Computable, strong, and uniform reductions

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Problems.

A problem is a Π^1_2 statement of second-order arithmetic, thought of as

$$\text{for every } X \in \mathit{Inst}(P), \text{ there is a } Y \in \mathit{Soln}(P, X),$$

where $\mathit{Inst}(P)$ and $\mathit{Soln}(P, X)$ are arithmetically-definable sets.

Examples.

RT^n_k. Every coloring $c : [\omega]^n \to k$ has an infinite homogeneous set.

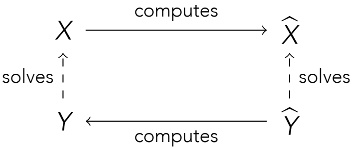
COH. Every family of sets $X = \langle X_0, X_1, \ldots \rangle$ has an infinite X-cohesive set Y, meaning that for each i, either $Y \cap X_i$ or $Y \cap \overline{X_i}$ is finite.

DNR_n: For every set X there is an $f : \omega \to n$ such that $f(e) \neq \Phi^X_e(e)$ for all e.

Reductions.

Let P and Q be problems.

P is strongly computably reducible to Q, written $P \leq_{sc} Q$, if every $X \in Inst(P)$ computes an $\widehat{X} \in Inst(Q)$, such that every $\widehat{Y} \in Soln(Q, \widehat{X})$ computes a $Y \in Soln(P, X)$.



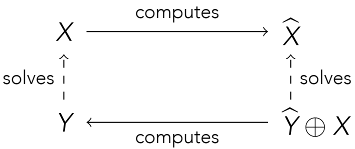
Reductions.

Let P and Q be problems.

P is computably reducible to Q, written $P \leq_c Q$, if

every $X \in Inst(P)$ computes an $\widehat{X} \in Inst(Q)$, such that

every $\widehat{Y} \in Soln(Q, \widehat{X})$, together with X, computes a $Y \in Soln(P, X)$.



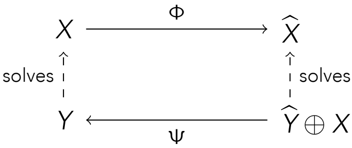
Reductions.

Let P and Q be problems.

P is uniformly reducible to Q, written $P \leq_u Q$, if

there are fixed functionals Φ and Ψ

witnessing the computations in both directions of a computable reduction.



$\leq_c$, $\leq_{sc}$ extend Muchnik reducibility. $\leq_u$ extends Medvedev reducibility. In the context where they are both defined, $\leq_u$ agrees with Weihrauch reducibility.

As a finer metric.

Most implications between combinatorial problems are actually formalizations of uniform and/or strong reductions.

Each of $\leq_u$ and $\leq_{sc}$ implies $\leq_c$, and $\leq_c$ implies provability in RCA.

Theorem (Cholak, Jockusch, and Slaman). $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{COH}$.

The proof is a formalization in RCA_0 of a strong uniform reduction.

These reducibilities offer a way to tease apart subtle differences between various principles that provability over RCA_0 alone does not see.

Theorem (Jockusch). If $n < m$, then $\mathsf{DNR}_n \equiv_c \mathsf{DNR}_m$ but $\mathsf{DNR}_n \not\leq_u \mathsf{DNR}_m$.

Theorem (Patey). If $j > k$, then $\mathsf{RT}^n_j \not\leq_c \mathsf{RT}^n_k$.

Two versions of Ramsey's theorem.

A coloring c, henceforth always $[\omega]^2 \to 2$, is stable if $\lim_y c(x, y)$ exists for all x.

SRT^2_2. Every stable coloring has an infinite homogeneous set.

A set L is limit-homogeneous for a stable coloring c if there is an $i \in \{0, 1\}$ such that $\lim_y c(x, y) = i$ for all $x \in L$.

D^2_2. Every stable coloring has an infinite limit-homogeneous set.

Observation. $\mathsf{SRT}^2_2 \equiv_c D^2_2$.

