# Reverse mathematics of combinatorial problems

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<u>Problem</u> (instance-solution problem) $P$

$(I, S)$

$I$ set of instances of $P$

for each $X \in I$ , $S(X)$ = set of solutions to $X$.

$k \geq 1$ $RT^1_k$

- If $\mathbb{N}$ is partitioned into $k$ parts, then at least one part is infinite.
- Given a partition of $\mathbb{N}$ into $k$ parts, there is a part that is infinite.

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Instances: $k$-partitions of $\mathbb{N}$

for each $k$-partition $A_0 \sqcup \dots \sqcup A_{k-1} = \mathbb{N}$, the solutions are all $i<k$ s.t. $A_i$ is infinite; all $A_i$ s.t. $A_i$ is infinite.

$2^{<\omega} = \{0,1\}^* =$ set of all finite binary ($\{0,1\}$-valued) strings, ordered by prefix (initial segment) relation.

$000101 \in 2^{<\omega}$

$0001011 \in 2^{<\omega}$

$000101 \prec 0001011$

A <u>binary tree</u> $T \subseteq 2^{<\omega}$ is a set closed downward under $\preceq$:
if $\sigma \in T$ and $\tau \preceq \sigma$ then $\tau \in T$.

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A tree $T$ is infinite (as a set) iff it contains strings of arbitrary large length iff it contains strings of every length.

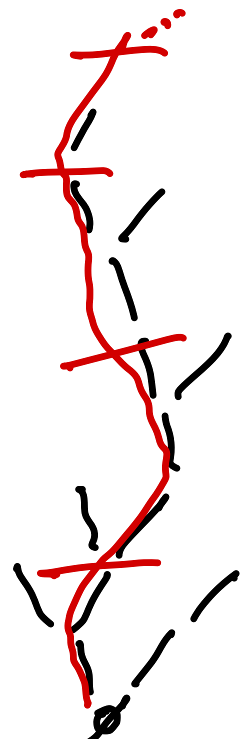
$2^\omega$ = set of all infinite binary sequences = functions $f : \omega \to \{0,1\}$

if $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$ then

$\sigma \prec X$ if the first length($\sigma$) many bits of $X$ agree with $\sigma$.

$X \upharpoonright \text{length}(\sigma) = \sigma$.

If $T$ is a tree, $X \in 2^\omega$ is a <u>path</u> through $T$ if every $\sigma \prec X$ belongs to $T$.



## Weak Königs' Lemma

If $T$ is an infinite tree then it has at least one path.

**instances**: all infinite trees $T \subseteq 2^{<\omega}$

**solutions to a given $T$**: all paths through $T$

## Jump problem

**instances**: all $X \subseteq \mathbb{N}$

**solutions to a given $X$**: $X' = TJ(X)$

$''$

the halting set relative to $X$

# Ramsey's theorem

Given $X \subseteq \mathbb{N}$, $n \geq 1$, $k \geq 1$

- $[X]^n = \{F \subseteq X : |F| = n\}$
- a <u>$k$-coloring of $[X]^n$</u> is a function $c : [X]^n \longrightarrow k = \{0, 1, \dots, k-1\}$
- $Y \subseteq X$ is <u>homogeneous for $c$</u> if $c$ is constant on $[Y]^n$.

$n = 1 \qquad X = \mathbb{N}$

$c : [\mathbb{N}]^1 \longrightarrow k$

$c : \mathbb{N} \to k$

$A_i = \{x \in \mathbb{N} : c(x) = i\}$

$A_0 \sqcup \cdots \sqcup A_{k-1} = \mathbb{N}$

$Y \subseteq \mathbb{N}$ is homogeneous if $c$ is constant on $Y$, i.e. if $Y \subseteq A_i$ for some $i < k$.

# Ramsey's theorem for k-colorings of $[\mathbb{N}]^n$

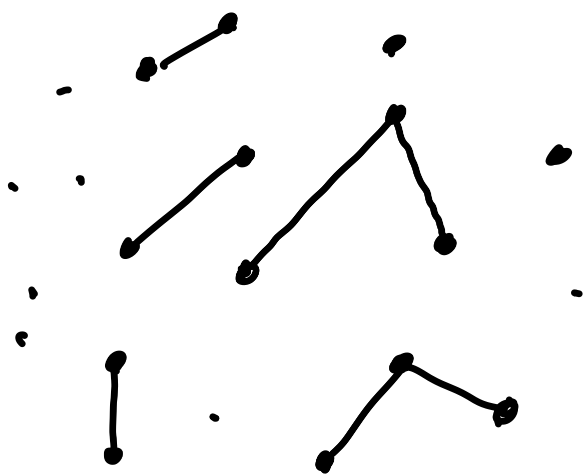
$(RT^n_k)$

For every $c : [\mathbb{N}]^n \to k$
there is an infinite set $Y$ that
is homogeneous for $c$.

For $n=1$, Ramsey's theorem is just the pigeon hole principle.

$c : [\mathbb{N}]^2 \to 2 = \{0,1\}$

For $n=2$,



given an infinite graph, there is an infinite subgraph which is either a clique, or an anti-clique.

$RT^n_k$ as a problem:

instances: all $c : [\mathbb{N}]^n \longrightarrow k$

solutions to a specific $c$: all the infinite homogeneous sets.

## $RT^1_k$ (before)

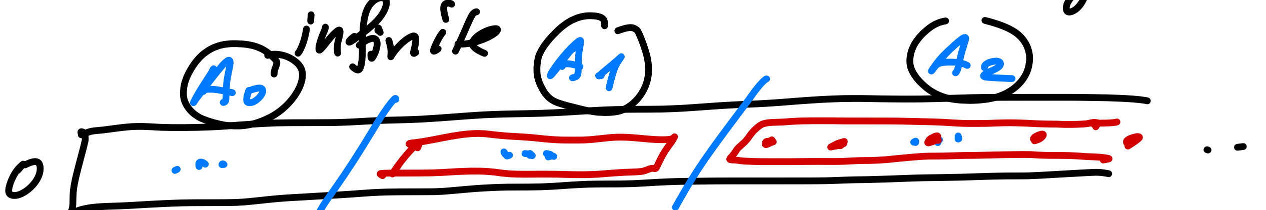
instances: $k$-partitions of $\mathbb{N}$

solutions: all pieces of the given partition that are infinite

## $RT^1_k$ (now)

instances: colorings $c : [\mathbb{N}]^1 \to k$

solutions: all the inf homogeneous sets for a given $c$.



# Computable combinatorics

Given a problem, what can we say about its solutions = complexity, definability relative to its instances?

$RT^1_k$ – computably true

$c : \mathbb{N} \to k$ given

$\exists i < k \quad c^{-1}(i)$ is infinite

$\{x \in \mathbb{N} : c(x) = i\}$ is a solution to $c$.

$\{x \in \mathbb{N} : c(x) = i\} \leq_T c$

$\rightsquigarrow$

computable from $c$

<u>WKL</u> (weak Königs Lemma)

is not computably true:

build a computable infinite tree $T \subseteq 2^{<\omega}$

that has no computable path.

(i.e., <u>not</u> every instance computes a solution to itself).

<u>Ensure</u> for each $\underline{e}$: $e^{th}$ Turing functional $\Phi_e$, if it is total and $\{0,1\}$-valued, then $\Phi_e$ is not a path through $T$.

