Weak irregular principles

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Background

Applied computability theory looks at computable instances of various problems and studies the complexity of their solutions. Usual questions:

1. Is there always a computable solution?
2. Is there always a solution computable in $\emptyset'$?

Reverse mathematics seeks to calibrate the strength of (miniaturizations of) theorems according to the minimal sets of axioms needed to prove them. In practice, we use subsystems of second-order arithmetic:

1. first, we find a subsystem strong enough to prove the theorem;
2. then, we obtain sharpness, by showing that the theorem implies (reverses to) this subsystem, over some weak base theory.

There is a fruitful interaction between these two endeavors.
Background

\textbf{RCA}_0: \text{ basic axioms of arithmetic, together with induction for } \Sigma_1^0 \text{ formulas, and comprehension for sets definable by both a } \Sigma_1^0 \text{ and } \Pi_1^0 \text{ formula.}

\textbf{WKL}_0: \text{ RCA}_0 \text{ plus every infinite binary tree has an infinite path.}

\textbf{ACA}_0: \text{ RCA}_0 \text{ plus comprehension for arithmetically-definable sets.}

\textbf{ATR}_0: \text{ RCA}_0 \text{ plus any arithmetically-defined functional } F : 2^\mathbb{N} \to 2^\mathbb{N} \text{ may be iterated along any countable well-ordering, starting with any set.}

\textbf{\Pi}_1^1\text{-CA}_0: \text{ RCA}_0 \text{ plus comprehension for } \Pi_1^1\text{-definable sets.}
Background

Remarkably, most theorems fall into one of the “big five”.

**Provable in RCA\(_0\):** Baire category theorem, intermediate value theorem, Urysohn’s lemma, Tietze extension theorem, soundness theorem.

**Equivalent to WKL\(_0\):** every countable commutative ring has a prime ideal, Gödel’s compactness theorem, separable Hahn/Banach theorem, Heine/Borel theorem.

**Equivalent to ACA\(_0\):** every countable commutative ring has a maximal ideal, every bounded sequence of real numbers has a least upper bound, Ascoli lemma, Bolzano/Weierstrass theorem, the range of every function exists, the Turing jump of every set exists.
Background

\[
\begin{array}{c}
Z_2 \\
\downarrow \\
\vdots \\
\downarrow \\
\Pi_1^1-CA_0 \\
\downarrow \\
ATR_0 \\
\downarrow \\
ACA_0 \\
\downarrow \\
WKL_0 \\
\downarrow \\
RCA_0
\end{array}
\]
Background
Ramsey’s theorem

Jockusch. Fix $n, k \geq 2$.

1. Every computable $f : [\omega]^n \to k$ has a $\Pi^0_n$ infinite homogeneous set, but not necessarily a $\Delta^0_n$ (or $\Sigma^0_n$) one.

2. Every computable $f : [\omega]^n \to k$ has an infinite homogeneous set $H$ with $H' \leq_T \emptyset^{(n)}$.

3. There exists a computable $f : [\omega]^n \to 2$ every infinite homogeneous set of which computes $\emptyset^{(n-2)}$.

Corollary. Over RCA$_0$, WKL$_0$ does not imply RT$_2^2$.

Corollary. For $n \geq 3$, RT$_2^n$ is equivalent to ACA$_0$ over RCA$_0$. 
Ramsey’s theorem for pairs

**Seetapun.** For any non-computable set $C$, every computable $f : [\omega]^2 \to 2$ has an infinite homogeneous set $H$ that does not compute $C$.

**Cholak, Jockusch, and Slaman.** Every computable $f : [\omega]^2 \to 2$ has a low$_2$ infinite homogeneous set $H$, i.e., $H'' \leq_T \emptyset''$.

**Dzhafarov and Jockusch.** Every computable $f : [\omega]^2 \to 2$ has a pair of infinite homogeneous sets $H$ whose degrees form a minimal pair.

