allowing finitely many mistakes.

It is known that $\mathsf{RT}(2,k) \leftrightarrow \mathsf{SRT}(2,k) + \mathsf{COH}$ over $\mathsf{RCA}_0$.

Open Question (Cholak, Jockusch, and Slaman). Does $\mathsf{SRT}(2,k)$ imply $\mathsf{RT}(2,k)$ in standard models of $\mathsf{RCA}_0$? I.e., does $\mathsf{SRT}(2,k)$ imply COH?

As a partial solution, lending credence to a negative answer:

Theorem (Dzhafarov). COH does not computably reduce to $\mathsf{SRT}(2,k)$.

Thus, if $\mathsf{SRT}(2,k)$ implies COH it is not via the typical argument. The proof is a forcing argument: $\mathsf{SRT}(2,k)$ does not generically imply COH.

# Different numbers of colors.

If $j < k$, then $\mathsf{RT}(n, j)$ is trivially uniformly reducible to $\mathsf{RT}(n, k)$, via the identity reductions. Over $\mathsf{RCA}_0$, $\mathsf{RT}(n, j)$ also implies $\mathsf{RT}(n, k)$.

Question. If $j < k$, is there a uniform reduction from $\mathsf{RT}(n, k)$ to $\mathsf{RT}(n, j)$?

Definition. Two instances of $\mathsf{RT}(n, j)$ are uniformly reducible to one if there are effective procedures $\Phi$ and $\Psi$ such that:

1. if $f_0, f_1 : [\mathbb{N}]^n \to j$ then $\Phi(f_0, f_1) = g : [\mathbb{N}]^n \to j$;
2. if $G$ is infinite and $g$-homogeneous then $\Psi(G) = \langle H_0, H_1 \rangle$ where $H_i$ is infinite and $f_i$-homogeneous.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). If two instances of $\mathsf{RT}(n, j)$ are uniformly reducible to one, then so are $\omega$ many.

By coding colors in terms of multiple instances, we obtain:

Corollary. If $j < k$, then $\mathsf{RT}(n, k)$ is not uniformly reducible to $\mathsf{RT}(n, j)$.

# Proof of the squashing theorem.

Fix $f_0, f_1, \ldots : [\mathbb{N}]^n \to j$, and computable reductions $\Phi, \Psi$.

We build new colorings $g_0, g_1, \ldots : [\mathbb{N}]^n \to j$ so as to satisfy

$$g_i(\vec{x}) = \Phi(f_i, g_{i+1})(\vec{x})$$

for large enough $\vec{x} \in [\mathbb{N}]^n$. Thus, $g_0 \approx \Phi(f_0, g_1)$.

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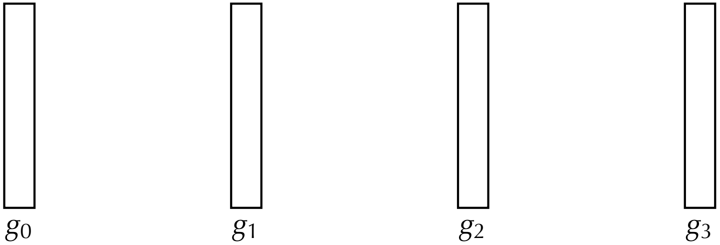

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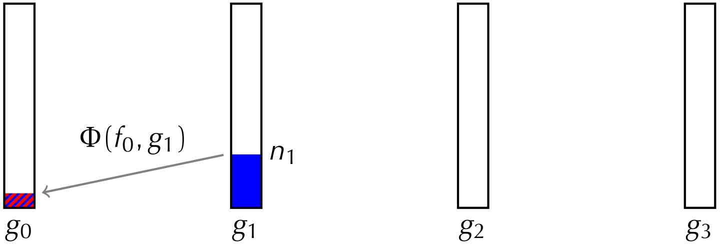

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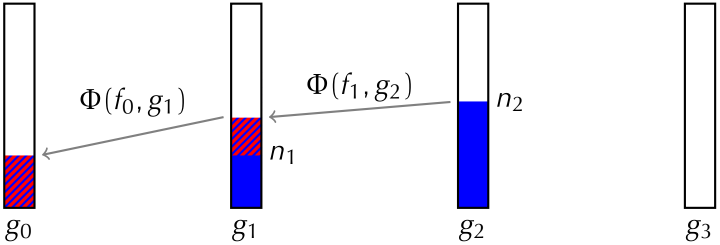

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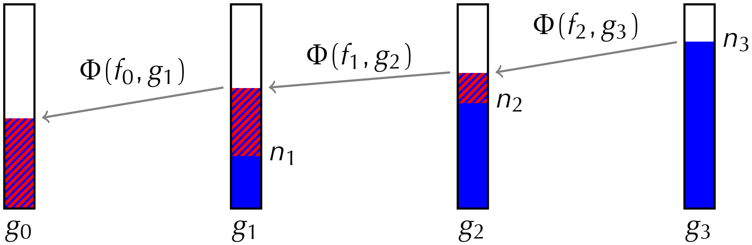

Thank you for your attention!