Pf. Let c be a coloring. Every infinite limit-homogeneous set L for c can be computable thinned to an infinite homogeneous set with the same color.

Theorem (Chong, Lempp, and Yang). $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \leftrightarrow \mathsf{D}^2_2$.

Two versions of Ramsey's theorem.

Theorem (Hirschfeldt and Jockusch). SRT^2_2 is uniformly reducible to two applications of D^2_2.

Question (Hirschfeldt and Jockusch). Can this be improved to $\leq_u$ or $\leq_{sc}$?

If L is limit-homogeneous, but we do not know what color $i \in \{0, 1\}$ the elements in it limit to, then thinning it to a homogeneous set seems difficult.

Theorem (Dzhafarov). There is a stable coloring c such that every other stable coloring d has an infinite limit-homogeneous set L that computes no infinite homogeneous set for c.

Corollary. $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{D}^2_2$.

Theorem (Dzhafarov). $\mathsf{SRT}^2_2 \not\leq_u \mathsf{D}^2_2$.

COH and D^2_2.

Open question (Chong, Slaman, and Yang). Does SRT^2_2 (or D^2_2) imply COH in ω-models of RCA_0? Is COH $\leq_c$ SRT^2_2? Equivalently, is COH $\leq_c$ D^2_2?

Theorem (Dzhafarov). COH $\not\leq_{sc}$ D^2_2.

The proof is a computable forcing argument. Any 3-generic yields a family $\langle X_0, X_1, \ldots \rangle$ witnessing the theorem, so we can find one computable in $\emptyset^{(3)}$.

Theorem (Hirschfeldt and Jockusch; Patey). There is a family of sets $X = \langle X_0, X_1, \ldots \rangle$ such that every stable coloring d has an infinite limit-homogeneous set L that computes no infinite X-cohesive set.

The X built by Hirschfeld and Jockusch is non-hyperarithmetical. Patey's is Δ^0_2.

Question. Given the differences between SRT^2_2 and D^2_2 under $\leq_u$ and $\leq_{sc}$, what relationships hold between COH and SRT^2_2?

COH and SRT^2_2: the u case.

Theorem (Dzhafarov). There is a computable family of sets $X = \langle X_0, X_1, \ldots \rangle$ such that for every stable coloring $d \leq_T X$ and every functional Ψ, there is an infinite homogeneous set H for d with Ψ^H not an infinite X-cohesive set.

Corollary. COH $\not\leq_u$ SRT^2_2. (Hence, also COH $\not\leq_u$ D^2_2.)

The proof involves uniformly computably building, for each pair Φ and Ψ, a certain coloring $c_{\langle \Phi, \Psi \rangle} : \omega \to 3$, and then pasting these colorings together.

Under a suitable coding, we can view $X = \langle c_n : n \in \omega \rangle$ as a family of sets.

The construction ensures that if $d = \Phi^X$ then for every Ψ there is an infinite homogeneous set H for d such that no finite modification of Ψ^H is homogeneous for $c_{\langle \Phi, \Psi \rangle}$. Thus, Ψ^H cannot be cohesive for $\langle c_n : n \in \omega \rangle$.

COH and SRT^2_2: the *sc* case.

The *sc* case appears quite close to the full *c* case.

Recall that whether $\mathrm{COH} \leq_c \mathrm{SRT}^2_2$ is equivalent to whether $\mathrm{COH} \leq_c \mathrm{D}^2_2$.

$\mathrm{COH} \leq_c \mathrm{D}^2_2$:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{d} \\ \uparrow & & \uparrow \\ Y & \longleftarrow & L \oplus X \end{array}$$

$\mathrm{COH} \leq_{sc} \mathrm{SRT}^2_2$:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{d} \\ \uparrow & & \uparrow \\ Y & \longleftarrow & H \end{array}$$

What could be the role of X in the reduction on the left?

An obvious guess is that X thins out L to a homogeneous set.

If that were all, we would get the reduction on the right.

COH and SRT^2_2: the *sc* case.