$\Phi_0(0)\downarrow = 1$

$\Phi_1(1)\downarrow = 0$ ✓

$\Phi_2(2)\downarrow = 0$

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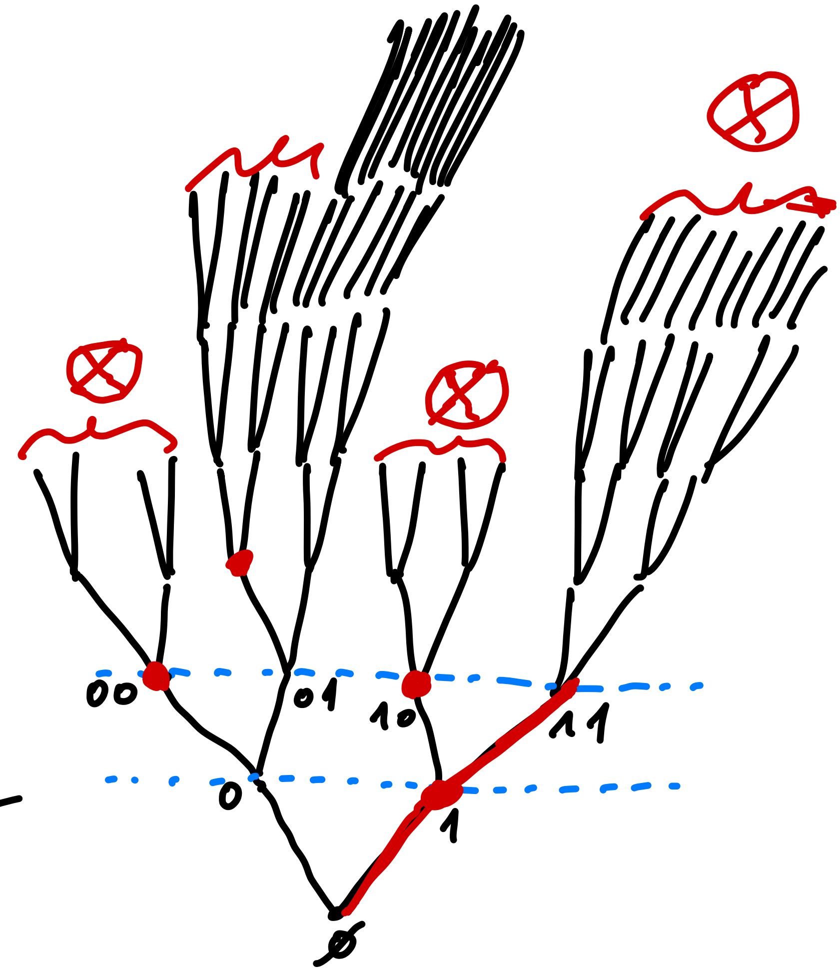
$T$ infinite ✓

$T$ computable ✓

no $\Phi_e$ is a path ✓



- WKL is not computably true.

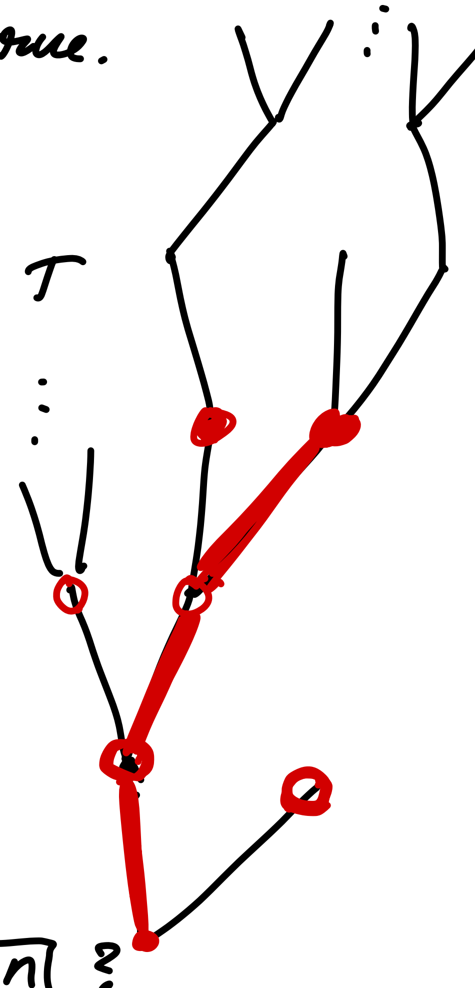
- Given infinite $T \subseteq 2^{<\omega}$, build a path $X$ through $T$ computable in $T'$.

$T'$ can repeatedly answer for each $\sigma \in T$ the question:

"is $T$ infinite above $\sigma$"

is $\{\tau \in T : \sigma \prec \tau\}$ infinite"?

$\forall n \boxed{\exists \sigma \in T : \text{length}(\sigma) = n}$ ?



## Complexity of WKL:

- not computably true
- always has solutions computable in the jump of the instance

The class of Turing degrees that can solve any computable instance of WKL is exactly the class of PA degrees.

## Low basis theorem (Jockusch & Soare)

Every computable instance of WKL has a solution that is low, i.e., a solution $X$ s.t. $X' \leq_T \emptyset'$.

# Cone-avoidance basis theorem (Jockusch-Soare)

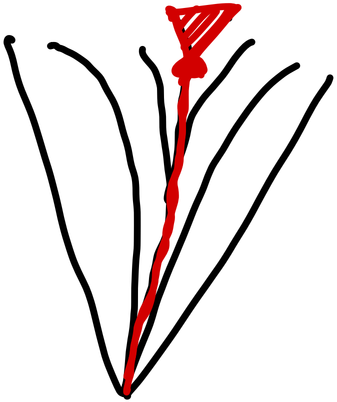
Suppose $C \not\leq_T \emptyset$. Every computable instance of WKL has a solution $X$ s.t. $C \not\leq_T X$.

avoids

$\{Y : Y \geq_T \emptyset'\}$

"cone above $\emptyset'$"

Proof is by forcing.



We work with infinite subtrees of $T$. ↖ given instance of WKL.

For each $e$, $\underbrace{\Phi_e^X \neq C}$ $\subseteq \mathbb{N}$

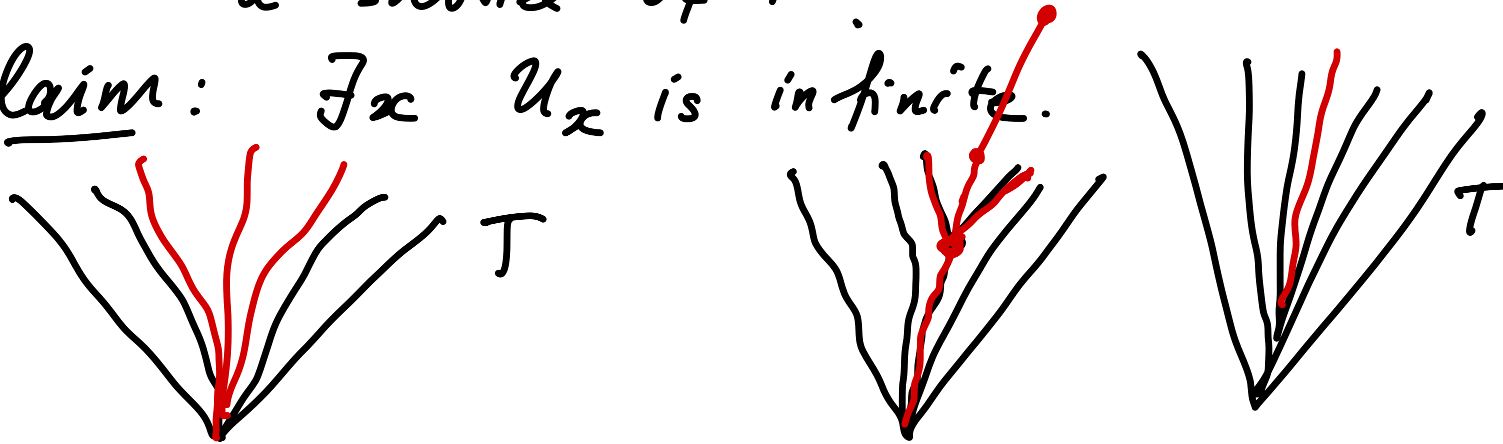
$$\exists x \quad \Phi_e^X(x)\uparrow \quad \text{or}$$