All proofs use some flavor of Mathias forcing.

**Corollary.** Over RCA$_0$, RT$_2^2$ does not imply ACA$_0$. 
Ramsey’s theorem for pairs
Ramsey’s theorem for pairs

\[ \text{RT}_2^2 \rightarrow \text{ACA}_0 \]

\[ \text{ACA}_0 \rightarrow \text{WKL}_0 \]

\[ \text{WKL}_0 \downarrow \text{WWKL}_0 \downarrow \text{DNR} \]

\[ \text{RCA}_0 \rightarrow \text{RT}_2^2 \]
Ramsey’s theorem for pairs

**Liu.** Every computable coloring $f : [\omega]^2 \to 2$ has an infinite homogeneous set not of PA degree.

**Corollary.** Over RCA$_0$, RT$_2^2$ does not imply WKL$_0$.

**Dzhafarov and Shore.** Every computable coloring $f : [\omega]^2 \to 2$ has a low$_3$ infinite homogeneous set $H$ not of PA degree.

**Question.** Does every computable coloring $f : [\omega]^2 \to 2$ have an infinite low$_2$ homogeneous set not of PA degree?
Stability and cohesiveness

A coloring $f : [\omega]^{2} \to 2$ is stable if for every $x$, $\lim_{s} f(\{x, s\})$ exists.

$\text{SRT}^{2}_{2}$: restriction of $\text{RT}^{2}_{2}$ to stable colorings.

$\text{COH}$: for every family of sets $\langle A_{0}, A_{1}, \ldots \rangle$ there is a set $X$ such that for all $i$, either $X \subseteq^{*} A_{i}$ or $X \subseteq^{*} \overline{A}_{i}$.

If $A_{0}, A_{1}, \ldots$ contain all computable sets, $X$ is $r$-cohesive; if $A_{0}, A_{1}, \ldots$ contain all c.e. sets, $X$ is cohesive.

Cholak, Jockusch, and Slaman; Mileti; Jockusch and Lempp. Over $\text{RCA}_{0}$, $\text{RT}^{2}_{2}$ is equivalent to $\text{SRT}^{2}_{2} + \text{COH}$.
Stability and cohesiveness

By the limit lemma, the computable content of $\text{SRT}_2^2$ is the same as that of the infinite subsets and co-subsets of $\Delta^0_2$ sets. In particular, every computable stable coloring has a $\Delta^0_2$ infinite homogeneous set.

Downey, Hirschfeldt, Lempp, and Solomon. There exists a computable stable coloring $f : [\omega]^2 \rightarrow 2$ with no low infinite homogeneous set.

Corollary. Over RCA$_0$, WKL$_0$ does not imply SRT$_2^2$.

Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp and Slaman. Over RCA$_0$, SRT$_2^2$ implies DNR, but COH does not.

Chong, Slaman, Yang. Over RCA$_0$, SRT$_2^2$ does not imply RT$_2^2$.

The proof uses a highly customized, non-standard model of RCA$_0$. 
Stability and cohesiveness
Stability and cohesiveness

\[ \text{SRT}_2^2 + \text{COH} \leftrightarrow \text{RT}_2^2 \rightarrow \text{WKL}_0 \]

\[ \text{COH} \leftrightarrow \text{SRT}_2^2 \rightarrow \text{WWKL}_0 \]

\[ \text{RCA}_0 \rightarrow \text{DNR} \]
Stability and cohesiveness

**Question.** Does $\text{SRT}_2^2$ imply $\text{COH}$ in $\omega$-models?

Another way to phrase the question:

**Definition (Mileti).** A degree $d$ is s-Ramsey if every $\Delta^0_2$ set has an infinite subset or cosubset in $d$.

**Question.** Is every s-Ramsey degree a cohesive degree?

**Mileti.** The only $\Delta^0_2$ degree is $0'$. There is no low$_2$ s-Ramsey degree.
Measure-theoretic approach

Question. Does SRT$_2^2$ imply COH typically?

