New directions in reverse mathematics

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A foundational motivation.

Reverse mathematics is motivated by a foundational question:

Question. Which axioms do we really need to prove a given theorem?

The question leads to the idea of the strength of a theorem. Which theorems does it imply? Which imply it? Which is it equivalent to?

Example. Over ZF, the axiom of choice is equivalent to Zorn's lemma.

Of course, set theory is too powerful to calibrate ordinary mathematics.

We would like results of the form

Over the theory T, theorem P implies/is equivalent to theorem Q,

where T is weak enough to not prove everything, yet robust enough to accommodate a decent amount of mathematics.

Subsystems of second-order arithmetic.

Second-order arithmetic, Z_2, is a two-sorted theory with variables for numbers and sets of numbers, and the usual symbols of arithmetic.

The axioms of Z_2 are those of Peano arithmetic, and the comprehension scheme: if φ is a formula (with set parameters) then $\{x : \varphi(x)\}$ exists.

Example. If $G_0, G_1, \ldots$ and G are groups, then $\{i \in \mathbb{N} : G_i \cong G\}$ exists.

We restrict which formulas φ we allow to obtain various subsystems:

RCA_0	Δ^0_1 formulas (definitions of computable sets).
WKL_0	Formulas defining paths through infinite binary trees.
ACA_0	Arithmetical formulas.

Two other subsystems, ATR_0 and Π^1_1-CA_0, complete the "big five".

Subsystems of second-order arithmetic.

$$Z_2$$

$$\Downarrow$$

$$\Pi^1_1\text{-CA}_0$$

$$\Downarrow$$

$$\text{ATR}_0$$

$$\Downarrow$$

$$\text{ACA}_0$$

$$\Downarrow$$

$$\text{WKL}_0$$

$$\Downarrow$$

$$\text{RCA}_0$$

Connections with computability theory.

A set A is computable from B, written $A \leq_T B$, if $A = \Phi(B)$ for some effective procedure Φ; i.e., knowing B, we can computably figure out A.

The halting set of A is $A' = \{i \in \mathbb{N} : \text{the } i\text{th program with oracle } A \text{ halts}\}$.

A model of Z_2 consists of (a possibly nonstandard version of) $\mathbb{N}$ and a subset of $\mathcal{P}(\mathbb{N})$ closed under certain computability-theoretic operations.

RCA_0	Models closed under $\leq_T$ and $\oplus$ (disjoint union).
ACA_0	Models closed under the jump operator, $A \mapsto A'$.

The strength of a theorem corresponds to how effectively solutions can be found to a given computable instance of that theorem.

Example. Does every computable commutative unit ring have a computable maximal ideal? Do all ideals compute $\emptyset'$?

Reverse mathematics.

Most (countable) classical mathematics can be developed within Z_2.

Theorem. The following are provable in RCA_0.

1 (Simpson). Baire category theorem, intermediate value theorem.

2 (Brown; Simpson). Urysohn's lemma, Tietze extension theorem.

3 (Rabin). Existence of algebraic closures of countable fields.

Work in reverse mathematics has revealed a striking fact: most theorems are provable in RCA_0, or equivalent to one of the other four systems.

Theorem. The following are equivalent to WKL_0 over RCA_0.

1 (Brown; Friedman). Heine-Borel theorem for $[0, 1]$.

2 (Orevkov; Shoji and Tanaka). Brouwer fixed-point theorem.

3 (Friedman, Simpson, and Smith). Prime ideal theorem.

Reverse mathematics.

Theorem. The following are equivalent to ACA_0 over RCA_0.

1 (Friedman). Bolzano-Weierstrass theorem.

2 (Dekker). Existence of bases in vector spaces.

3 (Friedman, Simpson, and Smith). Maximal ideal theorem.

The maximal ideal theorem is stronger than the prime ideal theorem.

Question. Are there irregular principles, lying outside the big five?

Remark. Induction in RCA_0 is limited to $I\Sigma^0_1$, i.e., to

$$[\varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1))] \to \forall x\, \varphi(x)$$

for Σ^0_1 formulas φ. RCA_0 cannot prove $I\Sigma^0_n$ for $n > 1$.

Beyond the big five: Ramsey's theorem.

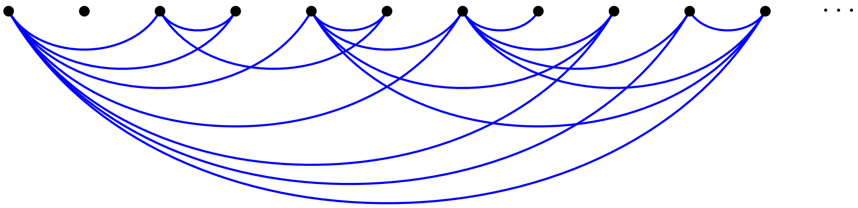
Definition. Let $[\mathbb{N}]^n$ denote the set of all n-element subsets of $\mathbb{N}$.