$$\Phi_e^X(x)\downarrow \neq C(x)$$

$T$ $\quad \Phi_0^X \neq C$

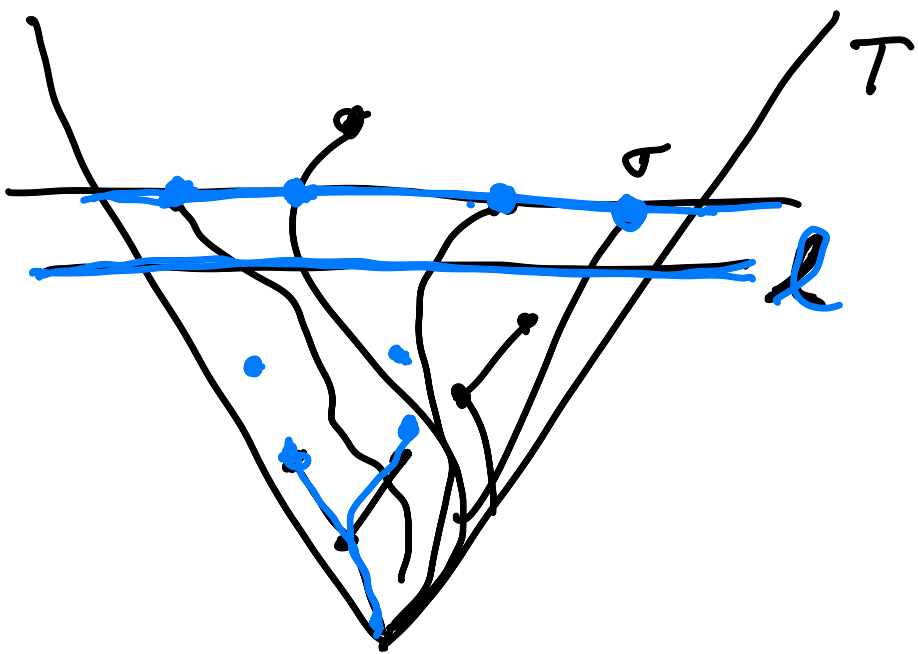
define for each $x \in \mathbb{N}$: $U_x = \{\sigma \in T : \neg\left(\Phi_{0,|\sigma|}^{\sigma}(x)\downarrow = C(x)\right)\}$.

- Each $U_x$ is a subtree of $T$.
- <u>Claim</u>: $\exists x$ $U_x$ is infinite.



If not, then we can compute $C$.

$C(x) = ?$ $\quad U_x$ is finite.




for every $\sigma$,

$\phi_0^\sigma(x)\downarrow$

& these values are all the same.

# Last time

- problems
  - $RT^1_k$
  - Jump
  - Weak Königs Lemma
  - Ramsey's theorem ($RT^n_k$)
- $RT^1_k$ is computably true
- WKL is not computably true, but always has solutions computable in the jump
- WKL has cone avoidance.

# Jump problem

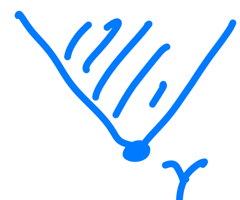
instances: $X \subseteq \mathbb{N}$

solutions to $X$: $X'$

$$\emptyset \longmapsto \emptyset'$$

does not have cone avoidance

$\{ Y \subseteq \mathbb{N} : Y \geq_T \emptyset' \}$



# Ramsey's theorem

$n = 2, \; k = 2$

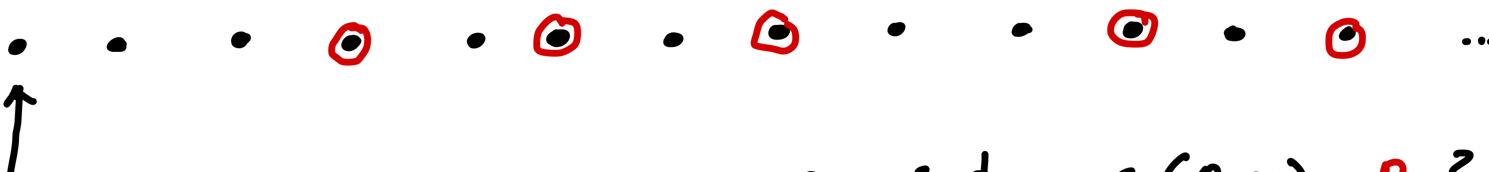
$c : [\mathbb{N}]^2 \to 2 = \{R, B\}$

define $d : A \to 2$

define $A \subseteq \mathbb{N}$

define $R \subseteq \mathbb{N}$

$0 \quad 1 \quad 2 \quad 3$



are there $\infty$ many $x > 0$ s.t. $c(0, x) = R$ ?

if yes, $d(0) = R$ put $0$ into $A$

if no, $d(0) = B$ put all $x > 0$ into $R$ s.t. $d(0) = c(0, x)$

$A = \{0\}$ $d(0)$

$R$ infinite

let $x_0 =$ min $R$ $(x_0 > 0)$ $c(0, x_0) = d(0)$

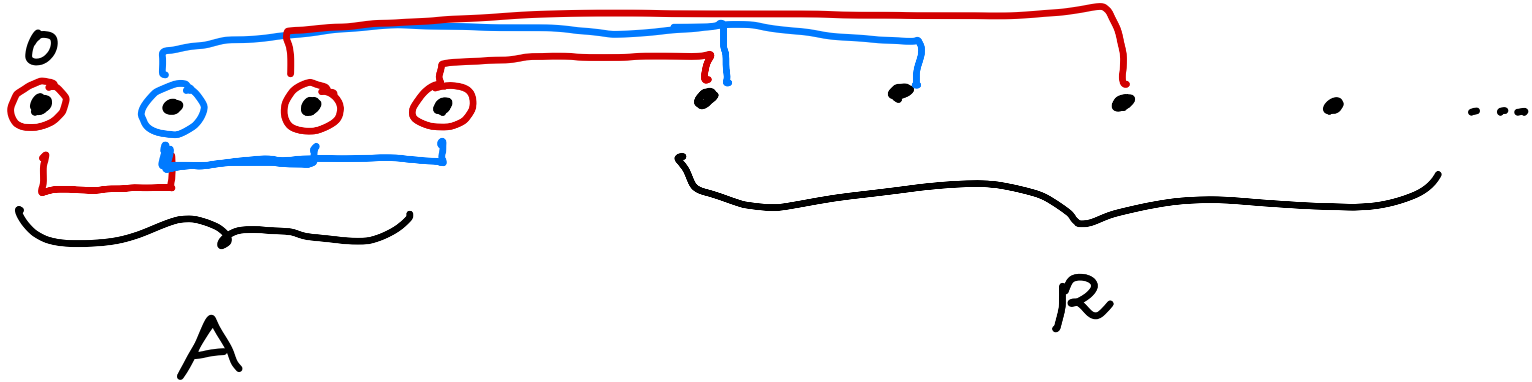
are there $\infty$ many $x > x_0$ in $R$

s.t. $c(x_0, x) = R$?

- if so, $d(x_0) = R$
- if no, $d(x_0) = B$

add $x_0$ to $A$

redefine $R$ to be all the $x > x_0$ s.t. $d(x_0) = c(x_0, x)$




Keep going, eventually define infinite $A$ and $d : A \to \{R, B\}$ s.t. if $x, y \in A$ with $x < y$ then $c(x,y) = d(x)$.

Consider this as a new instance of $RT^1_2$. $d \leq_T c''$, $A \leq_T c''$.

$d : A \to 2$ therefore has a $c''$-computable infinite homogeneous set. i.e., an infinite set $B$ and a color $i \in \{R, B\}$ s.t. $d(x) = i$ for all $x \in B$, meaning $c(x, y) = i$ for all $x, y \in B$.

So $B$ is a solution to $c$ (as an instance of $RT^2_2$). $B \leq_T c''$.

In general, we can do a similar inductive argument to see that $RT^n_k$ always has solutions computable in the $n^{\text{th}}$ jump.

(Jockusch)

<u>Theorem</u> $RT^n_2$ does not always have solutions computable in $(n-1)$st jump.

Jockusch 1972

- Build a <u>computable</u> $c : [\mathbb{N}]^2 \to 2$

- <u>Limit lemma</u>: $X \leq_T \emptyset'$ iff there is a prim. rec ~~computable~~ function $f : \mathbb{N}^2 \to 2$ s.t. $\forall n \quad X(n) = \lim_{s \to \infty} f(n,s)$.