Definition.

1 A martingale is a function $M : 2^{<\omega} \to \mathbb{Q}^\geq 0$ such that for every $\sigma \in 2^{<\omega}$, $2M(\sigma) = M(\sigma0) + M(\sigma1)$.

2 $M$ succeeds on a set $X$ if $\limsup_n M(X \upharpoonright n) = \infty$. $M$ succeeds on a class of sets $\mathcal{C}$ if it succeeds on every $X \in \mathcal{C}$.

Ville. A class of sets $\mathcal{C}$ has Lebesgue measure 0 if and only if there is a martingale that succeeds on it.
Measure-theoretic approach

**Definition.** A class $\mathcal{C}$ of $\Delta^0_2$ sets is $\Delta^0_2$ null if there is a $\emptyset'\text{-computable}$ martingale $M$ that succeeds on it.

Reasonable notion of nullity: additive, $\Delta^0_2$ is not $\Delta^0_2$ null, etc.

**Hirschfeldt and Terwijn.** The class of low sets is not $\Delta^0_2$ null.

**Definition (Dzhafarov).** A degree $d$ is almost s-Ramsey if the class of $\Delta^0_2$ set having an infinite subset or cosubset in $d$ is not $\Delta^0_2$ null.

**ASRT^2_2:** non-$\Delta^0_2$-null many $\Delta^0_2$ sets have an infinite subset or cosubset.
Measure-theoretic approach

Dzhafarov.

1. The only $\Delta^0_2$ almost s-Ramsey degree is $0'$. 
2. There exists an almost $s$-Ramsey degree $d \leq 0''$ that is not s-Ramsey. 
3. Over RCA$_0$, ASRT$_2$ is incomparable with COH. (And with WKL$_0$.) 
4. Over RCA$_0$, ASRT$_2$ is strictly stronger than DNR.

Proof idea for 2. Fix $A \in \Delta^0_2$ with no low infinite subset or cosset. 

Let $M_0, M_1, \ldots$ list all $\emptyset'$-computable martingales, and for every $i$, fix $L_i$ on which $M_i$ does not succeed with $\bigoplus_{j \leq i} L_i$ low. Let $D[0] = L_0$. 

If $\Phi^D_0$ is total, there must exist exist $n \notin A$ and a finite $F$ such that $F[0]$ agrees with $D[0]$ and $\Phi^F_0(n) \downarrow= 1$. Let $D[1]$ equal $L_1$ above $\varphi^F_0(n)$. 
Measure-theoretic approach

\[
\begin{array}{cccc}
\text{RT}_2^2 & \text{COH} & \text{SRT}_2^2 & \text{WWKL}_0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{ACA}_0 & \text{WKL}_0 & \text{WWKL}_0 & \text{DNR} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{RCA}_0 & \text{SRT}_2^2 & \text{WKL}_0 & \text{WWKL}_0 \\
\end{array}
\]
Measure-theoretic approach
Variations of $\text{RT}_2^2$

Definition (Erdős and Rado). Fix a coloring $f : [\omega]^n \to 2$.

1. A $p$-homogeneous set for $f$ is a sequence of infinite sets $\langle H_0, \ldots, H_{n-1} \rangle$ such that $f$ is constant on all $\{x_0, \ldots, x_{n-1}\}$ with $x_i \in H_i$ for each $i < n$.

2. If $f$ is merely constant on all such $\{x_0, \ldots, x_{n-1}\}$ with $x_0 < \cdots < x_{n-1}$, then $\langle H_0, \ldots, H_{n-1} \rangle$ is an increasing $p$-homogeneous set.

$\text{PT}^n_k$: every $f : [\omega]^n \to k$ has a $p$-homogeneous set.

$\text{IPT}^n_k$: every $f : [\omega]^n \to k$ has an increasing $p$-homogeneous set.

