A behavioral approach to the sorites paradox

Damir D. Dzhafarov
University of Chicago

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Sorites without vagueness I: Classificatory sorites

E. N. Dzhafarov and D. D. Dzhafarov, *Theoria*, 76 (2010), no. 1, 4–24.

Sorites without vagueness II: Comparative sorites

E. N. Dzhafarov and D. D. Dzhafarov, *Theoria*, 76 (2010), no. 1, 25–53.

The sorites

"Next in succession to Euclides, came Eubulides of Miletus, who handed down a great many arguments in dialectics; such as the Lying one; the Concealed one; the Electra; the Veiled one; the Sorites; the Horned one; the Bald one. [. . .] Eubulides had a quarrel with Aristotle, and was constantly attacking him."

Diogenes Laërtius, *Lives and Opinions of Eminent Philosophers*

Sorites, from the Greek *σωρός* [*soros*], meaning 'heap'.

The sorites

The “classificatory” sorites.

One grain of sand does not a heap make.

Removing one grain of sand from a heap leaves a heap.

Removing one grain of sand from a heap enough times eventually leaves only one grain.

Behavioral approach

"Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $(1 + \frac{i}{100})x$ grams of sugar, $i = 0, 1, \ldots,$ where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup $i + 1$, for any i, but by choice he is not indifferent between $i = 0$ and $i = 400$."

R. Duncan Luce, *Econometrica*, 1956

Behavioral approach

A set of stimuli S endowed with a closeness structure, allowing one, for every pair of stimuli $x, y \in S$, to characterize 'how close' x is to y.

A system (a human observer, a digital scale, a set of rules, or anything whatever) assigning a response to each stimulus $x \in S$.

Behavioral approach

Supervenience assumption. Everything else being equal, the system cannot have different responses to different instances (repeated applications) of one and the same stimulus. That is, there is a function π such that the response of the system to stimulus $x \in S$ is $\pi(x)$.

Tolerance assumption. The response function π is 'tolerant to microscopic changes' in stimuli: if stimuli $a, b \in S$ are chosen 'sufficiently close' then $\pi(a) = \pi(b)$.

Connectedness assumption. There is at least one pair of stimuli $a, b \in S$ such that $\pi(a) \neq \pi(b)$ and there is a finite chain of stimuli

$$a = x_1, x_2, \ldots, x_{n-1}, x_n = b$$

in which successive elements 'as close as desired' or 'as close as possible'.

Behavioral approach

Example.

Supervenience and connectedness, but not tolerance:

- $S = \mathbb{R}$,
- ordinary metric closeness,
- π any nonconstant function on $\mathbb{R}$.

Supervenience and tolerance, but not connectedness:

- $S = [0, 1] \cup [2, 3] \cup [4, 5] \cup \cdots$,
- ordinary metric closeness,
- $\pi(x) =$ least even integer $\leq x$.

Behavioral approach

Ensuring supervenience.

Our treatment is unrelated to issues of vagueness. Without supervenience, the sorites is simply unformulable.

Redefining stimuli.

- $\pi(x)$
- $\pi(x_i, x_{i-1})$
- $\pi(x_i, x_{i-1}, x_{i-2}, \ldots, x_1)$
- $\pi(x_i, x_{i-1}, r_{i-1}, x_{i-2}, r_{i-2}, \ldots, x_1, r_1)$

Redefining responses.

- $\pi(x) = \mathrm{Prob}(x \text{ grains of sand form a heap})$
- $\pi(x) = \begin{cases} 1 & \mathrm{Prob}(x \text{ grains of sand form a heap}) \geq \frac{1}{2} \\ 0 & \mathrm{Prob}(x \text{ grains of sand form a heap}) < \frac{1}{2} \end{cases}$

Formal treatment

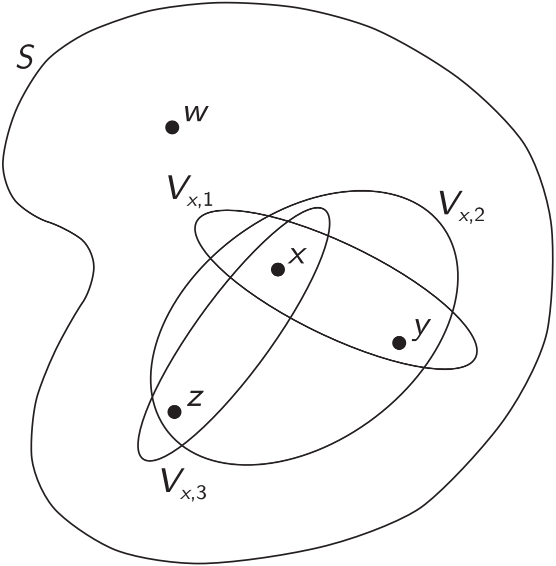
Definition (Fréchet, 1918). A V-space on a nonempty set S is a pair

$$\{S, \{\mathcal{V}_x : x \in S\}\},$$

where each $\mathcal{V}_x$ is a nonempty collection of subsets of S containing x.

