Computable, strong, and uniform reductions

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A problem is a $\Pi^1_2$ statement of second-order arithmetic, thought of as

for every $X \in \text{Inst}(P)$, there is a $Y \in \text{Soln}(P, X)$,

where $\text{Inst}(P)$ and $\text{Soln}(P, X)$ are arithmetically-definable sets.

Examples.

$\text{RT}^n_k$. Every coloring $c : [\omega]^n \rightarrow k$ has an infinite homogeneous set.

$\text{COH}$. Every family of sets $X = \langle X_0, X_1, \ldots \rangle$ has an infinite $X$-cohesive set $Y$, meaning that for each $i$, either $Y \cap X_i$ or $Y \cap \overline{X_i}$ is finite.

$\text{DNR}_n$: For every set $X$ there is an $f : \omega \rightarrow n$ such that $f(e) \neq \Phi^X_e(e)$ for all $e$. 
Reductions.

Let $P$ and $Q$ be problems.

$P$ is strongly computably reducible to $Q$, written $P \leq_{sc} Q$, if every $X \in \text{Inst}(P)$ computes an $\hat{X} \in \text{Inst}(Q)$, such that every $\hat{Y} \in \text{Soln}(Q, \hat{X})$ computes a $Y \in \text{Soln}(P, X)$.

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X   computes   \hat{X}
|    solves    |
|              |
Y ← computes   \hat{Y}
|    solves    |
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Reductions.

Let $P$ and $Q$ be problems.

$P$ is **computably reducible** to $Q$, written $P \leq_c Q$, if every $X \in \text{Inst}(P)$ computes an $\hat{X} \in \text{Inst}(Q)$, such that every $\hat{Y} \in \text{Soln}(Q, \hat{X})$, together with $X$, computes a $Y \in \text{Soln}(P, X)$.
Reductions.

Let $P$ and $Q$ be problems.

$P$ is uniformly reducible to $Q$, written $P \leq_u Q$, if there are fixed functionals $\Phi$ and $\Psi$ witnessing the computations in both directions of a computable reduction.

\[ X \xrightarrow[\Phi]{\text{solves}} \hat{X} \]
\[ Y \xleftarrow[\Psi]{\hat{Y} \oplus X} \]

$\leq_c$, $\leq_{sc}$ extend Muchnik reducibility. $\leq_u$ extends Medvedev reducibility. In the context where they are both defined, $\leq_u$ agrees with Weihrauch reducibility.
Most implications between combinatorial problems are actually formalizations of uniform and/or strong reductions.

Each of $\leq_u$ and $\leq_{sc}$ implies $\leq_c$, and $\leq_c$ implies provability in RCA.

**Theorem** (Cholak, Jockusch, and Slaman). RCA$_0 \vdash RT_2^2 \rightarrow COH$.

The proof is a formalization in RCA$_0$ of a strong uniform reduction.

These reducibilities offer a way to tease apart subtle differences between various principles that provability over RCA$_0$ alone does not see.

**Theorem** (Jockusch). If $n < m$, then DNR$_n \equiv_c$ DNR$_m$ but DNR$_n \not\leq_u$ DNR$_m$.

**Theorem** (Patey). If $j > k$, then RT$_j^n \not\leq_c$ RT$_k^n$.
Two versions of Ramsey's theorem.

A coloring $c$, henceforth always $[\omega]^2 \to 2$, is stable if $\lim_y c(x, y)$ exists for all $x$.

$\text{SRT}^2_2$. Every stable coloring has an infinite homogeneous set.

A set $L$ is limit-homogeneous for a stable coloring $c$ if there is an $i \in \{0, 1\}$ such that $\lim_y c(x, y) = i$ for all $x \in L$.

$\text{D}^2_2$. Every stable coloring has an infinite limit-homogeneous set.

**Observation.** $\text{SRT}^2_2 \equiv_c \text{D}^2_2$.

**Pf.** Let $c$ be a coloring. Every infinite limit-homogeneous set $L$ for $c$ can be computable thinned to an infinite homogeneous set with the same color.

**Theorem** (Chong, Lempp, and Yang). $\text{RCA}_0 \vdash \text{SRT}^2_2 \iff \text{D}^2_2$. 
Two versions of Ramsey's theorem.