A k-coloring of $[\mathbb{N}]^n$ is a map $f : [\mathbb{N}]^n \to k = \{0, 1, \ldots, k-1\}$.

RT(n, k). Every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

So RT$(1, k)$ is just the pigeonhole principle.

RT$(2, k)$ asserts the existence of infinite cliques or anticliques in graphs.



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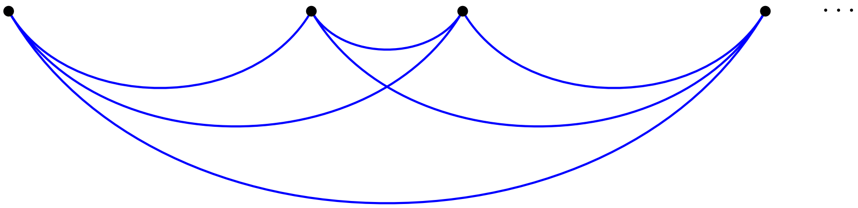

Beyond the big five: Ramsey's theorem.

$\mathrm{RT}(1,k)$ is proved in RCA_0. For $n \geq 2$, $\mathrm{RT}(n,k)$ is provable in ACA_0.

Theorem (Jockusch). For $n \geq 3$, $\mathrm{RT}(n,k)$ is equivalent to ACA_0.

How strong is $\mathrm{RT}(2,k)$? It can be shown that it is not provable in WKL_0.

Theorem. Let f be a computable k-coloring of $[\mathbb{N}]^2$.

1 (Seetapun). For any $C \not\leq_T \emptyset$, f has an infinite homogeneous set $H \not\geq_T C$.

2 (Cholak, Jockusch, and Slaman). f has a low_2 infinite homogeneous set.

3 (Dzhafarov and Jockusch). f has two low_2 infinite homogeneous sets H_0 and H_1 whose degree form a minimal pair.

This allows us to build a model of $\mathrm{RCA}_0 + \mathrm{RT}(2,k)$ not computing $\emptyset'$.

Corollary. $\mathrm{RT}(2,k)$ does not imply ACA_0 over RCA_0.

Beyond the big five: Ramsey's theorem.

$$\Pi^1_1\text{-CA}_0$$
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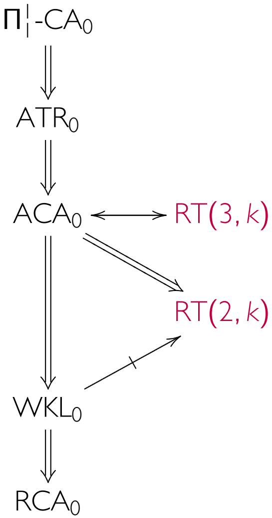

A zoo of combinatorial principles.

Elucidating the strength of $\mathrm{RT}(2, k)$ has given rise to an industry of research into the reverse mathematics of combinatorial principles.

Downey, Hirschfeldt, Lempp, and Solomon; Mileti; Liu; many others. Further investigation of the information content of $\mathrm{RT}(2, k)$.

Cholak, Jockusch, and Slaman. Stable Ramsey's theorem, cohesive principle.

Hirschfeldt and Shore. Principles about partial and linear orders.

Dzhafarov and Hirst. Polarized versions of Ramsey's theorem.

Bovykin and Weiermann; Lerman, Solomon, and Towsner. Tournaments.

Ambos-Spies et al.; Dzhafarov. Combinatorial principles related to notions of algorithmic randomness.

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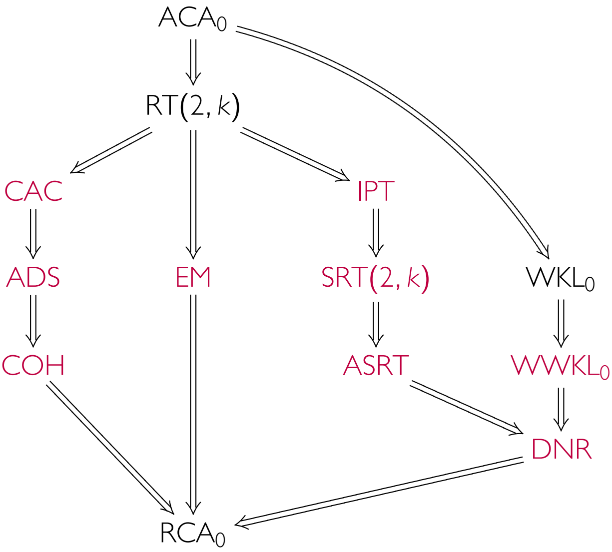


Beyond combinatorics: the atomic model theorem.