Theorem (Dzhafarov). There exists a family of sets $X = \langle X_0, X_1, \ldots \rangle$ and a collection $\mathcal{C}$ of subsets of ω such that:

– for all $Y \in \mathcal{C}$, there is no $(X \oplus Y)$-computable infinite X-cohesive set;

and for every stable coloring $d \leq_T X$, one of the following is true:

– d has an $(X \oplus Y)$-computable infinite homogeneous set for some $Y \in \mathcal{C}$;

– d has infinite homogeneous sets of both colors computing no X-cohesive set.

Corollary. COH $\not\leq_{sc}$ SRT^2_2.

The proof introduces a new method (tree labeling) to build homogeneous sets.

But it uses ω many iterates of the hyperjump, so X is quite complex. Also, the proof does not seem to work to show COH $\not\leq_{sc}$ SRT^2_k for any $k > 2$.

Hypothetical

Suppose there is a Δ^0_2 family X witnessing that COH $\not\leq_{sc}$ SRT^2_3. Then X has a self-modulus (i.e., a function $m \equiv_T X$ such that $X \leq_T f$ for every $f \geq^* m$.)

Let $c : [\omega]^2 \to 2$ be an arbitrary X-computable stable coloring.

Define $d : [\omega]^2 \to 3$ by $d(x, y) = c(x, y)$ if $y - x > m(x)$ and $d(x, y) = 2$ else.

Then c and d have the same limit-homogeneous sets. And every infinite homogeneous set for d dominates m and therefore computes X.

By assumption, let H be an infinite homogeneous set for d that computes no infinite X-cohesive set. Then also H is limit-homogeneous for c and $X \oplus H$ computes no infinite X-cohesive set.

We conclude that COH $\not\leq_c$ D^2_2.

COH and SRT^2_k for $k > 2$.

Question. Is it the case that COH $\leq_{sc}$ SRT^2_k for any $k > 2$?

For $k = 3$, this question was also asked by Hirschfeldt and Jockusch.

Theorem (Hirschfeldt and Jockusch). $\mathsf{RT}^1_3 \not\leq_{sc} \mathsf{D}^2_2$.

Question (Hirschfeldt and Jockusch). Is it the case that $\mathsf{RT}^1_3 \leq_{sc} \mathsf{SRT}^2_2$?

By simplifying the tree labeling method used to show that COH $\not\leq_{sc}$ SRT^2_2, we obtain a negative answer.

Theorem (Dzhafarov, Patey, Solomon, Westrick). If $j > k$ then $\mathsf{RT}^1_j \not\leq_{sc} \mathsf{SRT}^2_k$.

We now paste together various colorings $c : \omega \to j$ to obtain a family of sets.

Corollary. COH $\not\leq_{sc}$ SRT^2_k for all $k \geq 2$.

A nicer instance of COH.

Above, we wanted a family X witnessing that COH $\not\leq_{sc}$ SRT^2_3 which has a self-modulus. By a result of Solovay, all such sets are hyperarithmetical.

Question. Is there a hyperarithmetical family X witnessing that COH $\not\leq_{sc}$ SRT^2_3?

Theorem (Dzhafarov, Patey, Solomon, Westrick). For every $j > k$, there is a $\emptyset^{(\omega)}$-computable coloring $c : \omega \to j$ witnessing that $\mathsf{RT}^1_j \not\leq_{sc} \mathsf{SRT}^2_k$.

Corollary. There is a $\emptyset^{(\omega)}$-computable family X witnessing that COH $\not\leq_{sc}$ SRT^2_3.

Alas, not every hyperarithmetical set has a self-modulus. But $\emptyset^{(\omega)}$ does.

Proposition (Dzhafarov, Patey, Solomon, Westrick). There is a $\emptyset^{(\omega)}$-computable family X witnessing that COH $\not\leq_{sc}$ SRT^2_3 such that $\emptyset^{(n)} \leq_T X$ for all n.

Open question. Can X be chosen with $X \equiv_T \emptyset^{(\omega)}$ or with a self-modulus?

Thank you.