For each $x \in S$, each $V \in \mathcal{V}_x$ is called a vicinity of x.

Formal treatment



Formal treatment

Definition. Let $\{S, \{\mathcal{V}_x : x \in S\}\}$ be a V-space.

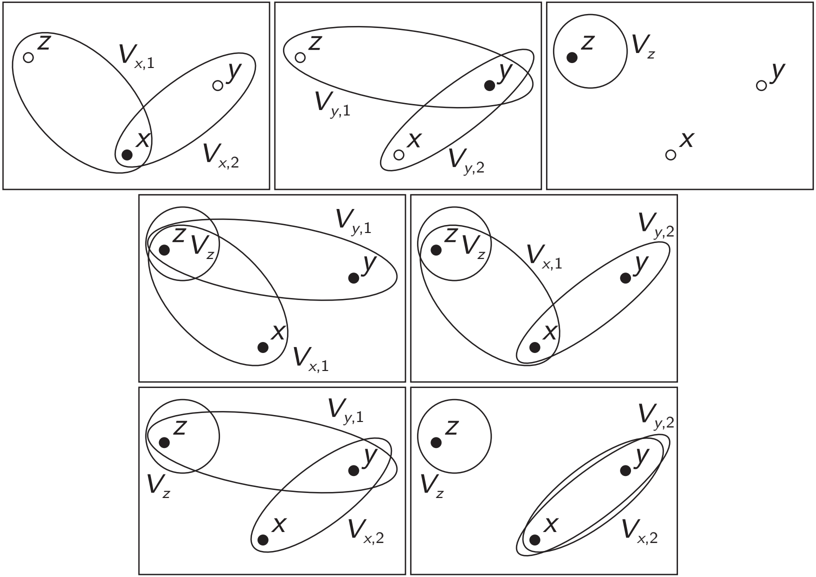
Any collection of vicinities obtained by choosing one $V \in \mathcal{V}_x$ for each $x \in S$ is called a V-cover of S.

We say a and b in S are V-connected if for any V-cover $\{V_x : x \in S\}$ of S there is a chain of points

$$x_1, x_2, \ldots, x_{n-1}, x_n \in S$$

such that $a = x_1$, $b = x_n$, and $V_{x_i} \cap V_{x_{i+1}} \neq \emptyset$ for $i = 1, \ldots, n-1$.

Formal treatment



Formal treatment

Theorem. Let $\{S, \{\mathcal{V}_x : x \in S\}\}$ be a V-space and π a function on S. If there exist V-connected $a, b \in S$ with $\pi(a) \neq \pi(b)$, then there is an $x \in S$ such that π is not constant on any $V \in \mathcal{V}_x$.

Another sorites

The “comparative” sorites.

In a row of cups of coffee, each may ‘match’ the next in sweetness, yet the first may not match the last.

Another sorites

Example.

For $x, y \in \mathbb{R}$, say x matches y if $|y - x| \leq \frac{1}{4}$.

Then 1 matches $\frac{3}{4}$, and $\frac{3}{4}$ matches $\frac{1}{2}$, but 1 does not match $\frac{1}{2}$.

Behavioral approach

"...if 'preference' is taken to mean which of two weights a person believes to be heavier after hefting them, and if 'adjacent' weights are properly chosen, say a gram difference in total weight of many grams, then a subject will be indifferent between any two 'adjacent' weights. If indifference were transitive, then he would be unable to detect any weight differences, however great, which is patently false."

R. Duncan Luce, *Econometrica*, 1956

Behavioral approach

Differential threshold property.

For every stimulus x, x always matches itself, and there is a vicinity V of x (called a differential threshold) such that x matches y for all $y \in V$.

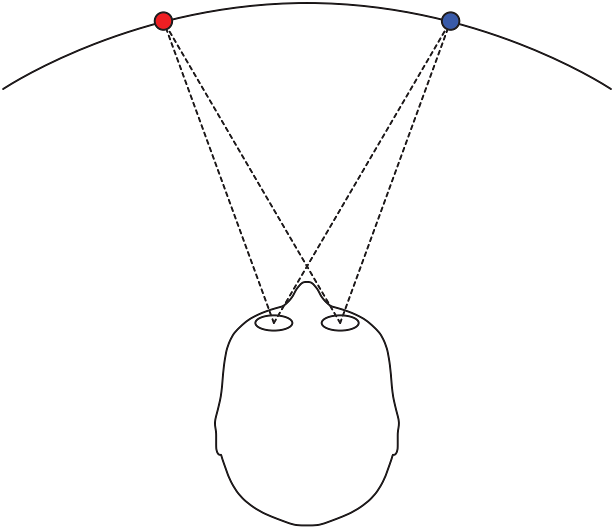
Behavioral approach

A set of stimulus values S, and a set of stimulus areas Ω. A stimulus is an element of $S \times \Omega$, i.e., a pair (x, α) where $x \in S$ and $\alpha \in \Omega$.