**Theorem** (Hirschfeldt and Jockusch). SRT$_2^2$ is uniformly reducible to two applications of D$_2^2$.

**Question** (Hirschfeldt and Jockusch). Can this be improved to $\leq_u$ or $\leq_{sc}$?

If $L$ is limit-homogeneous, but we do not know what color $i \in \{0, 1\}$ the elements in it limit to, then thinning it to a homogeneous set seems difficult.

**Theorem** (Dzhafarov). There is a stable coloring $c$ such that every other stable coloring $d$ has an infinite limit-homogeneous set $L$ that computes no infinite homogeneous set for $c$.

**Corollary.** SRT$_2^2 \nleq_{sc} D^2_2$.

**Theorem** (Dzhafarov). SRT$_2^2 \nleq_u D^2_2$. 
Open question (Chong, Slaman, and Yang). Does SRT$_2^2$ (or D$_2^2$) imply COH in $\omega$-models of RCA$_0$? Is COH $\leq_c$ SRT$_2^2$? Equivalently, is COH $\leq_c$ D$_2^2$?

Theorem (Dzhafarov). COH $\not\leq_{sc}$ D$_2^2$.

The proof is a computable forcing argument. Any 3-generic yields a family $\langle X_0, X_1, \ldots \rangle$ witnessing the theorem, so we can find one computable in $\emptyset^{(3)}$.

Theorem (Hirschfeldt and Jockusch; Patey). There is a family of sets $X = \langle X_0, X_1, \ldots \rangle$ such that every stable coloring $d$ has an infinite limit-homogeneous set $L$ that computes no infinite $X$-cohesive set.

The $X$ built by Hirschfeldt and Jockusch is non-hyperarithmetical. Patey's is $\Delta^0_2$.

Question. Given the differences between SRT$_2^2$ and D$_2^2$ under $\leq_u$ and $\leq_{sc}$, what relationships hold between COH and SRT$_2^2$?
**Theorem** (Dzhafarov). There is a computable family of sets $X = \langle X_0, X_1, \ldots \rangle$ such that for every stable coloring $d \leq_T X$ and every functional $\Psi$, there is an infinite homogeneous set $H$ for $d$ with $\Psi^H$ not an infinite $X$-cohesive set.

**Corollary.** COH $\not\leq_u \text{SRT}_2^2$. (Hence, also COH $\not\leq_u \text{D}_2^2$.)

The proof involves uniformly computably building, for each pair $\Phi$ and $\Psi$, a certain coloring $c_{\langle \Phi, \Psi \rangle} : \omega \to 3$, and then pasting these colorings together.

Under a suitable coding, we can view $X = \langle c_n : n \in \omega \rangle$ as a family of sets.

The construction ensures that if $d = \Phi^X$ then for every $\Psi$ there is an infinite homogeneous set $H$ for $d$ such that no finite modification of $\Psi^H$ is homogeneous for $c_{\langle \Phi, \Psi \rangle}$. Thus, $\Psi^H$ cannot be cohesive for $\langle c_n : n \in \omega \rangle$. 
COH and SRT$^2_2$: the sc case.

The sc case appears quite close to the full c case.

Recall that whether COH $\leq_c$ SRT$^2_2$ is equivalent to whether COH $\leq_c$ D$^2_2$.

\[
\begin{align*}
\text{COH } \leq_c \text{ D}^2_2: & \quad X \rightarrow d \\
& \quad Y \leftarrow L \oplus X
\end{align*}
\]

\[
\begin{align*}
\text{COH } \leq_{sc} \text{ SRT}^2_2: & \quad X \rightarrow d \\
& \quad Y \leftarrow H
\end{align*}
\]

What could be the role of $X$ in the reduction on the left?

An obvious guess is that $X$ thins out $L$ to a homogeneous set.

If that were all, we would get the reduction on the right.
COH and SRT$_2^2$: the sc case.

