New directions in reverse mathematics

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A foundational motivation.

Reverse mathematics is motivated by a foundational question:

**Question.** Which axioms do we really need to prove a given theorem?

The question leads to the idea of the strength of a theorem. Which theorems does it imply? Which imply it? Which is it equivalent to?

**Example.** Over ZF, the axiom of choice is equivalent to Zorn’s lemma.

Set theory is much too powerful to accurately calibrate strength.

We would like results of the form

Over the theory $T$, theorem $P$ implies/is equivalent to theorem $Q$, where $T$ is relatively weak, but also expressive enough to accommodate a decent amount of mathematics.
Subsystems of second-order arithmetic.

Second-order arithmetic, \( Z_2 \), is a two-sorted theory with variables for numbers and sets of numbers, and the usual symbols of arithmetic.

The axioms of \( Z_2 \) are those of Peano arithmetic, and the comprehenasion scheme: if \( \varphi \) is a formula (with set parameters) then \( \{x : \varphi(x)\} \) exists.

Example. If \( G_0, G_1, \ldots \) and \( G \) are groups, then \( \{i \in \mathbb{N} : G_i \cong G\} \) exists.

We restrict which formulas \( \varphi \) we allow to obtain various subsystems:

| \( \text{System} \) | \( \Delta^0_1 \) (computable) formulas. | Formulas defining paths through infinite binary trees. | Arithmetical formulas. |
|---------------------|----------------------------------------|----------------------------------------------------|
| \( \text{RCA}_0 \) |                                        |                                                    |
| \( \text{WKL}_0 \)  |                                        |                                                    |
| \( \text{ACA}_0 \)  |                                        |                                                    |

Two other subsystems, \( \text{ATR}_0 \) and \( \Pi^1_1\text{-CA}_0 \), complete the “big five”.
Subsystems of second-order arithmetic.
Connections with computability theory.

A set $A$ is computable from $B$, written $A \leq_T B$, if $A = \Phi(B)$ for some effective procedure $\Phi$; i.e., knowing $B$, we can computably figure out $A$. The halting set of $A$ is $A' = \{i \in \mathbb{N} : \text{the } i\text{th program with oracle } A \text{ halts}\}$.

A model of $\mathbb{Z}_2$ consists of (a possibly nonstandard version of) $\mathbb{N}$ and a subset of $\mathcal{P}(\mathbb{N})$ closed under certain computability-theoretic operations.

<table>
<thead>
<tr>
<th>$\text{RCA}_0$</th>
<th>Models closed under $\leq_T$ and $\bigoplus$ (disjoint union).</th>
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<tbody>
<tr>
<td>$\text{ACA}_0$</td>
<td>Models closed under the jump operator, $A \mapsto A'$.</td>
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</table>

The strength of a theorem corresponds to how effectively solutions can be found to a given computable instance of that theorem.

**Example.** Does every computable commutative unit ring have a computable maximal ideal? Do all ideals compute $\emptyset'$?
Reverse mathematics.

Most (countable) classical mathematics can be developed within $Z_2$.

**Theorem.** The following are provable in RCA$_0$.

1 (Simpson). Baire category theorem, intermediate value theorem.

2 (Brown; Simpson). Urysohn’s lemma, Tietze extension theorem.

3 (Rabin). Existence of algebraic closures of countable fields.

Work in reverse mathematics has revealed a **striking fact**: most theorems are provable in RCA$_0$, or equivalent to one of the other four systems.

**Theorem.** The following are equivalent to WKL$_0$ over RCA$_0$.

1 (Brown; Friedman). Heine-Borel theorem for $[0, 1]$.

2 (Orevkov; Shoji and Tanaka). Brouwer fixed-point theorem.

3 (Friedman, Simpson, and Smith). Prime ideal theorem.
Reverse mathematics.

Theorem. The following are equivalent to ACA$_0$ over RCA$_0$.

1 (Friedman). Bolzano-Weierstrass theorem.

2 (Dekker). Existence of bases in vector spaces.

3 (Friedman, Simpson, and Smith). Maximal ideal theorem.

The maximal ideal theorem is stronger than the prime ideal theorem.

Question. Are there irregular principles, lying outside the big five?

