The uniform content of partial and linear orders

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Joint work with Eric Astor, Reed Solomon, and Jacob Suggs.

Classical reverse mathematics.

Work with subsystems of second order arithmetic, Z_2:

- Base theory RCA_0;

- Stronger systems $WKL_0 < ACA_0 < ATR_0 < \Pi^1_1$-$CA_0$.

Initial focus was on a kind of zoological classification of theorems in terms of the “big five”. More recent focus has been on exceptions.

There is fruitful interplay between reverse mathematics and effective mathematics. In essence, they are two halves of a single endeavor.

RCA_0 has limited comprehension, but classical logic still applies.

- Non-uniform decisions in proofs over RCA_0 are allowed.

- Multiple appeals to a premise/hypothesis of a theorem are allowed.

Ramsey's theorem for pairs.

(RT^2_2) For very coloring $c : [\omega]^2 \to 2$, there exists an infinite homogeneous set for c.

Classical results:

- (Specker). There is a computable $c : [\omega]^2 \to 2$ with no computable homogeneous set; $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2$.
- (Jockusch). Every computable coloring $c : [\omega]^2 \to 2$ has a Π^0_2 infinite homogeneous set; $\mathsf{ACA}_0 \vdash \mathsf{RT}^2_2$.
- (Seetapun). Every computable coloring $c : [\omega]^2 \to 2$ has an infinite homogeneous set that does not compute $0'$; $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{ACA}$.
- (Liu). Every computable coloring $c : [\omega]^2 \to 2$ has an infinite homogeneous set whose degree is not PA; $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{WKL}$.

Weaker combinatorial principles.

(ADS) For every linear order $(L, \leqslant_L)$, there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending sequence in L.

(CAC) For every partial order $(P, \leqslant_P)$, there is either an infinite $\leqslant_P$-chain or infinite $\leqslant_P$-antichain in P.

Classical results:

- (Hirschfeldt and Shore). $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{CAC} \to \mathsf{ADS}$.

- (Hirschfeldt and Shore). $\mathsf{RCA}_0 + \mathsf{CAC} \nvdash \mathsf{ADS}$.

New classical results:

- (Lerman, Solomon, and Towsner). $\mathsf{RCA}_0 + \mathsf{ADS} \nvdash \mathsf{CAC}$.

The reverse mathematics zoo.

Instance-solution pairs.

A problem is a pair $(I, Soln)$, where $I \subseteq 2^\omega$ is a set of instances, and $Soln : I \to \wp(2^\omega)$ assigns to each $X \in I$ a set $S \subseteq 2^\omega$ of solutions to X.

If you like (?), a problem is a multifunction $I \rightrightarrows 2^\omega$.

All of the principles we look at typically have the form

$$(\forall X)[\phi(X) \to \exists Y[\theta(X, Y)]],$$

where ϕ and θ are arithmetical predicates.

These can be naturally regarded as problems:

- Let $I = \{X : \phi(X)\}$.
- Let $Soln(X) = \{Y : \theta(X, Y)\}$ for each $X \in I$.

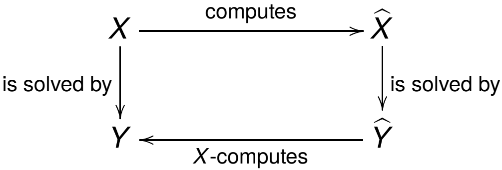
Stronger measures of strength.

Let P and Q be problems.

P is computbaly reducible to written Q, P $\leqslant_c$ Q, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, computes a P-solution Y to X.

So the following diagram commutes:



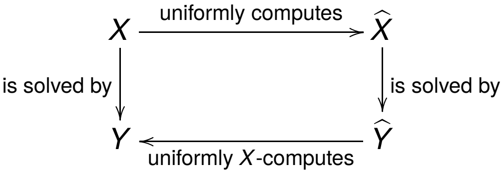
Stronger measures of strength.

Let P and Q be problems.

P is Weihrauch reducible to written Q, P $\leqslant_W$ Q, if

- every instance X of P uniformly computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, uniformly computes a P-solution Y to X.

So the following diagram commutes:



Stronger measures of strength.

Let P and Q be problems.

We have the following implications:

$$P \leqslant_W Q$$
$$\Downarrow$$
$$P \leqslant_c Q$$
$$\Downarrow$$
$$Q \models_c P$$

(Q computably entails P, i.e., every ω-model of Q is a model of P)

Because of induction issues, it does not follow that $RCA_0 \vdash Q \to P$.

ADS revisited.

Let $(L, \leqslant_L)$ be a linear order. What exactly is an ADS-solution?

Two possibilities:

(1) A set $S \subseteq L$ such that either for all $x, y \in S$, $x < y \to x <_L y$, or for all $x, y \in S$, $x < y \to x >_L y$.

(2) A set $C \subseteq L$ such that $(C, \leqslant_L)$ has order type ω or ω^*.