We can define stable versions, $\text{SPT}^2_2$ and $\text{SIPT}^2_2$, in the obvious way.
Variations of RT$_2^2$

Combinatorially, PT and RT$_2^2$ are rather different.

Define $f : [\omega]^2 \rightarrow 2$ by

$$f(x, y) = \begin{cases} 0 & x \equiv y \mod 2 \\ 1 & \text{else} \end{cases}$$

No infinite set is homogeneous for $f$ with color 1, but $\langle \text{evens, odds} \rangle$ is a p-homogeneous set for $f$ with color 1.
Variations of $\text{RT}_2^2$

It is easy to see that for all $n$, $\text{RT}_2^n \rightarrow \text{PT}_2^n \rightarrow \text{IPT}_2^n$ over RCA$_0$.

Dzhafarov and Hirst.

1. For $n \geq 3$, $\text{PT}_k^n$ and $\text{IPT}_k^n$ are equivalent to $\text{RT}_k^n$ over RCA$_0$.

2. Over RCA$_0$, $\text{PT}_2^2$ is equivalent to $\text{RT}_2^2$.

3. Over RCA$_0 + B\Sigma_2^0$, $\text{IPT}_2^2$ implies $\text{SRT}_2^2$.

4. Over RCA$_0 + B\Sigma_2^0$, $\text{SPT}_2^2$ and $\text{SIPT}_2^2$ are equivalent to $\text{SRT}_2^2$.

Proof outline of 2. To show: $\text{PT}_2^2 \rightarrow \text{RT}_2^2$. $\text{PT}_2^2$ implies $\text{SPT}_2^2$, hence $\text{SRT}_2^2$ over $B\Sigma_2^0$. Thus, it suffices to show that $\text{PT}_2^2$ implies $\text{COH} + B\Sigma_2^0$. Show $\text{PT}_2^2$ implies $\text{ADS}$, from which both $\text{COH}$ and $B\Sigma_2^0$ follow.

Chong, Lempp, and Yang. Parts 3 and 4 go through in $I\Sigma_1^0$. 
Variations of $RT^2_2$
Variations of $RT_2^2$
Consequence of Ramsey’s theorem

Hirschfeldt and Shore studied various consequences of $\text{RT}_2^2$:

**CAC:** every partial ordering on $\omega$ has an infinite chain or an infinite antichain.

**ADS:** every linear ordering on $\omega$ has an infinite ascending sequence or an infinite descending sequence.

They also considered:

1. stable and cohesive versions of these principles: **SCAC**, **SADS**, **CCAC**, **CADS**, with $\text{ADS} \iff \text{SADS} + \text{CADS}$ and $\text{CAC} \iff \text{SCAC} + \text{CCAC}$.

2. a stronger version of COH, **StCOH**, equivalent to it under $\text{B}\Sigma^0_2$. 
Consequence of Ramsey’s theorem
Consequence of Ramsey’s theorem
Model-theoretic principles

A (countable, consistent) theory \( T \) is **atomic** if every \( T \)-consistent formula is provably implied by some atom (complete formula) of \( T \).

A model of \( T \) is **atomic** if every type of \( T \) realized in it is principal.

Hirschfeldt, Shore, and Slaman studied the following:

**AMT**: every complete atomic theory has an atomic model.

**OPT**: for every theory \( T \) and every set \( S \) of partial types of \( T \), there is a model of \( T \) that omits all the non-principal members of \( S \).

**\( \Pi_1^0 G \)**: for every uniformly \( \Pi_1^0 \) sequence \( U_0, U_1, \ldots \) of dense subsets of \( 2^{<\omega} \), there exists a \( G \) that meets each \( U_i \).
Model-theoretic principles

Hirschfeldt, Shore, and Slaman. The following hold in RCA$_0$:

1. $\Pi^0_1 G \rightarrow$ AMT, strictly.
2. $\text{SADS} \rightarrow \text{AMT} \rightarrow \text{OPT}$, strictly.
3. $\text{COH} \rightarrow \text{OPT}$, strictly.
4. OPT is equivalent to the statement that for every $X$, there exists a set hyperimmune relative to $X$.