Remark. Induction in RCA$_0$ is limited to I$\Sigma^0_1$, i.e., to

$$((\varphi(0) \land (\forall x)[\varphi(x) \rightarrow \varphi(x+1)]) \rightarrow (\forall x)\varphi(x))$$

for $\Sigma^0_1$ formulas $\varphi$. RCA$_0$ cannot prove I$\Sigma^0_n$ for $n > 1$. 
Ramsey’s theorem.

Definition. Let $[\mathbb{N}]^n$ denote the set of all $n$-element subsets of $\mathbb{N}$.

A $k$-coloring of $[\mathbb{N}]^n$ is a map $f : [\mathbb{N}]^n \rightarrow k = \{0, 1, \ldots, k - 1\}$.

$\text{RT}(n, k)$. Every $k$-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

So $\text{RT}(1, k)$ is just the pigeonhole principle.

$\text{RT}(2, k)$ asserts the existence of infinite cliques or anticliques.
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**RT($n, k$).** Every $k$-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

So RT($1, k$) is just the pigeonhole principle.

RT($2, k$) asserts the existence of infinite cliques or anticliques.
Ramsey's theorem.

RT(1, k) is proved in RCA₀. For \( n \geq 2 \), RT(n, k) is provable in ACA₀.

Theorem (Jockusch). For \( n \geq 3 \), RT(n, k) is equivalent to ACA₀.

How strong is RT(2, k)? It can be shown that it is not provable in WKL₀.

Theorem. Let \( f \) be a computable \( k \)-coloring of \([\mathbb{N}]^2\).

1 (Seetapun). For any \( C \nsubseteq_T \emptyset \), \( f \) has an infinite homogeneous set \( H \nsubseteq_T C \).

2 (Cholak, Jockusch, and Slaman). \( f \) has a low₂ infinite homogeneous set.

3 (Dzhafarov and Jockusch). \( f \) has two low₂ infinite homogeneous sets \( H₀ \) and \( H₁ \) whose degree form a minimal pair.

This allows us to build a model of RCA₀ + RT(2, k) not computing \( \emptyset' \).

Corollary. RT(2, k) does not imply ACA₀ over RCA₀.
Ramsey’s theorem.

\[
\begin{align*}
\Pi_1^1 - \text{CA}_0 & \quad \hookrightarrow \\
\text{ATR}_0 & \quad \hookrightarrow \\
\text{ACA}_0 & \quad \hookrightarrow \\
\text{WKL}_0 & \quad \hookrightarrow \\
\text{RCA}_0
\end{align*}
\]
Ramsey's theorem.

\[
\begin{align*}
\Pi_1^1 - \text{CA}_0 & \quad \Downarrow \\
\text{ATR}_0 & \quad \Downarrow \\
\text{ACA}_0 & \quad \Rightarrow \quad \text{RT}(3, k) \quad \Leftarrow \\
\text{WKL}_0 & \quad \Rightarrow \\
\text{RCA}_0
\end{align*}
\]
A zoo of combinatorial principles.

Elucidating the strength of $\text{RT}(2, k)$ has given rise to an industry of research into the reverse mathematics of combinatorial principles.

Downey, Hirschfeldt, Lempp, and Solomon; Mileti; Liu. Further investigation of the information content of $\text{RT}(2, k)$.

Cholak, Jockusch, and Slaman. Stable Ramsey’s theorem, cohesive principle.

Hirschfeldt and Shore. Principles about partial and linear orders.

Dzhafarov and Hirst. Polarized versions of Ramsey’s theorem.

Bovykin and Weiermann; Lerman, Solomon, and Towsner. Tournaments.

Ambos-Spies et al.; Dzhafarov. Combinatorial principles related to notions of algorithmic randomness.
A zoo of combinatorial principles.
Beyond combinatorics: the atomic model theorem.

Hirschfeldt, Shore, and Slaman studied model-theoretic theorems concerning when a theory has an atomic model, one as small as possible. (Atomic models are initial objects in the category of models.)

AMT. Every atomic theory has an atomic model.

There are two variants, OPT and AST, which are special cases of AMT.

Theorem (Hirschfeldt, Shore, and Slaman.)

1. AMT is not provable in RCA\(_0\), but it is extremely weak: it is implied over RCA\(_0\) by virtually every combinatorial principle below RT\((2, k)\).

2. OPT is equivalent to the existence of hyperimmune sets, i.e., it can be characterized in terms of growth rates of computable functions.

3. AST is equivalent to the existence of noncomputable sets.
Beyond combinatorics: the atomic model theorem.
Intersection principles.