We call solutions as in (1) sequences, and solutions as in (2) chains.

Obviously, every infinite ascending/descending sequence is an ascending/descending chain.

By contrast, every infinite ascending/descending chain computes an infinite ascending/descending sequence. Is the proof uniform?

ADS revisited.

(ADS) For every linear order $(L, \leqslant_L)$, there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending sequence in L.

(ADC) For every linear order $(L, \leqslant_L)$, there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending chain in L.

By our observation, ADS $\equiv_c$ ADC, and indeed, $\mathsf{RCA}_0 \vdash$ ADS $\leftrightarrow$ ADC.

We have ADC $\leqslant_W$ ADS. Conversely, ADS is almost Weihrauch reducible to to ADC, modulo a single bit of non-uniform information.

Theorem (Astor, Dzhafarov, Solomon, and Suggs). ADS $\not\leqslant_W$ ADC.

Of course, we expect this result once we think to consider ADC. But it is a subtlety that classical reverse mathematics cannot express.

Stability and RT^2_2

Cholak, Jockusch, and Slaman introduced a method to get at the strength of RT^2_2 by looking at a simpler form of it called SRT^2_2.

Definition. $c : [\omega]^2 \to 2$ is stable if for every x, $\lim_y c(x, y)$ exists.

(SRT^2_2) For very stable coloring $c : [\omega]^2 \to 2$, there exists an infinite homogeneous set for c.

(D^2_2) For very stable coloring $c : [\omega]^2 \to 2$, there exists an infinite set L and a color $i < 2$ such that $\lim_y c(x, y) = i$ for all $x \in L$.

- (Cholak, Jockusch, and Slaman; Chong, Lempp, and Yang) $\mathsf{SRT}^2_2 \equiv_c \mathsf{D}^2_2$ and $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \leftrightarrow \mathsf{D}^2_2$.

- (Dzhafarov) $\mathsf{SRT}^2_2 \not\leq_W \mathsf{D}^2_2$ (so $\mathsf{D}^2_2 <_W \mathsf{SRT}^2_2$).

Stability and ADS

Hirschfeldt and Shore investigated stability for ADS and CAC.

Definition. $(L, \leqslant_L)$ is stable if every $x \in L$ is $\leqslant_L$-small or $\leqslant_L$-large, i.e., $(\forall^\infty y)[x <_L y]$ or $(\forall^\infty y)[x >_L y]$.

So an ordering is stable if and only if it has order type $\omega + \omega^*$, $\omega + k$, $k + \omega^*$ for some k.

But an ordering of type $\omega + k$ or $k + \omega^*$ has a trivial ADS-solution. So the reverse mathematics of ADS for these order types is uninteresting.

(SADS) For every linear order $(L, \leqslant_L)$ of order type $\omega + \omega^*$ there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending sequence in L.

Stability and ADS

Theorem (Hirschfeldt and Shore). $\mathsf{RCA}_0 + \mathsf{SADS} \nvdash \mathsf{ADS}$.

SADS has a model consisting entirely of low sets. ADS does not.

In a sense, there is a wider gap between sequences and chains than between stable orderings and non-stable ones.

Indeed, our result above that $\mathsf{ADC} \nleqslant_W \mathsf{ADS}$ can be sharpened:

Theorem (ADSS). $\mathsf{SADS} \nleqslant_W \mathsf{ADC}$ (so $\mathsf{SADS} \mid_W \mathsf{ADC}$).

We also have the following similar result. Note that $\mathsf{SADS} \leqslant_W \mathsf{SRT}^2_2$.

Theorem (ADSS). $\mathsf{SADS} \nleqslant_W \mathsf{D}^2_2$ (so $\mathsf{SADS} \mid_W \mathsf{D}^2_2$).

Stability and ADS

We introduce the following as a more natural version of SADS:

(G-SADS) For every stable linear order $(L, \leqslant_L)$ there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending sequence in L.

We have G-SADS $\equiv_c$ SADS and $\mathsf{RCA}_0 \vdash$ G-SADS $\leftrightarrow$ SADS.

We have G-SADS $\leqslant_W$ SADS.

In the other direction, SADS is almost Weihrauch reducible to G-SADS, modulo determining the order type of a linear order (which not just a single bit of information).

Proposition (ADSS). G-SADS $\nleqslant_W$ SADS (so SADS $<_W$ G-SADS).

Stability and ADC

We can combine our generalizations to obtain two further principles.

(SADC) For every linear order $(L, \leqslant_L)$ of order type $\omega + \omega^*$ there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending chain in L.

(G-SADC) For every stable linear order $(L, \leqslant_L)$ there is either an infinite $\leqslant_L$-ascending or infinite $\leqslant_L$-descending chain in L.

Proposition (ADSS). G-SADS $\nleqslant_W$ SADC.

The Weihrauch zoo.

Two forms of stable CAC

CAC and immunity

A local zoo.

Thank you for your attention.