Some of the weakest principles that are not computably true.

Hirschfeldt, Shore, and Slaman also identified a principle that can claim to be the weakest such principle: $\text{AST}$ is a weak form of OPT equivalent to the statement that for every $X$, there exists $Y \not\leq_T X$. 
Model-theoretic principles

Hirschfeldt, Shore, and Slaman.

1 AMT is $\Pi^1_1$ conservative over $B\Sigma^0_2$.

2 AMT and $\Pi^0_1G$ are $r-\Pi^1_2$ conservative over $\text{RCA}_0$; i.e., conservative for statements of the form

$$(\forall X)[\varphi(X) \rightarrow (\exists Y)\psi(X, Y)],$$

where $\varphi$ is arithmetical and $\psi$ is $\Sigma^0_3$.

3 Over $\text{RCA}_0 + B\Sigma^0_2$, $\Pi^0_1G \rightarrow I\Sigma^0_2$. Over $\text{RCA}_0 + I\Sigma^0_2$, AMT $\rightarrow \Pi^0_1G$.

$r-\Pi^1_2$ conservativity was first shown for $\text{COH}$. The same proof is used for all three principles, only the forcing notion differs: Cohen forcing for $\Pi^0_1G$ and AMT, Mathias forcing for $\text{COH}$. 
Model-theoretic principles

\[
\begin{array}{c}
\text{RT}_2 \downarrow \\
\text{CAC} \\
\downarrow \\
\text{ADS} \\
\downarrow \\
\text{StCOH} \\
\downarrow \\
\text{COH} \\
\downarrow \\
\text{CADS} \\
\downarrow \\
\text{RCA}_0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{IPT}_2 \\
\downarrow \\
\text{SCAC} \\
\downarrow \\
\text{SADS} \\
\downarrow \\
\text{RCA}_0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{SRAM} \\
\downarrow \\
\text{ASRAM} \\
\downarrow \\
\text{WWKL}_0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{WKL}_0 \\
\downarrow \\
\text{WWKL}_0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{DNR} \\
\end{array}
\]
Model-theoretic principles

\[\begin{align*}
\Pi_1^0 G &\quad \rightarrow \quad ACA_0 \\
\Pi_1^G &\quad \rightarrow \quad RT_2^2 \rightarrow \quad CAC \quad \rightarrow \quad ADS \rightarrow \quad StCOH \rightarrow \quad \Pi_1^0 G \\
\Pi_1^G &\quad \rightarrow \quad IPT_2^2 \rightarrow \quad SCAC \rightarrow \quad SADS \rightarrow \quad COH \\
\Pi_1^G &\quad \rightarrow \quad SRT_2^2 \rightarrow \quad ASRAM \rightarrow \quad WWKL_0 \rightarrow \quad DNR \\
\Pi_1^G &\quad \rightarrow \quad ASRAM \rightarrow \quad ASRT_2^2 \rightarrow \quad WWKL_0 \\
\Pi_1^G &\quad \rightarrow \quad WWKL_0 \rightarrow \quad DNR \\
\Pi_1^G &\quad \rightarrow \quad SADs \rightarrow \quad AMT \rightarrow \quad OPT \rightarrow \quad AST \\
\Pi_1^G &\quad \rightarrow \quad AMT \rightarrow \quad OPT \rightarrow \quad AST \\
\Pi_1^G &\quad \rightarrow \quad AST \\
\Pi_1^G &\quad \rightarrow \quad RCA_0
\end{align*}\]
Equivalents of choice

Fix $n \geq 2$. A family $A$ of sets has the

1. $F$ intersection property if $\bigcap F \neq \emptyset$ for all finite $F \subseteq A$.
2. $D_n$ intersection property if $\bigcap F = \emptyset$ for all $F \subseteq A$ of size $n$.
3. $\overline{D}_n$ intersection property if $\bigcap F \neq \emptyset$ for all $F \subseteq A$ of size $n$.