**Theorem** (Dzhafarov). There exists a family of sets $X = \langle X_0, X_1, \ldots \rangle$ and a collection $\mathcal{C}$ of subsets of $\omega$ such that:

– for all $Y \in \mathcal{C}$, there is no $(X \oplus Y)$-computable infinite $X$-cohesive set;

and for every stable coloring $d \leq_T X$, one of the following is true:

– $d$ has an $(X \oplus Y)$-computable infinite homogeneous set for some $Y \in \mathcal{C}$;

– $d$ has infinite homogeneous sets of both colors computing no $X$-cohesive set.

**Corollary.** $\text{COH} \not\preceq_{\text{sc}} \text{SRT}_2^2$.

The proof introduces a new method (tree labeling) to build homogeneous sets.

But it uses $\omega$ many iterates of the hyperjump, so $X$ is quite complex. Also, the proof does not seem to work to show $\text{COH} \not\preceq_{\text{sc}} \text{SRT}_k^2$ for any $k > 2$. 
Suppose there is a $\Delta^0_2$ family $X$ witnessing that $\text{COH} \not\leq_{sc} \text{SRT}^2_3$. Then $X$ has a self-modulus (i.e., a function $m \equiv_T X$ such that $X \leq_T f$ for every $f \geq^* m$.)

Let $c : [\omega]^2 \to 2$ be an arbitrary $X$-computable stable coloring.

Define $d : [\omega]^2 \to 3$ by $d(x, y) = c(x, y)$ if $y - x > m(x)$ and $d(x, y) = 2$ else.

Then $c$ and $d$ have the same limit-homogeneous sets. And every infinite homogeneous set for $d$ dominates $m$ and therefore computes $X$.

By assumption, let $H$ be an infinite homogeneous set for $d$ that computes no infinite $X$-cohesive set. Then also $H$ is limit-homogeneous for $c$ and $X \oplus H$ computes no infinite $X$-cohesive set.

We conclude that $\text{COH} \not\leq_c \text{D}^2_2$. 
Question. Is it the case that $\text{COH} \leq_{\text{sc}} \text{SRT}^2_k$ for any $k > 2$?

For $k = 3$, this question was also asked by Hirschfeldt and Jockusch.

**Theorem** (Hirschfeldt and Jockusch). $\text{RT}^1_3 \not\leq_{\text{sc}} D^2_2$.

**Question** (Hirschfeldt and Jockusch). Is it the case that $\text{RT}^1_3 \leq_{\text{sc}} \text{SRT}^2_2$?

By simplifying the tree labeling method used to show that $\text{COH} \not\leq_{\text{sc}} \text{SRT}^2_2$, we obtain a negative answer.

**Theorem** (Dzhafarov, Patey, Solomon, Westrick). If $j > k$ then $\text{RT}^1_j \not\leq_{\text{sc}} \text{SRT}^2_k$.

We now paste together various colorings $c : \omega \to j$ to obtain a family of sets.

**Corollary.** $\text{COH} \not\leq_{\text{sc}} \text{SRT}^2_k$ for all $k \geq 2$. 
A nicer instance of COH.

Above, we wanted a family $X$ witnessing that $\text{COH} \nleq_{sc} \text{SRT}^2_3$ which has a self-modulus. By a result of Solovay, all such sets are hyperarithmetical.

**Question.** Is there a hyperarithmetical family $X$ witnessing that $\text{COH} \nleq_{sc} \text{SRT}^2_3$?

**Theorem** (Dzhafarov, Patey, Solomon, Westrick). For every $j > k$, there is a $\emptyset(\omega)$-computable coloring $c : \omega \to j$ witnessing that $\text{RT}^1_j \nleq_{sc} \text{SRT}^2_k$.

**Corollary.** There is a $\emptyset(\omega)$-computable family $X$ witnessing that $\text{COH} \nleq_{sc} \text{SRT}^2_3$.

Alas, not every hyperarithmetical set has a self-modulus. But $\emptyset(\omega)$ does.

**Proposition** (Dzhafarov, Patey, Solomon, Westrick). There is a $\emptyset(\omega)$-computable family $X$ witnessing that $\text{COH} \nleq_{sc} \text{SRT}^2_3$ such that $\emptyset(n) \leq_T X$ for all $n$.

**Open question.** Can $X$ be chosen with $X \equiv_T \emptyset(\omega)$ or with a self-modulus?
Thank you.