Hirschfeldt, Shore, and Slaman studied model-theoretic theorems concerning when a theory has a an atomic model, one as small as possible. (Atomic models are initial objects in the category of models.)

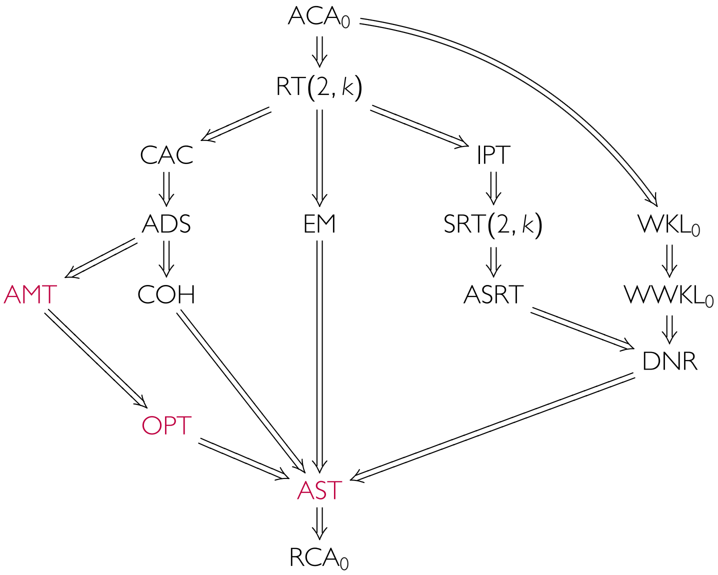
AMT. Every atomic theory has an atomic model.

There are two variants, OPT and AST, which are special cases of AMT.

Theorem (Hirschfeldt, Shore, and Slaman.)

1. AMT is not provable in RCA_0, but it is extremely weak: it is implied over RCA_0 by virtually every combinatorial principle below $RT(2, k)$.
2. OPT is equivalent to the existence of hyperimmune sets, i.e., it can be characterized in terms of growth rates of computable functions.
3. AST is equivalent to the existence of noncomputable sets.

Beyond combinatorics: the atomic model theorem.



Intersection principles.

A family of sets is said to have the finite intersection property (f.i.p.) if the intersection of any finitely many of its members is non-empty.

FIP. Every family of sets has a maximal subfamily with f.i.p.

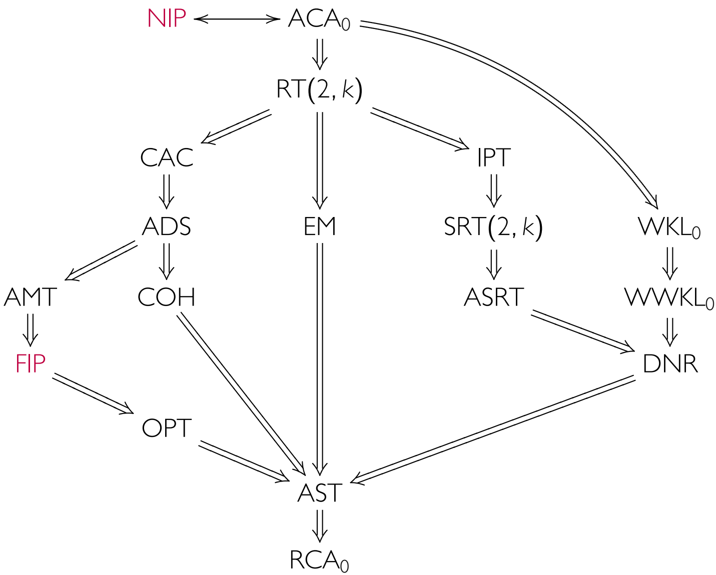
NIP. Every family of sets has a maximal pairwise disjoint subfamily.

Over ZF, these principles are equivalent to the axiom of choice.

Theorem (Dzhafarov and Mummert).

1. Over RCA_0, NIP is equivalent to ACA_0.
2. FIP is not provable in RCA_0, but it has solutions computable from any noncomputable c.e. set, as well as from any Cohen generic realw.
3. Over RCA_0, AMT implies FIP, which implies OPT.