$FIP$: every non-trivial family has a maximal subfamily with the $F$ intersection property.

$D_nIP$: every non-trivial family has a maximal subfamily with the $D_n$ intersection property.

$\overline{D}_nIP$: every non-trivial family has a maximal subfamily with the $\overline{D}_n$ intersection property.
Equivalents of choice

Classically, $FIP$, $D_nIP$, and $\overline{D}_nIP$ are equivalent to the axiom of choice.

Dzhafarov and Mummert.

1. For all $n$, $D_nIP$ is equivalent to ACA$_0$ over RCA$_0$.
2. For all $n$, $FIP \rightarrow D_{n+1}IP \rightarrow D_nIP$.
3. There is an $\omega$-model of $FIP$ consisting entirely of low sets.
4. $FIP$ is $r$-$\Pi^1_2$ conservative over RCA$_0$.

Proof of 3. Force with conditions $(\sigma, s) \in \omega^{<\omega} \times \omega$ such that some number $\leq s$ belongs to $\bigcap_{i<|\sigma|} A_{\sigma(i)}$, and $(\tau, t) \leq (\sigma, s)$ if $\sigma \preceq \tau$. Let $(\sigma_0, s_0) = (\emptyset, 0)$. At even stages, force the jump. Given $(\sigma_i, s_i)$ for $i$ odd, choose the least $s \geq s_i$ large enough to bound an element of $A_i \cap \bigcap_{j<|\sigma|} A_{\sigma(j)}$ if non-empty, and let $(\sigma_{i+1}, s_{i+1}) = (\sigma_i \triangleleft i, s)$. 

Equivalents of choice

$\overline{D}_2$IP is computably false. The obvious computable strategy fails:

Given $A = \langle A_0, A_1, \ldots \rangle$ non-trivial, define a subfamily $B = \langle B_0, B_1, \ldots \rangle$ as follows. Having defined $B_i$ for each $i < n$, search through $A$ to find the first non-empty $A_i$ not among the $B_i$ but intersecting each of them, and let this be $B_n$.

$B$ has the $\overline{D}_2$ intersection property, but it may fail to be maximal. It may be that $A_0$ intersects every set, but that it intersects $A_1, \ldots, A_n$ only after $A_{n+1}$ does. Then $B$ will equal $\langle A_1, A_2, \ldots \rangle$.

Can turn this problem into a proof that $\overline{D}_2$IP is not provable in RCA$_0$. 
Equivalents of choice

Dzhafarov and Mummert. There exists a computable non-trivial $A = \langle A_0, A_1, \ldots \rangle$ every maximal subfamily of which with the $\overline{D}_2$ intersection property has hyperimmune degree.

Proof sketch. We build $A$ along with finitely-branching trees $T_0, T_1, \ldots \subseteq \omega^{<\omega}$, and for every $J \in \omega^{\omega}$ a partial $J$-computable function $f_J$, so as to meet the following requirements:

$Q$ : if $J \in \omega^{\omega}$ defines a maximal subfamily of $A$ with the $\overline{D}_2$ intersection property then $f_J$ is total;

$R_e$ : if $J \in \omega^{\omega}$ defines a subfamily of $A$ with the $\overline{D}_2$ intersection property, and if $f_J$ is total and bounded by $\Phi_e$, then $J \in [T_e]$;

$S_e$ : no infinite path through $T_e$ defines a maximal subfamily of $A$ with the $\overline{D}_2$ intersection property.
Equivalents of choice

For each $e$, define sequence $P_e^0, P_e^1, \ldots \in A$ of $e$-prevention sets.