- So there is a uniformly computable sequence $\{f_e : e \in \mathbb{N}\}$ of <u>total</u> computable fns $f_e : \mathbb{N}^2 \to 2$ s.t. $\forall X \leq_T \emptyset' \ \exists e \ \forall n \ X(n) = \lim_s f_e(n,s)$.

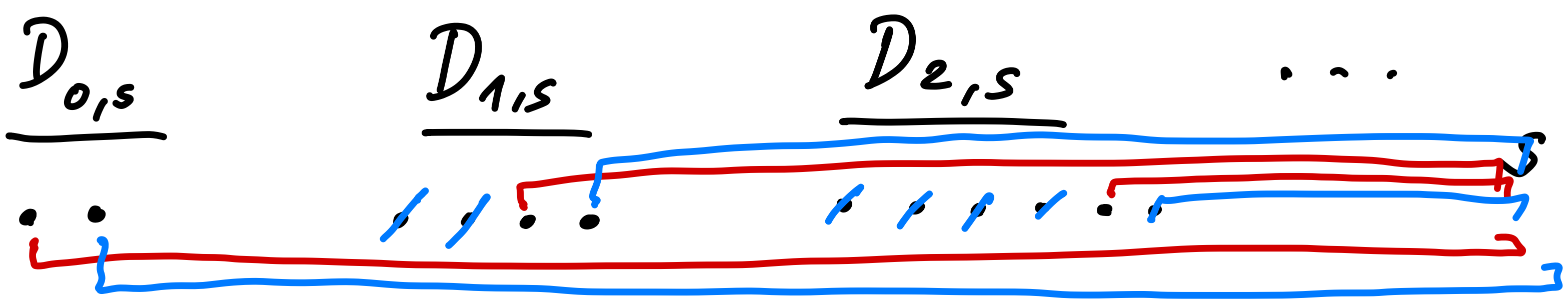
**Goal**: ensure that for all $e$, it is not the case that there is an inf homogeneous set $H$ for $c$ s.t.

$$\forall n \quad H(n) = \lim_s f_e(n,s).$$

Proceed by stages; at stage $s \in \mathbb{N}$ we define $c$ on $[0,s) \times \{s\}$.

At stage $s$, we work on $f_e$ for $e<s$.

Let $D_{e,s}$ be the least $2e+2$ many elements $x<s$ s.t. $f_e(x,s)=1$.



<u>Thm</u> (Jockusch) For all $n \geq 3$, there is a computable $RT^n_2$ all of whose solutions compute $\emptyset'$.

(Enough to show this for $n=3$.)

Construct $c : [\mathbb{N}]^3 \to 2$.

Fix a computable $f : \mathbb{N}^2 \to 2$ s.t.

$\forall n \ \emptyset'(n) = \lim_s f(n,s)$.

$$c(x, s, t) = \begin{cases} 1 & \text{if } (\forall y < x)\, f(y,s) = f(y,t) \\ 0 & \text{otherwise.} \end{cases}$$

$x < s < t$

Suppose $H \subseteq \mathbb{N}$ inf homogeneous set, for $c$. Claim: $\emptyset' \leq_T H$.

$\emptyset'(y) = ?$ Choose $x \in H$ $x > y$, $t > s > x$ in $H$.

$c$ restricted to $[H]^3$ must take the value 1. ✓

$c(x, s, t) = 1$ so $f(y, s) = f(y, t)$ $\forall t > s$ in $H$

so $\emptyset'(y) = \lim_u f(y, u) = f(y, s)$

**Thm** (Seetapun's theorem) For every $C \not\leq_T \emptyset$,
For every computable coloring $c : [\mathbb{N}]^2 \to 2$
there is an inf homogeneous set $H \not\geq_T C$.

"$RT^2_k$ has cone avoidance".

- Seetapun & Slaman (1995)
- Hummel & Jockusch (1994)
- Dzhafarov & Jockusch (2009)

$$\begin{tikzcd}
RT^2_2 \arrow[r, dash] \arrow[dr, dash] & SRT^2_2 \\
 & COH \\
\text{cone avoidance} \arrow[uu] \arrow[ur] \arrow[uur, bend right] &
\end{tikzcd}$$

**Definition** A coloring $c : [\mathbb{N}]^2 \to 2$ is stable if $\forall x \ \lim_y c(x,y)$ exists.

$SRT^2_2$ instances: stable colorings $c : [\mathbb{N}]^2 \to 2$

solutions to $c$: inf homogeneous sets.

$D^2_2$ : instances: all stable $c : [\mathbb{N}]^2 \to 2$

solutions to such a $c$:

($\Delta^0_2$ subset principle) all limit-homogeneous sets for $c$.

<u>Definition</u>. Given a stable $c : [\mathbb{N}]^2 \to 2$, a set $L \subseteq \mathbb{N}$ is <u>limit-homogeneous</u> if $(\forall x \in L) \lim_y c(x,y)$ is the same.

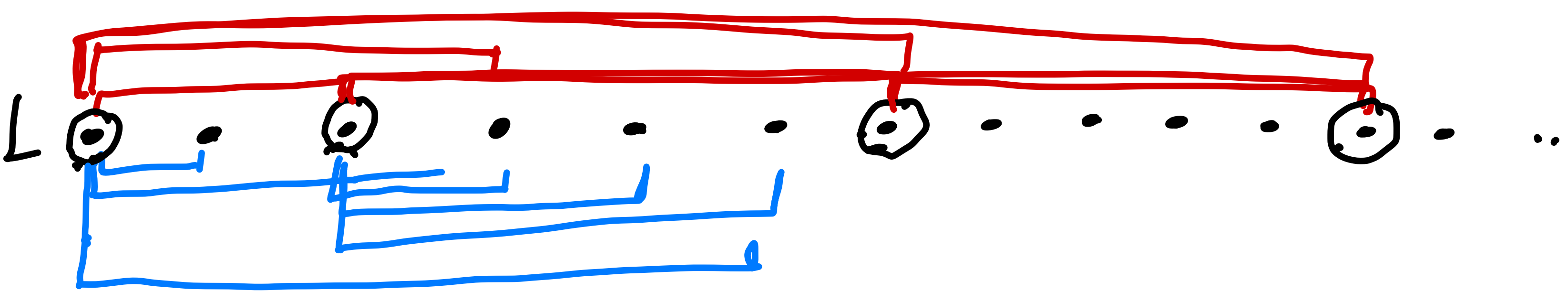
<u>Note</u>: every infinite homogeneous set <u>is</u> limit-homogeneous.

$x \in H$

"Reducing" $SRT^2_2$ to $D^2_2$:

Fix a stable $c : [\mathbb{N}]^2 \to 2$.

Let $L$ be an inf limit-homogeneous set for $c$. Say $\lim_y c(x,y) = i = R$ for all $x \in L$.



## Last time:

$RT^n_k$ - always has $\emptyset^{(n)}$-computable solutions

$n=1$: $RT^1_k$ is computably true

$n \geq 3$: $RT^n_k$ can code the halting problem

$n=2$: $RT^2_k$ admits cone avoidance

$C \not\leq_T \emptyset$ and computable $c : [\mathbb{N}]^2 \to k$

$\exists$ inf hom. set $H$ for $c$ s.t.

$C \not\leq_T H$.

$c : [\mathbb{N}]^2 \to k$ stable if $\forall x \lim_y c(x,y)$

SRT$^2_k$ : RT$^2_k$ restricted to stable colorings

D$^2_k$ : for every stable $c : [\mathbb{N}]^2 \to k$ $\exists$ inf limit-homogeneous set.

If we can "solve" D$^2_k$ we can solve SRT$^2_k$.

$\underline{D^2_k}$ : given stable $c : [\mathbb{N}]^2 \to k$

$\underline{d : \mathbb{N} \to k}$ $\quad d(x) = \lim_y c(x,y)$.

instance of $RT^1_k$

! every solution to $d$ is a solution to $c$

note: $d$ is not computable from $c$; $d \leq_T c'$.

<u>COH</u> (<u>cohesive</u> principle)

<u>instances:</u> $\vec{R} = (R_0, R_1, R_2, \ldots)$, $R_i \subseteq \mathbb{N}$

$\vec{R} = \{\langle x, i\rangle : x \in R_i\}$.