Intersection principles.



Beyond RCA_0: computable and uniform reductions.

An implication $P \to Q$ in RCA_0 may use P several times to obtain Q, or involve non-uniform decisions about how to proceed in a construction.

But in most cases, the implication is actually a computable reduction.

Example. P = "every infinite binary tree has an infinite branch".
Q = "every commutative ring with unity has a prime ideal".

	P	Q
Instances		
Solutions		

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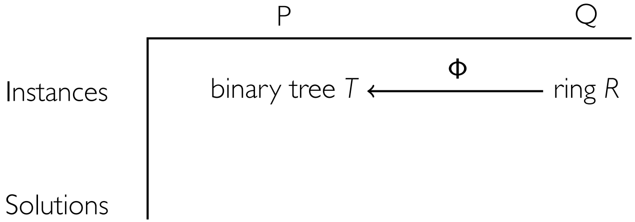

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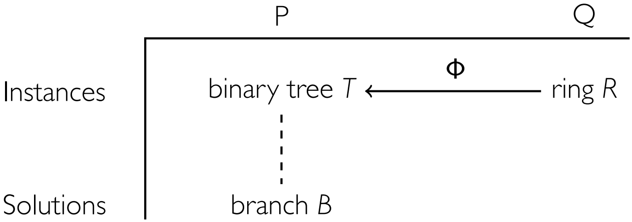

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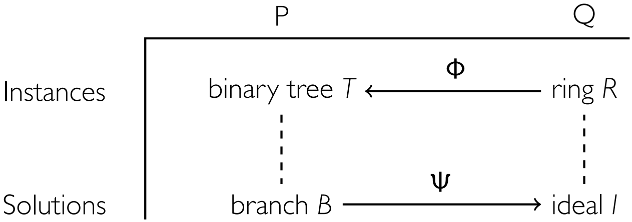

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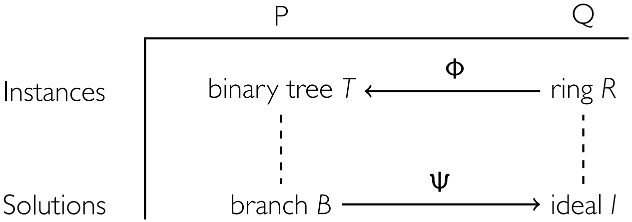

In computable analysis, this is called (uniform) Weihrauch reducibility.

Stable Ramsey's theorem.

Two versions of $RT(2,k)$ aimed at simplifying the problem:

SRT$(2,k)$. RT$(2,k)$ for stable colorings: $f(x,y) = f(x,z)$ for almost all y, z.

COH. Sequential form of RT$(1,k)$, but allowing finitely many mistakes.

Theorem (Cholak, Jockusch, and Slaman). $RT(2,k) \leftrightarrow SRT(2,k) + COH$.

Open Question (Chong, Slaman, and Yang). Does SRT$(2,k)$ imply RT$(2,k)$ in standard models of RCA_0? I.e., does SRT$(2,k)$ imply COH?

As a partial solution, lending credence to a negative answer:

Theorem (Dzhafarov; Lerman, Solomon, and Towsner). COH does not computably reduce to SRT$(2,k)$.

Thus, if SRT$(2,k)$ implies COH it is not via the typical argument. The proof is a forcing argument: SRT$(2,k)$ does not generically imply COH.

Different numbers of colors.

If $j < k$, then $\mathsf{RT}(n, j)$ is trivially uniformly reducible to $\mathsf{RT}(n, k)$, via the identity reductions. Over RCA_0, $\mathsf{RT}(n, j)$ also implies $\mathsf{RT}(n, k)$.

Question. If $j < k$, is there a uniform reduction from $\mathsf{RT}(n, k)$ to $\mathsf{RT}(n, j)$?

Definition. Two instances of $\mathsf{RT}(n, j)$ are uniformly reducible to one if there are effective procedures Φ and Ψ such that:

1. if $f_0, f_1 : [\mathbb{N}]^n \to j$ then $\Phi(f_0, f_1) = g : [\mathbb{N}]^n \to j$;
2. if G is infinite and g-homogeneous then $\Psi(G) = \langle H_0, H_1 \rangle$ where H_i is infinite and f_i-homogeneous.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). If two instances of $\mathsf{RT}(n, j)$ are uniformly reducible to one, then so are ω many.

By coding colors in terms of multiple instances, we obtain:

Corollary. If $j < k$, then $\mathsf{RT}(n, k)$ is not uniformly reducible to $\mathsf{RT}(n, j)$.

Thank you for your attention!