For some $e$, also define a missing set $M_e \in A$.

Goal: if $J$ defines a maximal subfamily and $f_J$ is total and bounded by $\Phi_e$, then $M_e$ intersects every $A_{J(i)}$, yet the $P_e^m$ prevent $M_e$ being in $J$.

Each $P_e^m$ will be a prevention set for some $\sigma \in \omega^{<\omega}$, representing that any maximal subfamily containing all $A_{\sigma(i)}$ should also contain $P_e^m$.

At stage $s$:

1. we consider each $\sigma \in \omega^{<\omega}$ bounded by $s$, and define a new $P_e^m$ for it;
2. for each $P_e^m$ defined at a previous stage for some $\sigma$, look at each $\tau \succ \sigma$ bounded by $s$ that only intersect $M_e$ if $\sigma$ does, and intersect $P_e^m$ with every set enumerated by $\tau$. 
Equivalents of choice

Say $\sigma$ e-extends $\tau$ if $\sigma$ enumerates a $P^m_\tau$ for some $\tau \leq \rho < \sigma$.

For $J \in \omega^\omega$, $f_J$ is defined as follows:

1. let $r_{-1} = 0$;
2. given $r_{i-1}$ let $r_i$ be least so that $J \upharpoonright r_i$ $j$-extends $J \upharpoonright r_{i-1}$ for each $j \leq i$;
3. let $f(0)$ be the least number that bounds $\bigcap_{j < r_i} A_J(j)$. 
Equivalents of choice

Say $\sigma$ e-extends $\tau$ if $\sigma$ enumerates a $P^m_\epsilon$ for some $\tau \preceq \rho < \sigma$.

For $J \in \omega^\omega$, $f_J$ is defined as follows:

1. let $r_{-1} = 0$;
2. given $r_{i-1}$ let $r_i$ be least so that $J \upharpoonright r_i$ j-extends $J \upharpoonright r_{i-1}$ for each $j \leq i$;
3. let $f(0)$ be the least number that bounds $\bigcap_{j<r_i} A_J(j)$.

\[
\begin{array}{c}
P^7_0 \\
\hline
J \upharpoonright r_0
\end{array}
\]
Equivalents of choice

Say $\sigma$ e-extends $\tau$ if $\sigma$ enumerates a $P^m_\rho$ for some $\tau \leq \rho < \sigma$.

For $J \in \omega^\omega$, $f_J$ is defined as follows:

1. let $r_{-1} = 0$;
2. given $r_{i-1}$ let $r_i$ be least so that $J \upharpoonright r_i$ j-extends $J \upharpoonright r_{i-1}$ for each $j \leq i$;
3. let $f(0)$ be the least number that bounds $\bigcap_{j < r_i} A_{J(j)}$. 

\[ P_7^0 \quad P_{11}^0 \quad P_2^1 \]

\[ J \upharpoonright r_0 \quad J \upharpoonright r_1 \]
Equivalents of choice

Say $\sigma$ e-extends $\tau$ if $\sigma$ enumerates a $P^m_\tau$ for some $\tau \preceq \rho < \sigma$.

For $J \in \omega^\omega$, $f_J$ is defined as follows:

1. let $r_{-1} = 0$;
2. given $r_{i-1}$ let $r_i$ be least so that $J \upharpoonright r_i$ $j$-extends $J \upharpoonright r_{i-1}$ for each $j \leq i$;
3. let $f(0)$ be the least number that bounds $\bigcap_{j < r_i} A_{J(j)}$.

\[
P_0^7 \quad P_0^{11} \quad P_1^2 \quad P_0^{14} \quad P_1^4 \quad P_2^8 \quad \ldots
\]
\[
J \upharpoonright r_0 \quad \quad \quad \quad \quad \quad \quad J \upharpoonright r_1 \quad \quad \quad \quad \quad \quad \quad J \upharpoonright r_2
\]
Equivalents of choice

Definition of $T_e$ via approximations $T_e[0], T_e[1], \ldots$:

1. Let $T_e[0] = \emptyset$.

2. Given $T_e[s - 1]$, suppose $s$ is least such that $\Phi_{e,s}(i) \downarrow$.

3. Let $T_e[s]$ consist of all $\sigma$ for which there exists a leaf $\tau$ of $T_e[s - 1]$ such that $\sigma$ $j$-extends $\tau$ for each $j \leq i$.