<u>solutions to $\vec{R}$</u>: all sets $X$ s.t.

$$\forall i \, [\underbrace{|X \cap R_i| < \infty}_{X \subseteq^* \overline{R_i}} \text{ or } \underbrace{|X \cap \overline{R_i}| < \infty}_{X \subseteq^* R_i}].$$

($X$ is <u>cohesive</u> for $\vec{R}$.)

# Obtaining $RT^2_2$ from $SRT^2_2$ and COH:

$\overset{\leq_T \emptyset'}{c : [\mathbb{N}]^2 \to 2}$ (not necessarily stable)

$\vec{R} \overset{\leq_T \emptyset'}{=} \langle R_x : x \in \mathbb{N} \rangle$ $\qquad R_x = \{ y > x : c(x,y) = 0 \}$

Let $X \not\geq_T \emptyset'$ be cohesive for $\vec{R}$.

$c \upharpoonright [X]^2$. is stable. Choose $x \in X$.

- either $X \subseteq^* R_x \implies \lim_{y \in X} c(x,y) = 0$.
- or $X \subseteq^* \overline{R_x} \implies \lim_{y \in X} c(x,y) = 1$.

Now apply $SRT^2_2$ to $c \upharpoonright [X]^2$.

1. Cone avoidance of COH.

for every $C \not\leq_T \emptyset$, every computable instance of COH has a solution that does not compute $C$.

2. Strong cone avoidance of $D^2_2$.

for every $C \not\leq_T \emptyset$, every instance of $D^2_2$ has a solution that does not compute $C$.

Mathias forcing constructions

Fix a computable instance $\vec{R} = \langle R_0, R_1, \dots \rangle$ of COH.

$C \not\leq_T \emptyset$.

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We build a cohesive set $G$ by forcing:

$(D, E)$ ← condition

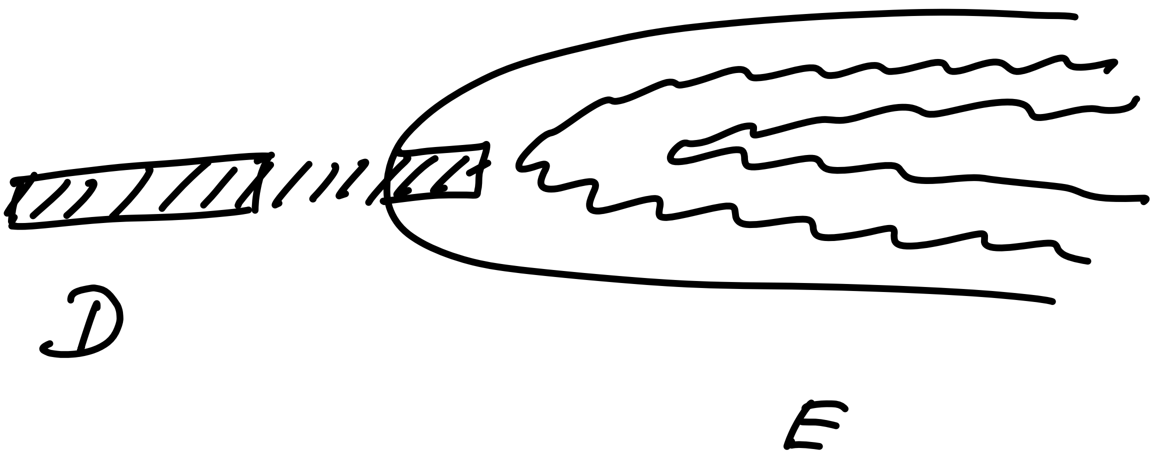
$D$: finite set

$E$: infinite set

$\max D < \min E$

$(\hat{D}, \hat{E})$ <u>extends</u> $(D, E)$ if

- $D \subseteq \hat{D} \subseteq D \cup E$ $\Big\langle$ $D \subseteq \hat{D}$, $\hat{D} \setminus D \subseteq E$
- $\hat{E} \subseteq E$



**For COH:** assume all reservoirs $\underline{E}$ in our conditions are computable

**Requirements:** I) $\forall e$ $G \subseteq^* R_e$ or

$G \subseteq^* \overline{R_e}$

II) $\forall e$ $\Phi_e^G \neq C$.

$(\emptyset, \mathbb{N})$ ← starting condition.

Stage $s = 2e$:

assume our condition is $(D, E)$.

Consider $R_e$: if $|E \cap R_e| = \infty$,

set $\hat{E} = E \cap R_e$.

otherwise, set $\hat{E} = E \cap \overline{R_e}$.

set $\hat{D} = D$. We take $(\hat{D}, \hat{E})$ as our new condition.

Stage $s = 2e+1$. Say our condition is $(D,E)$.

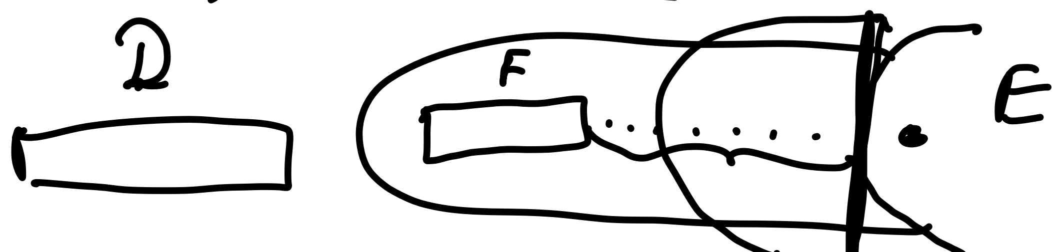
Ask: $\exists F_0, F_1 \subseteq E \ \exists x$ s.t.

$$\Phi_e^{D \cup F_0}(x)\downarrow \neq \Phi_e^{D \cup F_1}(x)\downarrow. \quad \text{(e-split)}$$

If so, $\exists i \ \ \Phi_e^{D \cup F_i}(x) \neq C(x)$.

Let $\hat{D} = D \cup F_i$, $\hat{E} = E \setminus [0, \mathrm{use}\, \Phi_e^{D \cup F_i}(x)]$

$(\hat{D}, \hat{E})$



<u>If not</u>: Let $\hat{D} = D$, $\hat{E} = E$.
Take $(\hat{D}, \hat{E})$ as our extension.

If for every $x$ we could find an $F \subseteq E$ s.t. $\Phi_e^{D \cup F}(x)\downarrow = C(x)$, then we could compute $C$.

$(\emptyset, \mathbb{N}) = (D_{-1}, E_{-1})$

At stage $s$, we defined $(D_s, E_s)$.

$G = \bigcup_s D_s$.

By construction, $G$ is cohesive for $\vec{R}$, and $C \not\leq_T G$.

Fix $d: \mathbb{N} \to 2$ ; $C \not\leq_T \emptyset$

goal: build a homogeneous set $H \not\geq_T C$.

Assume not.

$(D_0, D_1, E)$ s.t.

- $(D_i, E)$ is a Mathias condition
- $\forall i\ \forall x \in D_i$ $d(x) = i$
- $E \not\geq_T C$.

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$(\hat{D}_0, \hat{D}_1, \hat{E})$ extends $(D_0, D_1, E)$ if $\forall i$

$D_i \subseteq \hat{D}_i \subseteq D_i \cup E$

$\hat{E} \subseteq E$

**Lemma** If $(D_0, D_1, E)$ is a condition then for each $i$ $E \cap \{x : d(x) = i\}$ is infinite.

If not, then $E \subseteq^* \{x : d(x) = 1-i\}$.

So, some finite modification of $E$ is an infinite subset of $\{x : d(x) = 1-i\}$.

Requierements:

I) $\forall e \quad |G_0| > e$ & $|G_1| > e$

II) $\forall e \quad \phi_e^{G_0} \neq C$

OR

$\forall e \quad \phi_e^{G_1} \neq C.$

Start with $(\emptyset, \emptyset, \mathbb{N})$.

**Stage** $s = 2e$, given $(D_0, D_1, E)$.

Choose $x_0, \dots, x_{e-1} \in E$ $\quad d(x_i) = 0$

$y_0, \dots, y_{e-1} \in E$ $\quad d(y_i) = 1$

which exist by the lemma.

