The reverse sorites

Damir D. Dzhafarov
University of Connecticut

March 24, 2019

The sorites

Sorites, from the Greek *σωρος* [*soros*], meaning heap.

The classificatory sorites

- Removing one grain of sand from a heap leaves a heap.
- Repeatedly removing one grain from a heap eventually leaves only one.
- One grain of sand does not a heap make.

”Next … came Eubulides of Miletus, who handed down a great many arguments in dialectics; such as the Lying one; the Concealed one; the Electra; the Veiled one; the Sorites; the Horned one; the Bald one.”

— Diogenes Laërtius, *Lives and Opinions of Eminent Philosophers*, 3rd c. CE

The sorites

"Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $(1 + \frac{i}{100})x$ grams of sugar, $i = 0, 1, \ldots, 400$, where x is the weight of one cube of sugar. ... [The subject] will be indifferent between cup i and cup $i + 1$, for any i, but by choice not indifferent between $i = 0$ and $i = 400$."

— R. Duncan Luce, *Econometrica*, 1956

The behavioral approach (Dzhafarov and D., 2010)

- A set of stimuli S endowed with a closeness structure, allowing one, for every pair of stimuli $x, y \in S$, to characterize 'how close' x is to y.

- A system (a human observer, a digital scale, a set of rules, etc.) assigning a response to each stimulus $x \in S$.

Three assumptions

Supervenience. Everything else being equal, the system cannot have different responses to different instances (repeated applications) of one and the same stimulus. That is, the response is a function $\pi(x)$ of the stimulus $x \in S$.

Tolerance. The response function π is 'tolerant to microscopic changes' in stimuli: if stimuli $a, b \in S$ are chosen 'sufficiently close' then $\pi(a) = \pi(b)$.

Connectedness. There is at least one pair of stimuli $a, b \in S$ such that $\pi(a) \neq \pi(b)$ and there is a finite chain of stimuli

$$a = x_1, x_2, \ldots, x_{n-1}, x_n = b$$

in which x_i is 'as close as needed' or 'as close as possible' to x_{i+1}.

A note on supervenience

This treatment is unrelated to issues of vagueness.

Redefining stimuli

- $\pi(x)$.
- $\pi(x_i, x_{i-1})$.
- $\pi(x_i, x_{i-1}, x_{i-2}, \ldots, x_0)$.
- $r_i = \pi(x_i, x_{i-1}, r_{i-1}, x_{i-2}, r_{i-2}, \ldots, x_0, r_0)$.

Redefining responses

- $\pi(x) = \text{Prob}(x \text{ grains of sand form a heap})$.
- $\pi(x) = \begin{cases} 1 & \text{Prob}(x \text{ grains of sand form a heap}) \geq 1/2, \\ 0 & \text{Prob}(x \text{ grains of sand form a heap}) < 1/2. \end{cases}$

Formal treatment

Definition (Fréchet, 1918). A *V-space* on a nonempty set S is a pair

$$\{S, \{\mathcal{V}_x : x \in S\}\},$$

where each $\mathcal{V}_x$ is a nonempty collection of subsets of S containing x.

For each $x \in S$, each $V \in \mathcal{V}_x$ is called a vicinity of x.

Definition. Let $\{S, \{\mathcal{V}_x : x \in S\}\}$ be a V-space. Any collection of vicinities obtained by choosing one $V \in \mathcal{V}_x$ for each $x \in S$ is called a V-cover of S.

We say $a, b \in S$ are V-connected if for any V-cover $\{V_x : x \in S\}$ of S there is a chain of points

$$a = x_0, x_1, \ldots, x_{n-1}, x_n = b \in S$$

such that $V_{x_i} \cap V_{x_{i+1}} \neq \emptyset$ for all $i < n$.

Dissolving the paradox

Non-tolerance principle. Let $\{S, \{\mathcal{V}_x : x \in S\}\}$ be a V-space and π a function on S. If there exist V-connected $a, b \in S$ with $\pi(a) \neq \pi(b)$, then there is an $x \in S$ such that π is not constant on any $V \in \mathcal{V}_x$.

An alternative viewpoint

Fix a function π on S. For each $x \in S$, let $P_x = \{y \in S : \pi(x) = \pi(y)\}$.

Then $P_\pi = \{S, \{\{P_x\} : x \in S\}$ is a V-space, and $\{P_x : x \in S\}$ is its only V-cover.

Non-connectedness principle. Let π be a function on S, and P_π the V-space on S defined as above. Then two elements $a, b \in S$ are V-connected in P_π if and only if $\pi(a) = \pi(b)$.

Question.

Is one of these a better ‘resolution’ of the sorites than the other?

A case for logic

"One might want to argue that ... one should adopt the weakest logic as one's generally valid logic. ... Mastering a particular domain essentially involves mastering its logical laws."

— Stenning and van Lambalgen, *Human Reasoning and Cognitive Science*

Examples of domains and their logics

- **Marcone, 2007.** Human preference and decision-making.

  Theorems about interval orders lie at levels RCA_0, WKL_0, ACA_0.

- **D., 2011.** Knowledge spaces and learning theory.

  Theorems about saturated orders lie at levels RCA_0 and ACA_0.

A reverse math result

Recall the two outcomes of the formal treatment of the sorites:

Non-tolerance. If $a, b \in S$ are V-connected and $\pi(a) \neq \pi(b)$ then there is an $x \in S$ such that π is not constant on any vicinity of x.

Non-connectedness. Two elements $a, b \in S$ are V-connected in the associated V-space P_π if and only if $\pi(a) = \pi(b)$.

Theorem (D., 2019).

1. Over RCA_0, the non-tolerance theorem is equivalent to ACA_0.

There is a computable space S and a computable V-space on S such that every V-cover that consists of only 'tolerant' vicinities computes $\emptyset'$.

2. The non-connectedness theorem is provable in RCA_0.

Conclusion

Disciplines outside of mathematics are a source of interesting mathematical problems.

Often, these discplines themselves propose different competing mathematical (logical and non-logical) solutions to these problems.

Reverse mathematics can (should!) be applied to these questions, and thus offer a new way to compare these competing solutions.

The resulting analysis ought to inform our understanding of the original problem.

E.g., solutions requiring weaker axioms (weaker subsystems) can be viewed as being more general.

References

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Thanks for your attention!