$T_e$ is the intersection of the upward closure of the $T_e[s]$.

If $i = e$ above, define $M_e$.

If $M_e$ is defined, and there is an $r$ so that each leaf $\sigma$ of $T_e[s]$ $e$-extends $\sigma \upharpoonright r$ with witness $P^m_e$ disjoint from $M_e$ and not enumerated by any string of length $r$, intersect $A_{\sigma(i)}$ with $M_e$ for all $i < r$. 
Equivalents of choice
Equivalents of choice

\[
\text{define } M(e) \downarrow
\]
Equivalents of choice

\[ \Phi(e) \downarrow \ ]
Equivalents of choice

\[ \Phi_e(e) \downarrow \text{ define } M_e \]
Equivalents of choice

\[ M \Phi (\ell) \]
Equivalents of choice
Equivalents of choice

\[ P_e^3 \quad P_e^1 \quad P_e^8 \]

\(\Phi(e)\)
Equivalents of choice

$P^3_e, P^8_e, P^1_e$ do not have $P^3_e, P^8_e, P^1_e$ intersect with $M_e$
Equivalents of choice

The diagram shows a structure $M_e$ with equivalence classes $P^3_e, P^8_e, P^1_e$ that do not intersect with $M_e$. The structure is defined by the downward arrow $\Phi(e)$.
Equivalents of choice

$P^3_e, P^8_e, P^1_e$ do not have $\Phi(e)$ intersect with $M_e$.

Ensures that $M_e$ cannot be on by any path.
Equivalents of choice

Corollary. Over RCA₀, \( \overline{D}_2 \text{IP} \) implies OPT.

Dzhafarov and Mummert.
1 Over RCA₀, \( \Pi_1^0 \text{G} \) implies FIP.
2 Every non-zero c.e. set can compute a maximal subfamily with the \( F \) intersection property of any computable non-trivial family of sets.

(Most other choice principles live above ATR₀, some even above \( Z_2 \).)

Csima, Hirschfeldt, Knight, and Soare. No \text{low}_2 \Delta_2^0 set can compute an atomic model for every complete atomic decidable theory.

Corollary. There exists an \( \omega \)-model of RCA₀ + FIP + \( \neg \text{AMT} \).

Question. Does OPT imply \( \overline{D}_2 \text{IP} \)? Does AMT imply FIP?
Equivalents of choice

\[ \Pi_1^0 G \quad \rightarrow \quad ACA_0 \quad \rightarrow \quad RT_2^2 \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad CAC \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad ADS \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad StCOH \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad COH \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad CADS \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad CADS \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \rightarrow \quad \Pi_1 G \quad \rightarrow \quad ACA_0 \]
Equivalents of choice

\[ \begin{array}{c}
\text{ACA}_0 \\
\text{RT}_2^2 \\
\text{CAC} \\
\text{ADS} \\
\text{StCOH} \\
\Pi_1^0 G \\
\text{FIP} \\
D_2 \text{IP} \\
\text{COH} \\
\text{CADS} \\
\text{SCAC} \\
\text{SRT}_2^2 \\
\text{IPT}_2^2 \\
\text{SRAM} \\
\text{ASRAM} \\
\text{WKL}_0 \\
\text{WWKL}_0 \\
\text{DNR} \\
\text{OPT} \\
\text{AST} \\
\text{RCA}_0
\end{array} \]
Thank you for your attention.