A family of sets is said to have the finite intersection property (f.i.p.) if the intersection of any finitely many of its members is non-empty.

**FIP.** Every family of sets has a maximal subfamily with f.i.p.

**NIP.** Every family of sets has a maximal disjoint subfamily.

Over ZF, these principles are equivalent to the axiom of choice.

Theorem (Dzhafarov and Mummert).

1. Over RCA\(_0\), NIP is equivalent to ACA\(_0\).

2. FIP is not provable in RCA\(_0\), but it has solutions computable from any noncomputable c.e. set, as well as from any Cohen generic set.

3. Over RCA\(_0\), AMT implies FIP, which implies OPT.
Intersection principles.
Computable and uniform reductions.

An implication $P \to Q$ in RCA$_0$ may use $P$ several times to obtain $Q$, or involve non-uniform decisions about how to proceed in a construction. But in most cases, the implication is actually a **computable reduction**.

**Example.** $P = “$every infinite binary tree has an infinite path”.
$Q = “$every commutative ring with unity has a prime ideal”.

```
  P                       Q
 /-----------------------
 Instances
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\text{Instances} & \text{Solutions} \\
\text{binary tree } T & \Phi \\
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 instances

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 P  Q

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 branch $B$

 $\Phi$
```
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Here, the reduction is also **uniform**: $\Phi$ and $\Psi$ do not depend on $R$. 
Stable Ramsey’s theorem.

Two versions of RT(2, k) aimed at simplifying the problem:

SRT(2, k). RT(2, k) for stable colorings: \( f(x, y) = f(x, z) \) for almost all \( y, z \).

COH. Sequential form of RT(1, k), but allowing finitely many mistakes.

It is known that RT(2, k) \( \iff \) SRT(2, k) + COH over RCA_0.

**Open Question (Cholak, Jockusch, and Slaman).** Does SRT(2, k) imply RT(2, k) in standard models of RCA_0? I.e., does SRT(2, k) imply COH?

As a partial solution, lending credence to a **negative** answer:

**Theorem (Dzhafarov).** COH does not computably reduce to SRT(2, k).

Thus, if SRT(2, k) implies COH it is not via the **typical** argument. The proof is a forcing argument: SRT(2, k) does not **generically** imply COH.
Different numbers of colors.

If $j < k$, then $RT(n, j)$ is trivially uniformly reducible to $RT(n, k)$, via the identity reductions. Over $RCA_0$, $RT(n, j)$ also implies $RT(n, k)$.

**Question.** If $j < k$, is there a uniform reduction from $RT(n, k)$ to $RT(n, j)$?

**Definition.** Two instances of $RT(n, j)$ are uniformly reducible to one if there are effective procedures $\Phi$ and $\Psi$ such that:

1. if $f_0, f_1 : [\mathbb{N}]^n \to j$ then $\Phi(f_0, f_1) = g : [\mathbb{N}]^n \to j$;
2. if $G$ is infinite and $g$-homogeneous then $\Psi(G) = \langle H_0, H_1 \rangle$ where $H_i$ is infinite and $f_i$-homogeneous.

**Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer).** If two instances of $RT(n, j)$ are uniformly reducible to one, then so are $\omega$ many.

By coding colors in terms of multiple instances, we obtain:

**Corollary.** If $j < k$, then $RT(n, k)$ is not uniformly reducible to $RT(n, j)$. 
Proof of the squashing theorem.

Fix $f_0, f_1, \ldots : [\mathbb{N}]^n \to j$, and computable reductions $\Phi, \Psi$.
We build new colorings $g_0, g_1, \ldots : [\mathbb{N}]^n \to j$ so as to satisfy

$$g_i(\vec{x}) = \Phi(f_i, g_{i+1})(\vec{x})$$

for large enough $\vec{x} \in [\mathbb{N}]^n$. Thus, $g_0 \approx \Phi(f_0, g_1)$. 
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\begin{align*}
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\[ \Phi(f_0, g_1) \quad n_1 \quad g_0 \quad g_1 \quad g_2 \quad g_3 \]
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\[ g_0 \quad g_1 \quad g_2 \quad g_3 \]

\[ \Phi(f_0, g_1) \quad \Phi(f_1, g_2) \quad \Phi(f_2, g_3) \quad n_3 \]

\[ n_1 \quad n_2 \]
Thank you for your attention!