Let $\hat{D}_0 = D_0 \cup \{x_0, \dots, x_{e-1}\}$

$\hat{D}_1 = D_1 \cup \{y_0, \dots, y_{e-1}\}$

$\hat{E} = E \setminus [0, \max\{x_0, \dots, x_{e-1}, y_0, \dots, y_{e-1}\}]$.

Take $(\hat{D}_0, \hat{D}_1, \hat{E})$ as our extension.

<u>Stage</u> $s = 2\langle e_0, e_1\rangle + 1$, given $(D_0, D_1, E)$.

work to achieve: $\Phi_{e_0}^{G_0} \neq C$ <u>OR</u> $\Phi_{e_1}^{G_1} \neq C$.

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Let $\mathcal{A} = \{ \langle X_0, X_1 \rangle \in 2^\omega : X_0 \sqcup X_1 = E$

$\wedge (\forall i < 2)(\forall F_{0,i}, F_{1,i} \subseteq X_i)(\forall x)$

$\left[ \neg \left( \Phi_{e_i}^{D_i \cup F_{0,i}}(x)\downarrow \neq \Phi_{e_i}^{D_i \cup F_{1,i}}(x)\downarrow \right) \right] \}$.

$\mathcal{A}$ is $\Pi^0_1(E)$ class; set of paths through an $E$-computable binary tree.

$\mathcal{A} = \emptyset.$ $\qquad X_i = E \cap \{x : d(x) = i\}$

$$X_0 \sqcup X_1 = E$$

$$(X_0, X_1) \notin \mathcal{A}.$$

So: $\exists i < 2 \quad \exists F_{0,i}, \overset{\subseteq X_i}{F_{1,i}} \quad \exists x$

$$\underset{\substack{\neq \\ C(x)}}{\phi_{e_i}^{D_i \cup F_{0,i}}(x)\downarrow} \neq \phi_{e_i}^{D_i \cup F_{1,i}}(x)\downarrow.$$

$$\rightsquigarrow (\hat{D}_0, \hat{D}_1, \hat{E})$$

$$\hat{D}_i = D_i \cup F_{0,i} \qquad \hat{D}_{1-i} = D_{1-i}.$$

$$\hat{E} = E \setminus [0, \text{use } \phi_{e_i}^{D_i \cup F_{0,i}}(x)].$$

$\mathcal{A} \neq \emptyset$. $\quad C \not\leq_T E$

$\mathcal{A}$ was a $\Pi^0_1(E)$

By cone-avoidance basis thm (1st day) we get $\langle X_0, X_1 \rangle \in \mathcal{A}$ s.t. $E \oplus \langle X_0, X_1 \rangle \not\geq_T C$.

Say $X_i$ is infinite. $\hat{D}_0 = D_0$, $\hat{D}_1 = D_1$

$\hat{E} = X_i$.

Now $\Phi_{e_i}^{D_i \cup F} \neq C \quad \forall F \subseteq \hat{E}$.

$(D_0, D_1, E)_0 = (\emptyset, \emptyset, \mathbb{N})$

At stage $s$, we build $(D_{0,s}, D_{1,s}, E)_s$.

$G_0 = \bigcup_s D_{0,s}$ $\qquad$ $G_1 = \bigcup_s D_{1,s}$.

By construction, $|G_0| = |G_1| = \infty$.

$G_i$ is homogeneous for $d$ with color $i$.

Suppose $\Phi_{e_0}^{G_0} = \Phi_{e_1}^{G_1} = C$.

But at stage $s = 2\langle e_0, e_1\rangle + 1$ we ensured this was impossible.

Last time: Fix $Z$.

Cone avoidance of COH: fix $C \not\leq_T \overset{Z}{\cancel{\emptyset}}$. every $Z$-computable $\vec{R} = \langle R_0, R_1, \dots \rangle$ has an infinite $\vec{R}$-cohesive set $X$ s.t. $C \not\leq_T X \oplus Z$

Strong cone avoidance of $RT^1_k$: fix $C \not\leq_T \overset{Z}{\cancel{\emptyset}}$. every $c : \mathbb{N} \to k$ has an infinite homogeneous set $H$ s.t. $C \not\leq_T H \oplus Z$.

## Proof of cone avoidance of $RT^2_2$.

Fix $C \not\leq_T \emptyset$.

Fix a computable $c : [\mathbb{N}]^2 \to 2$.

Define a computable instance of COH as before: $R_x = \{ y > x : c(x,y) = 0 \}$. By cone avoidance of COH, choose a cohesive set $X \not\geq_T C$. As we saw, $c \restriction [X]^2$ is stable. Define $d : X \to 2$ by $d(x) = \lim_{y \in X} c(x,y)$. (Note: $d \leq_T X'$.) Since $C \not\leq_T X$, apply strong cone avoidance of $RT^1_2$, to get a set $H \subseteq X$ homogeneous for $d$ and s.t. $X \oplus H \not\geq_T C$. $H$ is <u>limit-homogeneous</u> for $c$. Thin out to get a $c \oplus H \oplus X$-comp. hom. set.

# Reverse Math

Second-order arithmetic, $Z_2$

language - two-sorted / two kinds of variables

variables: $x, y, z, \dots$ $\quad X, Y, Z, \dots$

arithmetical symbols: from first-order arithmetic $\quad 0, 1, +, \cdot, <, =$

$0 \in \mathcal{O}$

first-order $\quad$ second-order

$\underline{\underline{PA^-}}$ - algebraic axioms from Peano arithmetic

$$0 \neq 1$$

$$\neg \exists x \quad x < 0$$

$$\forall x \ (x \neq 0 \rightarrow \exists y \quad x = y + 1)$$

$$\vdots$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

axioms for the natural numbers
as an ordered semi-rings

## Comprehension

full comprehension

may have parameters

$\varphi(x)$ is a formula of our language

$$\exists Z \ \forall x \ (x \in Z \Leftrightarrow \varphi(x)).$$

# Induction

- suppose $X$   set induction

$$(0 \in X \wedge \forall x\,(x \in X \to x+1 \in X)) \to \forall x\,(x \in X).$$

- $\varphi(x)$   full induction

$$(\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1))) \to \forall x\,(\varphi(x)).$$

$$Z_2 = PA^- + \text{(full) comprehension} + \text{(full) induction}$$

$$= PA^- + \text{(full) comprehension} + \text{set induction}$$

**$RCA_0$** – recursive comprehension axiom

$= PA^- + \Delta^0_1$-comprehension $+ \Sigma^0_1$-induction

$\Delta^0_1$-CA: for every $\Sigma^0_1$ formulas $\varphi, \psi$

$$\forall x\,(\varphi(x) \leftrightarrow \neg\psi(x))$$
$$\rightarrow \exists Z\ \forall x\,(x \in Z \leftrightarrow \varphi(x))$$

I-$\Sigma^0_1$: for every $\Sigma^0_1$ formula $\varphi(x)$

$\Sigma^0_1$-IND: $\big(\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1)\big) \rightarrow \forall x\, \varphi(x).$

* still have <u>set induction</u>.

$\underline{\underline{ACA_0}}$ - arithmetical comprehension axiom

- $PA^-$ + arithmetical comprehension + arithmetical induction

- $RCA_0$ + arithmetical comprehension

<u>arithmetical comprehension</u>: for every $\Sigma^0_n$-formula $\varphi$, $\exists Z \, \forall x \, (x \in Z \Leftrightarrow \varphi(x))$.

## Big Five

Take a thm

Formalize it in $L_2$

See if it's provable in $RCA_0$,

and if not, which of the other 4 subsystems it's provable from / equivalent to over $RCA_0$.

$$
\begin{array}{ccc}
 & & Z_2 \\
 & & / \\
 & & \Pi^1_1\text{-}CA_0 \\
 & & / \\
 & & ATR_0 \\
 & & / \\
T & \leftrightarrow & ACA_0 \\
 & & / \\
 & & WKL_0 \\
 & & / \\
 & & \underline{RCA_0}
\end{array}
$$

# Semantics

$$\mathcal{M} = (M, \mathcal{S}^{\subseteq \mathcal{P}(M)}, \ldots) \models Z_2 \qquad \cdot \in \bullet$$

- $M$ is a model of $PA^- + \ldots$
- $M$ is thus some kind of (possibly nonstandard) model of arithmetic.