A system assigning a binary response (match/not match) to pairs of stimuli, with matching occurring only across different stimulus areas.

A notion of equivalence on $S \times \Omega$.

Behavioral approach



Behavioral approach

Regular mediality/minimality.

Let (x, α) be any stimulus. Then for each stimulus area we can choose a stimulus value such that any two of the resulting stimuli match each other, and each also matches (x, α).

If two stimuli (x, α) and (y, α) match one and the same stimulus (z, β), then they are equivalent.

Formal treatment

Definition. Let S and Ω be nonempty sets, $|\Omega| \geq 2$, and let M be a binary relation on $S \times \Omega$ such that $\neg(x, \alpha)M(y, \alpha)$ for all $x, y \in S$.

A sequence

$$(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_{n-1}, \alpha_{n-1}), (x_n, \alpha_n) \in S \times \Omega$$

is soritical if

- $(x_i, \alpha_i)M(x_{i+1}, \alpha_{i+1})$ for all $i = 1, 2, \ldots, n-1$,
- $\alpha_1 \neq \alpha_n$,
- and $\neg(x_1, \alpha_1)M(x_n, \alpha_n)$.

Formal treatment

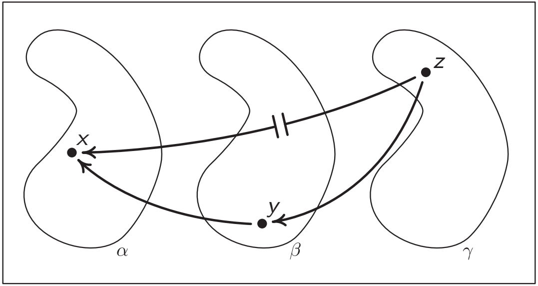
Lemma. Every soritical sequence has either a triadic soritical subsequence

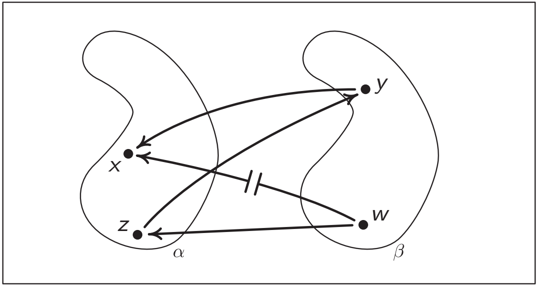
$$(x, \alpha), (y, \beta), (z, \gamma),$$

or a tetradic soritical subsequence

$$(x, \alpha), (y, \beta), (z, \alpha), (w, \beta).$$

Formal treatment





Formal treatment

Definition. Define a relation E on $S \times \Omega$ by setting $(x, \alpha)E(y, \beta)$ if and only if for all $z \in S$ and all $\gamma \in \Omega$,

$$(x, \alpha)M(z, \gamma) \iff (y, \beta)M(z, \gamma).$$

The space (S, Ω, M) is regular if for any $\alpha \neq \beta$ and any $x, y, z \in S$, if $(x, \alpha)M(z, \beta)$ and $(y, \alpha)M(z, \beta)$ then $(x, \alpha)E(y, \alpha)$.

The space (S, Ω, M) is well-matched if for any $\alpha_1, \alpha_2, \alpha_3 \in \Omega$ and any $x_1 \in S$, there exist $x_2, x_3 \in S$ such that for all $i, j = 1, 2, 3$, if $\alpha_i \neq \alpha_j$ then $(x_i, \alpha_i)M(x_j, \alpha_j)$.

Formal treatment

Theorem. Let (S, Ω, M) be a regular and well-matched space. Then there are no soritical sequences in this space.

Conclusion

The classificatory sorites, viewed behaviorally, is unrelated to issues of vagueness.

The paradox itself, when it can be formulated, is based on the false belief that responses to two microscopically different stimuli must agree.

In fact, whenever a stimulus space and responses can be defined in such a way that the responses supervene on stimuli and stimuli are connectable by microscopic steps, the stimulus space must contain stimuli in any vicinity of which, however small, the responses vary.

Conclusion

Comparative sorites is very different from classificatory sorites. Most notably, it is logically consistent.

Contrary to the concept of a differential threshold, where human comparative judgments are concerned, the comparative sorites contradicts a certain regularity principle supported by empirical evidence.

Thus, in systems resembling human comparative judgements, the comparative sorites does not obtain.

Thank you for your attention!