Fix $\mathcal{M} \models RCA_0$, $\mathcal{M} = (M, \mathcal{S})$.

If $M = \mathbb{N}$, then $\mathcal{M}$ is an $\omega$-model.

For $\omega$-models, we can thus identify $\mathcal{M}$ with $\mathcal{S}$.

For $RCA_0$, the $\omega$-models are precisely those $\mathcal{S}$ that are closed under $\leq_T$ and $\oplus$, i.e. a <u>Turing ideals</u>.

$$\left(\varphi(x, A, B, C) \Longleftrightarrow \quad \leq_T A \oplus B \oplus C\right)$$

For $\mathcal{M} \models ACA_0$, $\mathcal{M} = (\mathbb{N}, \mathcal{S})$

the $\omega$-models of $ACA_0$ are those $\mathcal{S}$ that are Turing ideal (closed under $\leq_T$, $\oplus$) closed under $X \mapsto X'$, i.e. jump ideals.

<u>Cor</u>. $RCA_0$ is strictly weaker than $ACA_0$.

<u>Pf</u>. $\{X : X \leq_T \emptyset\} \models RCA_0 + \neg ACA_0$.

$\Pi^1_2$ statement $\quad \forall X \, \exists Y \, (\ldots)$

$$\forall X \left( \phi(X) \longrightarrow \exists Y \, \psi(X, Y) \right).$$

$\forall X$ ( $X$ is a set of pairs that defines a function $[\mathbb{N}]^2 \to 2$

$\longrightarrow \exists Y$ ( $X$ as a function on $[Y]^2$ is constant)).

<u>As a problem</u>:
instances are those $X$ s.t. $\phi(X)$ holds.
solutions are those $Y$ s.t. $\psi(X, Y)$ holds

**Thm** If $P$ is a $\Pi^1_2$ theorem that, as a problem satisfies cone avoidance, then there is a $\omega$-model of $RCA_0 + P + \neg ACA_0$. (so, $RCA_0 \nvdash P \to ACA_0$)

**Pf.** We build $\emptyset = Z_0 \leq_T Z_1 \leq_T Z_2 \leq_T \cdots$

and take $\mathcal{S} = \{X : \exists i \; X \leq_T Z_i\}$.

$\mathcal{S}$ is a Turing ideal.

Ensure: $\emptyset' \notin \mathcal{S}$. Hence, $\mathcal{S} \not\models ACA_0$.

$Z_0 = \emptyset$.

Suppose $Z_0, \dots Z_s$ defined $\quad s = \langle e, i \rangle$; $(e, i < s)$.

Assume inductively that

$\emptyset' \not\leq_T Z_s$. If $\Phi_e^{Z_i}$ is not an instance of $P_1$, let $Z_{s+1} = Z_s$.

By cone avoidance of $P_1$ there is a solution $Y$ to $\Phi_e^{Z_i}$ s.t. $\emptyset' \not\leq_T Z_s \oplus Y$.

Let $Z_{s+1} = Z_s \oplus Y$.

$\emptyset' \not\leq_T Z_i$ for all $i$, so $\emptyset' \notin \mathcal{S}$.

Now suppose $X$ is any instance of $\mathsf{P}$ in $\mathcal{S}$. $X \leq_T Z_i$ for some $i$, say $\Phi_e^{Z_i} = X$. But then a solution to $X$ is computable from $Z_{\langle e,i\rangle+1}$. So, $\mathcal{S} \models \mathsf{P}$.

$\forall X(\cdots \to \exists Y \cdots)$

<u>Corollary</u>. $RCA_0 \nvdash RT^2_2 \rightarrow ACA_0$.

We also know that $RT^2_2$ has a computable instance with no computable solution.

<u>Corollary</u>. $RCA_0 \nvdash RT^2_2$

(Take $\{X : X \text{ is computable}\}$).

**Exercise**

Over $RCA_0$,

$RT^3_2 \to ACA_0$.

(at least over $\omega$-models)

$ACA_0$ - arithmetic operations

$RT^2_2$ ← strictly

$RCA_0$ - computable mathematics

$$\begin{tikzcd}
RT_2^n, \; n \geq 3 \arrow[r, leftrightarrow] & ACA_0 \arrow[d] \arrow[dr] & \\
 & RT^2_2 \arrow[d] \arrow[r, shift left, "/"] & WKL_0 \arrow[l, shift left, "/"] \arrow[dl] \\
 & RCA_0 &
\end{tikzcd}$$

<u>Lu Liu</u> (2013)

$RCA_0 + RT^2_2 \nvdash WKL_0$

* <u>Hirschfeldt</u> (201?)

Slicing the truth

$\underline{WKL}_0$ : $RCA_0$

+ Weak Königs Lemma

= $RCA_0$ +

"For every infinite tree $T \subseteq 2^{<\mathbb{N}}$, there exists an infinite path"

- there is a computable instance of WKL with no computable solution
- WKL satisfies cone avoidance

**Low basis thm** every computable infinite tree $T \subseteq 2^{<\mathbb{N}}$ has a low infinite path, i.e., a path $X$ s.t. $X' \leq_T \emptyset'$.

**Exercise.** Show that $WKL_0$ has an $\omega$-model consisting entirely of low sets.

**Corollary.** $WKL_0 \nvdash RT^2_2$.

**Pf.** There is a comp. inst. of $RT^2_2$ with no $\emptyset'$-computable solutions.

$\mathsf{SRT}^2_2$ every stable $c : [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set

$\mathsf{D}^2_2$ every stable $c : [\mathbb{N}]^2 \to 2$ has an infinite limit-hom. set.

$$\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \to \mathsf{D}^2_2$$

$$\mathsf{RCA}_0 \overset{?}{\vdash} \mathsf{D}^2_2 \to \mathsf{SRT}^2_2$$

$c : [\mathbb{N}]^2 \to 2$ stable.

apply $D^2_2$ to get an infinite lim-hom. set $L$.

say $L = \{ x_0 < x_1 < \dots \}$, of color $i$.

Build an inf subset $H$ of $L$.

Put $x_0$ into $H$, call it $x_{n_0}$.

Assume $x_{n_0} < \dots < x_{n_s}$ have been put into $H$.

For all $t \leq s$, $\lim_y c(x_{n_t}, y) = i$

* Choose $N$ s.t. $\forall t \leq s \ \forall y > N \ c(x_{n_t}, y) = i$.

Let $x_{n_{s+1}}$ be the least element $y \in L$, $y > N$.

Chong, Lempp, Yang: Over $RCA_0$, $D^2_2$ does imply $SRT^2_2$. (Really: $RCA_0 + D^2_2 \vdash B\Pi^0_1$).

---

Let $\Gamma$ be a class of formulas. (of $L_2$).

$B\Gamma$ (bounding for $\Gamma$) is the following scheme:

for each formula $\varphi \in \Gamma$

$\boxed{RCA_0 \nvdash B\Pi^0_1}$

$$\forall n \left( \forall i<n \, \exists y \, \varphi(i,y) \longrightarrow \exists b \, \forall i<n \, \exists y<b \, \varphi(i,y) \right).$$

(FUF) A finite union of finite sets is finite.

(Marcone - Frittaion) FUF $\leftrightarrow$ $B\Sigma^0_2$ over $RCA_0$.

**Thm** (Hirst) Over $RCA_0$, $B\Sigma^0_2 \longleftrightarrow \forall k\, RT^1_k$.

(**Exercise**: $B\Sigma^0_2 \longleftrightarrow B\Pi^0_1$)

**Pf.** ($B\Sigma^0_2 \to \forall k\, RT^1_k$.) Fix $c : \mathbb{N} \to k$.
Suppose there is no infinite set on which $c$ is constant. Then $\forall i < k\ \exists y\ \underbrace{\forall x > y\ c(x) \neq i}_{\Pi^0_1}$.
By $B\Pi^0_1$, $\exists b\ \forall i < k\ \exists y < b\ \forall x > y\ c(x) \neq i$.
So $\forall x > b\ \forall i < k\ c(x) \neq i$. Contradiction.

$(\forall k\, RT^1_k \to B\Sigma^0_2)$. Fix a $\Pi^0_1$ formula $\varphi$. Suppose $\exists n (\forall b\, \exists i < n\, \forall y < b\, \underbrace{\neg \varphi(i,y)}_{\Sigma^0_1}.)$

$c : \mathbb{N} \to n$ $\quad c(b) =$ least $i < n$ as above.

By $\forall k\, RT^1_k$, there is an infinite homogeneous set $H$ for $c$, say of color $i$.

For infinitely many $b$, $\forall y < b\, \neg\varphi(i,y)$.

So for all $y$, $\neg\varphi(i,y)$. So we showed:

$\exists i\, \forall y\, \neg\varphi(i,y)$.

Marcone - Gherardi 2009

Dorais, Dzhafarov, Hirst, Mileti, Shafer 2016

Weihrauch reducibility:

Let $P, Q$ be problems.

$P \leq_W Q$ if there are Turing functionals $\Phi, \Psi$,

s.t. $\forall$ $P$-instance $X$ $\qquad \Phi(X)$ is $Q$-instance

$\forall$ $\hat{Y}$ $Q$-solution to $\Phi(X)$ $\qquad \Psi(X, \hat{Y})$ is a $P$-solution to $X$.

## $\underline{P \leq_W Q}$

$$\begin{tikzcd}
\underline{\underline{P}} & & \underline{\underline{Q}} \\
X \arrow[rr, "\Phi"] \arrow[d, dashed, "\text{is solved by}"'] & & \hat{X} \arrow[d, dashed, "\text{is solved by}"] \\
Y & & \hat{Y} \arrow[ll, "\Psi(X, \cdot)"']
\end{tikzcd}$$

$$\underline{\underline{P \leq_c Q}}$$

$$\begin{tikzcd}
\underline{P} & & \underline{Q} \\
X \arrow[rr, "\text{computes}"] \arrow[d, dashed, "\text{is solved by}"'] & & \hat{X} \arrow[d, dashed, "\text{is solved by}"] \\
Y & & \hat{Y} \arrow[ll, "X\text{-computes}"']
\end{tikzcd}$$

if $P \leq_W Q$ then $P \leq_c Q$

if $P \leq_c Q$ then every $\omega$-model of $Q$ is an model of $P$ (and often, $RCA_0 \vdash Q \to P$).

$$\leq_W \Longrightarrow \leq_c \Longrightarrow \Rightarrow_\omega$$

$$\leq_c \rightsquigarrow \vdash_{RCA_0}$$

$RCA_0 \vdash D^2_2 \leftrightarrow SRT^2_2$ $\qquad D^2_2 \equiv_c SRT^2_2$

<u>Clearly</u>: $D^2_2 \leq_W SRT^2_2$

<u>Claim</u>: $SRT^2_2 \not\leq_W D^2_2$

<u>Thm</u> (Downey, Hirschfeldt, Lempp, Solomon)
There is a computable instance of $SRT^2_2$ with no low solution.

<u>Pf</u>. Priority argument.

**To show:** $SRT^2_2 \not\leq_W D^2_2$

$$\begin{tikzcd} c \arrow[r, "\Phi"] \arrow[d] & d \arrow[d] \\ H & L \arrow[l, "\Psi"'] \end{tikzcd}$$

Fix $\Phi, \Psi$.

Build a stable coloring $c : [\mathbb{N}]^2 \to 2$.

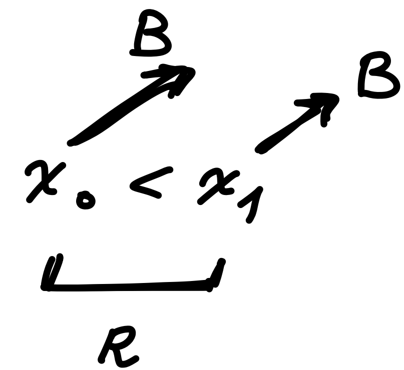
If $\Phi(c)$ is a stable coloring $d : [\mathbb{N}]^2 \to 2$, build a solution to $d$, a limit-homogeneous set $L$, s.t. $\Psi(c \oplus L)$ is not a homogeneous set for $c$.

We begin building $c$ $\phi(c)$

We want to find a finite set $F$

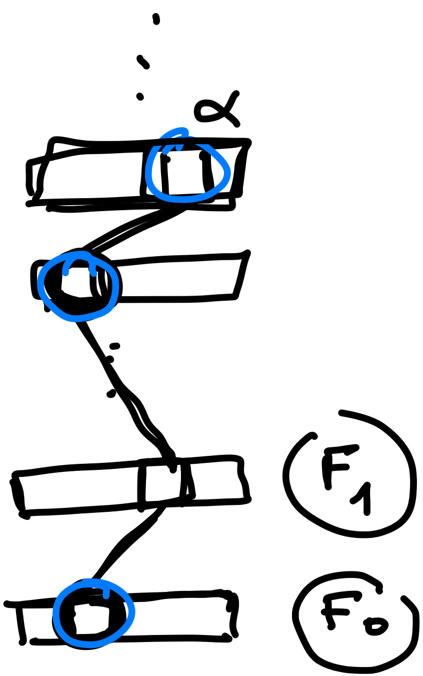
s.t. $\psi_0(c \oplus F)(x_0)\downarrow = 1$

$\psi(c \oplus F)(x_1)\downarrow = 1$



Make everything color BLUE in c.

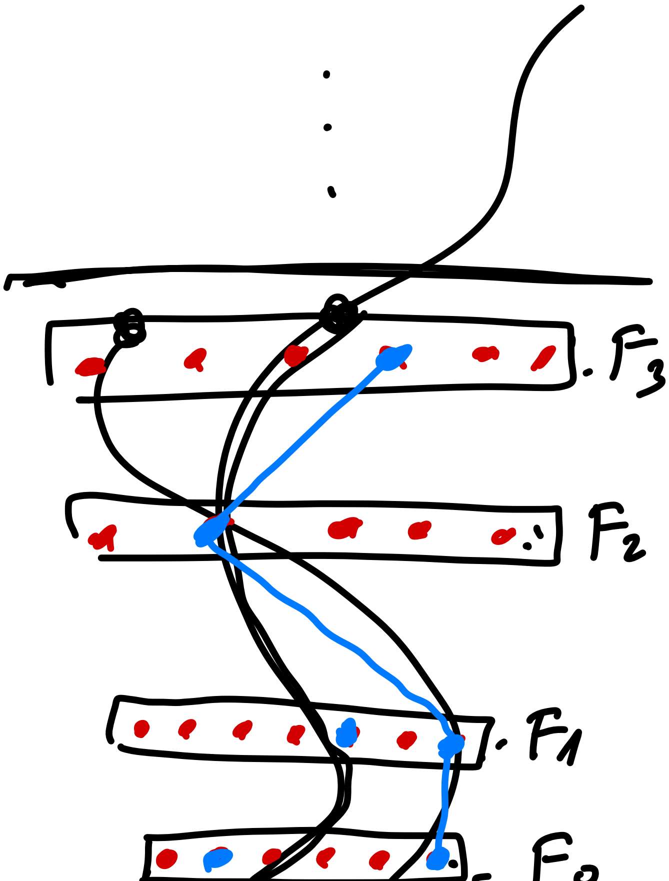
Looking for a seetapun configuration:




$\Psi(c \oplus F)\downarrow = 1$ on two inputs for some finite $F \subseteq \mathrm{range}(\alpha)$.

$\Psi(c \oplus F_1)\downarrow = 1$ on two inputs

$\Psi(c \oplus F_0)\downarrow = 1$ on two inputs




Look at the tree of all $\alpha$ with $\alpha(i) \in F(i)$ s.t. $\nexists F \subseteq \text{range } F$ $\psi(c \oplus F)\downarrow = 1$ on two inputs.

By Seetapun's argument, get a $\Phi(c)$-lim-hom set $F$ s.t. $\psi(c \oplus F)\downarrow = 1$ on two inputs.